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C*-Algebras
Volume 2: Banach Algebras and Compact Operators

C. Constantinescu

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C*-ALGEBRAS

VOLUME 2:
BANACH ALGEBRAS AND COMPACT OPERATORS

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Preface

Functional analysis plays an important role in the program of studies at the Swiss Federal Institute of Technology. At present, courses entitled Functional Analysis I and II are taken during the fifth and sixth semester, respectively. I have taught these courses several times and after a while typewritten lecture notes resulted that were distributed to the students. During the academic year 1987/88, I was fortunate enough to have an eager enthusiastic group of students that I had already encountered previously in other lecture courses. These students wanted to learn more in the area and asked me to design a continuation of the courses. Accordlingly, I proceeded during the academic year, following, with a series of special lectures, Functional Analysis III and IV, for which I again distributed typewritten lecture notes. At the end I found that there had accumulated a mass of textual material, and I asked myself if I should not publish it in the form of a book. Unfortunately, I realized that the two special lecture series (they had been given only once) had been badly organized and contained material that should have been included in the first two portions. And so I came to the conclusion that I should write everything anew – and if at all – then preferably in English. Little did I realize what I was letting myself in for! The number of pages grew almost impercepetibly and at the end it had more than doubled. Aslo, the English language turned out to be a stumbling block for me; I would like to take this opportunity to thank Prof. Imre Bokor and Prof. Edgar Reich for their help in this regard. Above all I must thank Mrs. Barbara Aquilino, who wrote, first a WordMARC™, and then a LaTeX™ version with great competence, angelic patience, and utter devotion, in spite of illness. My thanks also go to the Swiss Federal Institute of Technology that generously provided the infrastructure for this extensive enterprise and to my colleagues who showed their understanding for it.

Corneliu Constantinescu

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Introduction

This book has evolved from the lecture course on Functional Analysis I had given several times at the ETH. The text has a strict logical order, in the style of "Definiton – Theorem – Proof – Example – Exercises". The proofs are rather thorough and there are many examples.

The first part of the book (the first three chapters, resp. the first two volumes) is devoted to the theory of Banach spaces in the most general sense of the term. The purpose of the first chapter (resp. first volume) is to introduce those results on Banach spaces which are used later or which are closely connected with the book. It therefore only contains a small part of the theory, and several results are stated (and proved) in a diluted form. The second chapter (which together with Chapter 3 makes the second volume) deals with Banach algebras (and involutive Banach algebras), which constitute the main topic of the first part of the book. The third chapter deals with compact operators on Banach spaces and linear (ordinary and partial) differential equations – applications of the theory of Banach algebras.

The second part of the book (the last four chapters, resp. the last three volumes) is devoted to the theory of Hilbert spaces, once again in the general sense of the term. It begins with a chapter (Chapter 4, resp. Volume 3) on the theory of C^*–algebras and W^*–algebras which are essentially the focus of the book. Chapter 5 (resp. Volume 4) treats Hilbert spaces for which we had no need earlier. It contains the representation theorems, i.e. the theorems on isometries between abstract C^*–algebras and the concrete C^*–algebras of operators on Hilbert spaces. Chapter 6 (which together with Chapter 7 makes Volume 5) presents the theory of $\mathcal{L}^p$–spaces of operators, its application to the self–adjoint linear (ordinary and partial) differential equations, and the von Neumann algebras. Finally, Chapter 7 presents examples of C^*–algebras defined with the aid of groups, in particular the Clifford algebras. Many important domains of C^*–algebras are ignored in the present book. It should be emphasized that the whole theory is constructed in parallel for the real and for the complex numbers, i.e. the C^*–algebras are real or complex.

In addition to the above (vertical) structure of the book, there is also a second (horizontal) division. It consists of a main strand, eight branches, and additional material. The results belonging to the main strand are marked with (0). Logically speaking, a reader could restrict himself/herself to these and ignore the rest. Results on the eight subsidiary branches are marked with (1), (2), (3), (4), (5), (6), (7), and (8). The key is

1. Infinite Matrices
2. Banach Categories
3. Nuclear Maps
4. Locally Compact Groups
5. Differential Equations
6. Laurent Series
7. Clifford Algebras
8. Hilbert C^*–Modules

These are (logically) independent of each other, but all depend on the main strand. Finally, the results which belong to the additional material have no marking and – from a logical perspective – may be ignored. So the reader can shorten for himself/herself this very long book using the above marks. Also, since the proofs are given with almost all references, it is possible to get into the book at any level and not to read it linearly.

We assume that the reader is familiar with classical analysis and has rudimentary knowledge of set theory, linear algebra, point–set topology, and integration theory. The book addresses itself mainly to mathematicians, or to physicists interested in C^*–algebras.

I would like to apologize for any omissions in citations occasioned by the fact that my acquaintance with the history of functional analysis is, unfortunately, very restricted. For this history we recommand the following texts.

1. BIRKHOFF, G. and KREYSZIG, E., The Establishment of Functional Analysis, Historia Mathematica 11 (1984), 258–321.
2. BOURBAKI, N., Elements of the History of Mathematics, (21. Topological Vector Spaces), Springer–Verlag (1994).
3. DIEUDONNÉ, J., History of Functional Analysis, North–Holland (1981).
4. DIEUDONNÉ, J., A Panorama of Pure Mathematics (Chapter C III: Spectral Theory of Operators), Academic Press (1982).
5. HEUSER, H., Funktionalanalysis, 2. Auflage (Kapitel XIX: Ein Blick auf die werdende Functionalanalysis), Teubner (1986), 3. Auflage (1992).
6. KADISON, R.V., Operator Algebras, the First Forty Years, in: Proceedings of Symposia in Pure Mathematics 38 I (1982), 1–18.
7. MONNA, A.F., Functional Analysis in Historical Perspective, John Whiley & Sons (1973).

8. STEEN, L.A., Highlights in the History of Spectral Theory, Amer. Math. Monthly 80 (1973), 359–382.

There is no shortage of excellent books on C^*–algebras. Nevertheless, we hope that this book will be also of some utility to the mathematics commutity.

2. Banach Algebras

Banach algebras are the cornerstone of this book. In fact it would be fair to say that the first chapter merely presented the pre–requisites and that the book begins properly with this chapter.

2.1 Algebras

This section collects definitions and facts from algebra in preparation for the rest of the chapter. The concept of a spectrum is introduced. It plays a central rôle in this book, particularly in two special cases. The first concerns compact operators, where the spectrum is simply the set of eigenvalues of the operator, (with the possible exception of 0). The second case concerns the normal (self-fadjoint) elements of a complex (real) C^*–algebra, where the spectrum is the image set of an associated scalar valued function on a locally compact space.

2.1.1 General Results

Definition 2.1.1.1 (0) *A* ***multiplication*** *on a vector space E is a bilinear map*

$$E \times E \longrightarrow E\,, \quad (x,y) \longmapsto xy$$

which is associative, i.e.

$$(xy)z = x(yz)$$

for all $x, y, z \in E$. xy is called ***the product of x and y***. *An* ***algebra*** *is a vector space endowed with a multiplication. An algebra is called* ***real (complex)*** *if the underlying vector space is real (complex). An algebra E is called* ***degenerate*** *if $E = \{0\}$.*

Let E be an algebra. A ***subalgebra (a left ideal, a right ideal)*** *of E is a vector subspace F of E such that*

$$xy \in F$$

for every $x, y \in F$ *(for every* $(x, y) \in E \times F$*, for every* $(x, y) \in F \times E$*). An* ***ideal of*** E *is a subset of* E *which is at the same time a left and a right ideal. A* ***proper (left, right) ideal of*** E *is a (left, right) ideal of* E *different from* E*. A* ***maximal proper (left, right) ideal of*** E *is a proper (left, right) ideal* F *of* E*, such that* F *is the only proper (left, right) ideal of* E *containing* F*. We define* x^n *inductively for* $x \in E$ *and* $n \in \mathbb{N}$ *by*

$$x^1 := x, \quad x^{n+1} := xx^n = x^n x.$$

An element x *of* E *is called* ***nilpotent*** *if* $x^n = 0$ *for some* $n \in \mathbb{N}$ (B. Pierce, 1870)*. An element* x *of* E *is called a* ***zero-divisor*** *if there is a* $y \in E \backslash \{0\}$ *such that*

$$xy = 0 \quad or \quad yx = 0.$$

A subset of E *is called* ***commutative*** *if*

$$xy = yx$$

for every $x, y \in A$*. A* ***unit of*** E *is an element* e *of* E *such that*

$$ex = xe = x$$

for every $x \in E$*.*

Every complex algebra becomes a real algebra if we restrict the scalar multiplication to $\mathbb{R}$. Every left (right) ideal of an algebra is a subalgebra.

There is at most one unit in an algebra for if e and e' are units, then

$$e = ee' = e'.$$

Because of the associativity, in expressions like $x(yz)$ and $(xy)z$ we usually omit the parantheses. If follows from the bilinearity of the multiplication that

$$x0 = 0x = 0$$

for every element x of an algebra.

Every subalgebra of an algebra is an algebra with respect to the induced structure.

In commutative algebras the notions ideal, left ideal, and right ideal coincide.

Proposition 2.1.1.2 (0) *Let E be an algebra. The intersection of a nonempty family of (left, right) ideals of E is a (left, right) ideal of E. In particular, every $A \subset E$ is contained in a smallest (left, right) ideal of E, called the (left, right) ideal of E generated by A.* ■

Definition 2.1.1.3 (0) *A **unital algebra** is an algebra endowed with a unit which is different from 0. The unit is generally denoted by 1. Define*

$$\mathbb{K}1 := \{\alpha 1 \mid \alpha \in \mathbb{K}\}$$

and

$$x^0 := 1$$

*for every element x of a unital algebra. A **unital subalgebra** of a unital algebra is a subalgebra containing the unit of the algebra.*

We emphasize that a unital algebra is not merely an algebra which has a neutral element for its multiplication. To be unital, such a neutral element must be distinguished in the sense that the neutral element in question is an essential component of the algebra structure. We are obliged to make this distinction since in practice we may be confronted with a subalgebra F of a unital algebra E, which has a unit different from the unit of E. This unit of F is somehow accidental and has not the meaning of a true unit for the problems formulated on E. So in this case F has to be treated only as an algebra and not as a unital algebra. In particular we make the convention that F endowed with the induced structure is unital iff E is unital and F is a unital subalgebra of E.

An example to clarify this is provided by the algebra $\mathbb{K}_{2,2}$ of 2×2–matrices with entries from $\mathbb{K}$, which is clearly a unital algebra over $\mathbb{K}$ with $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ as unit (Proposition 2.1.4.24 b),c)). $\mathbb{K}$ can be identified with

$$E := \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix} \;\middle|\; \alpha \in \mathbb{K} \right\}$$

and is clearly also a unital algebra, with $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as unit. Since the unit of E and that of $\mathbb{K}_{2,2}$ do not conincide, we cannot consider E to be a subalgebra of $\mathbb{K}_{2,2}$ "in the sense of unital algebras", but only in the sense of algebras in general, i.e. algebras not required to admit a unit.

$\mathbb{K}$ will be considered as a unital algebra with respect to the usual multiplication.

0 is not excluded in the above definition of x^0, i.e. we define

$$0^0 := 1 .$$

Proposition 2.1.1.4 (0) *Let E be a (unital) algebra. The intersection of a nonempty family of (unital) subalgebras of E is a (unital) subalgebra of E. In particular, every $A \subset E$, is contained in a smallest (unital) subalgebra of E called* ***the (unital) subalgebra of E generated by A****. Given $x, y \in E$, the (unital) subalgebra of E generated by $\{x\}$ (by $\{x, y\}$) is called* ***the (unital) subalgebra of E generated by x (by x and y)****.* ■

Proposition 2.1.1.5 (0) *If E is a unital algebra then*

$$(-1)x = x(-1) = -x$$

for every $x \in E$.

We have that

$$x + (-1)x = 1x + (-1)x = (1 - 1)x = 0x = 0 ,$$

$$x + x(-1) = x1 + x(-1) = x(1 - 1) = x0 = 0 .$$ ■

Definition 2.1.1.6 (0) *Let $u : E \to F$ be a linear map between the (unital) algebras, E and F. Then u is called* ***(unital) algebra homomorphism*** *if given $x, y \in E$,*

$$u(xy) = (ux)(uy)$$

(and $u1 = 1$). A bijective (unital) algebra homomorphism is called a ***(unital) algebra isomorphism****. Two (unital) algebras E, F are called* ***isomorphic*** *if there is a (unital) algebra isomorphism $E \to F$.*

Let E, F be algebras and $u : E \to F$ be a surjective algebra homomorphism. If E has a unit 1, then $u1$ is a unit of F. In particular, if E and F are unital algebras, u is also unital.

Proposition 2.1.1.7 (0) *Let E be an algebra and F be the vector space $\mathbb{K} \times E$ endowed with the bilinear map*

$$F \times F \longrightarrow F , \quad ((\alpha, x), (\beta, y)) \longmapsto (\alpha\beta, \alpha y + \beta x + xy) .$$

Then F is a unital algebra with $(1,0)$ as unit. $\{0\} \times E$ is an ideal of F of codimension one. F is commutative iff E is commutative. If G is a unital algebra and $u : E \to G$ is an algebra homomorphism, then

$$F \longrightarrow G\,, \quad (\alpha, x) \longmapsto \alpha 1 + ux$$

is the unique unital algebra homomorphism extending u (here E is identified with $\{0\} \times E$).

The proof is a simple verification. ■

Definition 2.1.1.8 (0) *Let E be an algebra. If E is not unital then, up to an isomorphism of unital algebras keeping the points of E fixed, there is a unique unital algebra for which E is an ideal of codimension one (Proposition 2.1.1.7). This unital algebra is called* ***the unital algebra associated to*** *E. If E is unital then* ***the unital algebra associated to*** *E will be E itself.*

Convention *We treat every algebra as imbedded in its associated unital algebra. Many results formulated for unital algebras then extend automatically to all algebras. In particular, the symbol 1 will be used for any algebra and is to be understood as the unit of the associated unital algebra.*

Proposition 2.1.1.9 (0) *Let E, F be (unital) algebras and $u : E \to F$ a (unital) algebra homomorphism. Let G be a (unital) subalgebra of E and A a commutative set of E. Then $u(G)$ is a (unital) subalgebra of F and $u(A)$ is a commutative set of F.* ■

Proposition 2.1.1.10 (0) *Let E, F be algebras, $u : E \to F$ an algebra homomorphism, and G a subalgebra (an ideal) of F. Then $\overset{-1}{u}(G)$ is a subalgebra (an ideal) of E. Hence if $\{0\}$ is the only proper ideal of E, then either $u = 0$ or u is injective.* ■

Proposition 2.1.1.11 (0) *Let E, F be unital algebras, $u : E \to F$ a unital algebra homomorphism, and G a proper ideal of F. Then $\overset{-1}{u}(G)$ is a proper ideal of E.*

$\overset{-1}{u}(G)$ is an ideal of E such that $1 \notin \overset{-1}{u}(G)$ ■

Proposition 2.1.1.12 (0) *Let E be a unital algebra and $x' : E \to \mathbb{K}$ an algebra homomorphism. If x' is not identically zero, then it is a unital algebra homomorphism and $\operatorname{Ker} x'$ is a maximal proper ideal of E.*

Now

$$x'(1) = x'(1^2) = x'(1)^2 .$$

Since

$$x'(1) \neq 0$$

it follows that

$$x'(1) = 1 .$$

Hence x' is a unital algebra homomorphism.

By Proposition 2.1.1.11, Ker x' is a proper ideal of E. Since it has codimension 1, it must be maximal. ■

Proposition 2.1.1.13 (0) *Let E be an algebra, F an ideal of E, and $q : E \to E/F$ the quotient map.*

a) *Given $X, Y \in E/F$ and $x_1, x_2 \in X$, $y_1, y_2 \in Y$,*

$$q(x_1 y_1) = q(x_2 y_2) .$$

Define

$$XY := q(x_1 y_1) .$$

b) *The map*

$$E/F \times E/F \longrightarrow E/F , \quad (X, Y) \longmapsto XY$$

is a multiplication. Equip E/F with this multiplication.

c) *q is an algebra homomorphism.*

d) *If E is a unital algebra and F is proper, then E/F equipped with $q1$ is a unital algebra and q is a unital algebra homomorphism.*

e) *If E is commutative, then E/F is commutative.*

a) We have

$$x_1 y_1 - x_2 y_2 = x_1 y_1 - x_1 y_2 + x_1 y_2 - x_2 y_2 =$$

$$= x_1 (y_1 - y_2) + (x_1 - x_2) y_2 \in F + F \subset F .$$

b) Take $X, Y, Z \in E/F$ and $\alpha, \beta \in \mathbb{K}$. Take further $x \in X$, $y \in Y$, and $z \in Z$. Then

$$\alpha x + \beta y \in \alpha X + \beta Y ,$$

$$(\alpha x + \beta y)z \in (\alpha X + \beta Y)Z ,$$

$$xz \in XZ , \quad yz \in YZ ,$$

$$(\alpha x + \beta y)z = \alpha xz + \beta yz \in \alpha XZ + \beta YZ .$$

Thus

$$(\alpha X + \beta Y)Z = \alpha XZ + \beta YZ .$$

Similarly,

$$Z(\alpha X + \beta Y) = \alpha ZX + \beta ZY .$$

From

$$x(yz) \in X(YZ) , \quad (xy)z \in (XY)Z$$

$$x(yz) = (xy)z$$

it follows that

$$X(YZ) = (XY)Z .$$

c) is trivial.

d) Take $X \in E/F$ and $x \in X$. Then

$$Xq(1) = q(x1) = q(x) = X ,$$

$$q(1)X = q(1x) = q(x) = X .$$

$q(1)$ is therefore a unit of E/F and q is a unital algebra homomorphism.

e) follows from c) and Proposition 2.1.1.9. ■

Proposition 2.1.1.14 (0) *Let E be a unital algebra.*

a) *A (left,right) ideal F of E is proper iff $1 \notin F$.*

b) *The set of proper (left,right) ideals of E is inductively ordered with respect to inclusion.*

c) *Every proper (left, right) ideal of E is contained in a maximal proper (left, right) ideal of E.*

a) If $1 \in F$, then

$$x = x1 \in F \quad (x = 1x \in F)$$

for every $x \in E$, and so F is not proper. The converse is trivial.

b) follows from a).

c) follows from a),b), and Zorn's Lemma. ■

Proposition 2.1.1.15 (0) *Let E be a unital algebra, F a maximal proper left ideal of E and take $x \in E \backslash F$. Then there is a pair $(y,z) \in E \times F$ such that*

$$yx + z = 1 .$$

Put

$$G := \{yx + z \mid (y,z) \in E \times F\} .$$

G is a left ideal of E strictly containing F. Hence $G = E$, proving the assertion. ■

Definition 2.1.1.16 (0) *Let E be an algebra. Given $A \subset E$, define*

$$A^c := \{x \in E \mid y \in A \Longrightarrow xy = yx\} ,$$

$$A^{cc} := (A^c)^c , \quad A^{ccc} := (A^{cc})^c .$$

*A^c is called the **commutant of** A and A^{cc} the **bicommutant of** A.*

For every $x, y \in E$,

$$xy = yx \Longleftrightarrow x \in \{y\}^c \Longleftrightarrow y \in \{x\}^c .$$

Proposition 2.1.1.17 (0) *Let E be a (unital) algebra and take A a subset of E.*

a) *A^c is a (unital) subalgebra of E.*

b) *$B \subset A \Rightarrow A^c \subset B^c$.*

c) *$A \subset A^{cc}$.*

d) *$A^c = A^{ccc}$.*

e) *A commutative $\Leftrightarrow A \subset A^c \Leftrightarrow A^{cc}$ commutative.*

f) *A commutative $\Rightarrow$ the (unital) subalgebra of E generated by A is commutative.*

g) *If F is an ideal of E and 1_F is a unit of F then $1_F \in E^c$.*

a) Take $x, y \in A^c$ and $\alpha, \beta \in \mathbb{K}$. Then

$$(\alpha x + \beta y)z = \alpha xz + \beta yz = \alpha zx + \beta zy = z(\alpha x + \beta y),$$

$$xyz = xzy = zxy$$

for every $z \in A$. Thus $\alpha x + \beta y$, $xy \in A^c$ so that A^c is a subalgebra of E.

If E is a unital algebra, then $1 \in A^c$, so that A^c is a unital subalgebra of E.

b) and c) are trivial.

d) By c),

$$A \subset A^{cc}$$

so by b) and c),

$$A^{ccc} \subset A^c \subset A^{ccc},$$

i.e.

$$A^c = A^{ccc}.$$

e) The first equivalence is obvious. Assume $A \subset A^c$. By b) and c),

$$A^{cc} \subset A^c \subset A^{ccc}.$$

Hence A^{cc} is commutative.

By c) A^{cc} commutative implies A commutative.

f) follows from a), c), and e).

g) Let $x \in E$. Then $x1_F$, $1_F x \in F$, so

$$x1_F = 1_F x 1_F = 1_F x.$$

Hence $1_F \in E^c$. ■

Example 2.1.1.18 (0) *Let* $\mathbb{K}[t]$ *(resp.* $\mathbb{K}[s,t]$*) denote the commutative unital algebra of polynomials in* t *(resp. in* s *and* t*) with coefficients in* $\mathbb{K}$ *(the algebra operations are the usual ones for polynomials). Given* $n \in \mathbb{N}$*, the set of polynomials with the property that the coefficients of the terms of degree strictly smaller then* n *vanish, is a proper ideal.* ■

Proposition 2.1.1.19 (0) *Let E be a unital algebra and take $x, y \in E$ with $xy = yx$. The maps*

$$\mathbb{K}[t] \longrightarrow E\,, \quad P \longmapsto P(x)\,,$$

$$\mathbb{K}[s,t] \longrightarrow E\,, \quad Q \longmapsto Q(x,y)$$

are unital algebra homomorphism and

$$\{P(x) \mid P \in \mathbb{K}[t]\}$$

$$(\{Q(x,y) \mid Q \in \mathbb{K}[s,t]\})$$

is the unital subalgebra of E generated by x (by x and y). It is commutative. ■

Corollary 2.1.1.20 *Let E be an algebra and take $x, y \in E$ with $xy = yx$. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\,,$$

$$x^n - y^n = (x-y) \sum_{k=0}^{n-1} x^k y^{n-k-1}$$

for every $n \in \mathbb{N}$. ■

2.1.2 Invertible Elements

Definition 2.1.2.1 (0) *Let E be a unital algebra. Then $x \in E$ is called* ***invertible in*** *E if there is a $y \in E$ such that*

$$xy = yx = 1 .$$

E is called ***division algebra*** *if every element of $E\backslash\{0\}$ is invertible in E.*

Let E be a unital algebra, F a unital subalgebra of E, and $x \in F$. If x is invertible in F then it is obviously invertible in E. But x can be invertible in E without being invertible in F.

Proposition 2.1.2.2 (0) *The zero-divisors are not invertible.*

Let E be a unital algebra and x an invertible element of E. Take $y, z \in E$ with

$$xy = yx = 1 ,$$

$$xz = 0 \quad (\text{or } zx = 0) .$$

Then

$$z = 1z = yxz = y0 = 0$$

$$(\text{or } z = z1 = zxy = 0y = 0) .$$

Hence x is not a zero-divisor. ■

Proposition 2.1.2.3 (0) *Let E be a unital algebra and take $x, y, z \in E$ with*

$$xy = zx = 1 .$$

Then $y = z$ and x is invertible.

We have

$$z = z1 = zxy = 1y = y .$$ ■

Corollary 2.1.2.4 (0) *Let E be a unital algebra and x an invertible element of E. Then there is a unique element $x^{-1} \in E$ with*

$$x^{-1}x = xx^{-1} = 1\,.$$

x^{-1} is called the inverse of x in E. Moreover

$$(x^{-1})^{-1} = x\,.$$ ■

Convention *If we write x^{-1} for an element x of a unital algebra, we implicitly assert that x is invertbile.*

Remark. The set of invertible elements of a unital algebra is a group with respect to the induced multiplication with 1 as neutral element.

Definition 2.1.2.5 (0) *Let x be an invertible element of a unital algebra. Given $n \in \mathbb{N}$, define*

$$x^{-n} := (x^{-1})^n\,.$$

Corollary 2.1.2.6 (0) *Let E be a unital algebra and take $x, y \in E$. x and y are invertible iff xy and yx are invertible and in this case*

$$(xy)^{-1} = y^{-1}x^{-1}\,.$$

Suppose that x and y are invertible. Then

$$(y^{-1}x^{-1})(xy) = y^{-1}x^{-1}xy = y^{-1}y = 1$$

$$(xy)(y^{-1}x^{-1}) = xyy^{-1}x^{-1} = xx^{-1} = 1\,,$$

so xy is invertible and

$$(xy)^{-1} = y^{-1}x^{-1}\,.$$

By symmetry, yx is also invertible.

For the converse, assume that xy and yx are invertible. Set

$$u := xy\,, \quad v := yx\,.$$

Then

$$x(yu^{-1}) = (xy)u^{-1} = 1\,,$$

$$(v^{-1}y)x = v^{-1}(yx) = 1\,.$$

By Proposition 2.1.2.3, x is invertible. By symmetry, y is also invertible. ■

Corollary 2.1.2.7 (0) *Let E be a unital algebra and $(x_\iota)_{\iota\in I}$ a commutative finite family in E. The product $\prod_{\iota\in I} x_\iota$ of this family is invertible iff each x_ι $(\iota \in I)$ is invertible.*

The assertion follows from Corollary 2.1.2.6, by complete induction. ■

Proposition 2.1.2.8 (0) *Let E be a unital algebra. If x is an invertible element of E, then*

$$\{x\}^c = \{x^{-1}\}^c .$$

Given $y \in \{x\}^c$,

$$x^{-1}y = x^{-1}yxx^{-1} = x^{-1}xyx^{-1} = yx^{-1} , \quad y \in \{x^{-1}\}^c .$$

Hence

$$\{x\}^c \subset \{x^{-1}\}^c , \quad \{x\}^c = \{x^{-1}\}^c . \qquad ■$$

Proposition 2.1.2.9 (0) *If F is a proper left (right) ideal of a unital algebra, then the elements of F are not invertible.*

Take $x \in F$. If x were invertible, then

$$1 = x^{-1}x \in F \quad (1 = xx^{-1} \in F) ,$$

so $E = F$, contradicting the hypothesis. ■

Corollary 2.1.2.10 *Let E, F be unital algebras and $u : E \to F$ a unital algebra homomorphism. If E is a division algebra and $F \neq \{0\}$, then $\{0\}$ is the only proper ideal of E, and u is injective.*

By Proposition 2.1.2.9, $\{0\}$ is the only proper ideal of E and by Proposition 2.1.1.10, u is injective. ■

Corollary 2.1.2.11 (0) *Let E be a commutative unital algebra. An element of E is not invertible iff it belongs to a maximal proper ideal of E.*

Let x be a non-invertible element of E. Then $\{yx \mid y \in E\}$ is a proper ideal of E (Proposition 2.1.1.14 a)), so it is contained in a maximal proper left (right) ideal of E (Proposition 2.1.1.14 c)), which obviously contains x.

The converse follows from Proposition 2.1.2.9. ■

Proposition 2.1.2.12 (0) *If E is a unital algebra and F is a proper ideal of E, then:*

a) *If E/F is a division algebra then F is maximal.*

b) *If E is commutative and F is maximal then E/F is a division algebra.*

Let $q : E \to E/F$ be the quotient map.

a) Let G be an ideal of E strictly containing F and take $x \in G \backslash F$. Then qx is invertible, and so there is a $Y \in E/F$ with

$$(qx)Y = q1 .$$

Take $y \in Y$. Then $xy \in G$ and

$$q(1 - xy) = q1 - (qx)(qy) = q1 - q1 = 0 ,$$

$$1 - xy \in F \subset G , \quad 1 = (1 - xy) + xy \in G , \quad G = E .$$

Hence F is maximal.

b) Take $X \in (E/F)\backslash\{0\}$. Take $x \in X$. Then x does not belong to F. By Proposition 2.1.1.15, there is a $(y, z) \in E \times F$ with

$$xy + z = 1 .$$

Thus

$$X(qy) = q(xy + z) = q1 .$$

Hence X is invertible and so E/F is a division algebra. ■

Proposition 2.1.2.13 *Let E be a unital, finite-dimensional algebra and $x \in E$. If there is an $y \in E$ such that $xy = 1$ (resp. $yx = 1$), then x is invertible.*

Put

$$\tilde{z} : E \longrightarrow E , \quad \xi \longmapsto z\xi$$

for every $z \in E$. $\tilde{z}$ is linear and

$$\tilde{z_1} \circ \tilde{z_2} = \widetilde{z_1 z_2}$$

for all $z_1, z_2 \in E$. We have

$$\tilde{x} \circ \tilde{y}(\xi) = \tilde{x}(y\xi) = xy\xi = \xi$$

for every $\xi \in E$, so that $\tilde{x} \circ \tilde{y}$ is the identitiy map of E. Since E is finite-dimensional, $\tilde{y} \circ \tilde{x}$ is also the identity map of E. It follows

$$yx = yx1 = \widetilde{yx}1 = \tilde{y} \circ \tilde{x}(1) = 1 .$$

Hence x is invertible. ■

2.1.3 The Spectrum

Definition 2.1.3.1 (0) *Let E be an algebra, $\widetilde{E}$ its associated unital algebra, and take $x \in E$. Put*

$$\sigma(x) := \sigma_E(x) := \{\alpha \in \mathbb{K} \mid \alpha 1 - x \text{ is not invertible in } \widetilde{E}\},$$

$$r(x) := r_E(x) := \inf\{r \in \mathbb{R}_+ \mid \sigma_E(x) \subset U_r^{\mathbb{K}}(0)\}.$$

*We call $\sigma_E(x)$ **the spectrum of x (in E)*** (Hilbert, 1906) *and $r_E(x)$ **the spectral radius of x (in E)**. The map*

$$\mathbb{K}\backslash\sigma(x) \longrightarrow \widetilde{E}, \quad \alpha \longmapsto (\alpha 1 - x)^{-1}$$

*is called **the resolvent of x (in E)**.*

x is invertible in $\widetilde{E}$ if $0 \notin \sigma(x)$. If E has no unit, then $0 \in \sigma(x)$ (Proposition 2.1.2.9). The spectrum of an element of E remains unchanged if we reverse the multiplication.

Proposition 2.1.3.2 (0) *If E is a unital algebra, then*

$$\sigma(\alpha 1) = \{\alpha\}$$

for every $\alpha \in \mathbb{K}$.

Take $\beta \in \mathbb{K}$. It follows from

$$\beta 1 - \alpha 1 = (\beta - \alpha)1$$

that $\beta 1 - \alpha 1$ is invertible iff $\beta \neq \alpha$. ■

Proposition 2.1.3.3 *If E is a division algebra, then*

$$\sigma(x) = \emptyset$$

for every $x \in E\backslash\mathbb{K}1$.

Since for every $\alpha \in \mathbb{K}$,

$$\alpha 1 - x \neq 0,$$

$\alpha 1 - x$ is invertible. ■

Theorem 2.1.3.4 (0) *Let E be a unital algebra. Take $x \in E$ and $P \in \mathbb{K}[t]$.*

a) $P(\sigma(x)) \subset \sigma(P(x))$.

b) *If P is constant and $\sigma(x) \neq \emptyset$ or if P is not constant and $\mathbb{K} = \mathbb{C}$, then*

$$P(\sigma(x)) = \sigma(P(x)) .$$

a) Take $\alpha \in \sigma(x)$ and put

$$P(t) := \sum_{k=0}^{n} \alpha_k t^k .$$

Then

$$P(\alpha) - P(t) = \sum_{k=0}^{n} \alpha_k \alpha^k - \sum_{k=0}^{n} \alpha_k t^k = \sum_{k=0}^{n} \alpha_k (\alpha^k - t^k) =$$

$$= (\alpha - t) \sum_{k=0}^{n} \alpha_k \left(\sum_{j=0}^{k-1} \alpha^j t^{k-j-1} \right) .$$

Hence

$$P(\alpha)1 - P(x) = (\alpha 1 - x) \sum_{k=0}^{n} \alpha_k \left(\sum_{j=0}^{k-1} \alpha^j x^{k-j-1} \right) .$$

Since $\alpha 1 - x$ is not invertible, $P(\alpha)1 - P(x)$ is not invertible either (Corollary 2.1.2.6). Hence

$$P(\alpha) \in \sigma(P(x)) \quad \text{and} \quad P(\sigma(x)) \subset \sigma(P(x)) .$$

b) Assume that P is constant, i.e. $P = \alpha$ with $\alpha \in \mathbb{K}$. Then

$$P(x) = \alpha 1$$

and

$$\sigma(P(x)) = \sigma(\alpha 1) = \{\alpha\} = P(\sigma(x))$$

(Proposition 2.1.3.2).

If P is not constant and $\mathbb{K} = \mathbb{C}$, then take $\beta \in \sigma(P(x))$. There is an $\alpha \in \mathbb{C} \backslash \{0\}$ and a finite family $(\alpha_i)_{i \in I}$ in $\mathbb{C}$, such that

$$\beta - P(t) = \alpha \prod_{\iota \in I} (\alpha_\iota - t) \ .$$

Then

$$\beta 1 - P(x) = \alpha \prod_{\iota \in I} (\alpha_\iota 1 - x) \ .$$

Since $\beta 1 - P(x)$ is not invertible, there is an $\iota \in I$ such that $\alpha_\iota 1 - x$ is not invertible (Corollary 2.1.2.7). Thus $\alpha_\iota \in \sigma(x)$ and

$$\beta - P(\alpha_\iota) = 0$$

i.e.

$$\beta = P(\alpha_\iota) \in P(\sigma(x)) \, ,$$

$$\sigma(P(x)) \subset P(\sigma(x)) \, . \qquad \blacksquare$$

Remark. a) In the notation $P(\sigma(x))$, we identified P with the map

$$\mathbb{K} \longrightarrow \mathbb{K} \, , \quad \alpha \longmapsto P(\alpha) \, .$$

b) does not hold if the supplementary conditions are not fulfilled. Assume P is constant and $\sigma(x) = \emptyset$. Let E be the algebra of rational functions in the variable t. Then

$$\sigma(t) = \emptyset$$

(Proposition 2.1.3.3) and

$$P(\sigma(t)) = \emptyset \neq \sigma(P(t))$$

(Proposition 2.1.3.2). Assume now that $\mathbb{K} = \mathbb{R}$. Let

$$P(t) := t^2 \in \mathbb{R}[t] \, .$$

$\mathbb{C}$ is a real algebra,

$$\sigma(i) = \emptyset$$

(Proposition 2.1.3.3), and

$$\sigma(P(i)) = \sigma(-1) = \{-1\} \neq \emptyset = P(\sigma(i))$$

(Proposition 2.1.3.2).

Corollary 2.1.3.5 (0) *Let E be an algebra. Take $x \in E$ and $P \in \mathbb{K}[t]\backslash\{0\}$. We assume that $P(x) = 0$ and $Q(x) \neq 0$ for every $Q \in \mathbb{K}[t]$ dividing P with*

$$\text{degree}\, Q < \text{degree}\, P\,.$$

Then $\sigma(x)$ is the set of the zeros of P. In particular, if E is unital and finite-dimensional, then

$$\text{Card}\, \sigma(x) \leq \text{Dim}\ E$$

for every $x \in E$.

We have that

$$P(\sigma(x)) \subset \sigma(P(x)) = \sigma(0) = \{0\}$$

(Theorem 2.1.3.4 a), Proposition 2.1.3.2), i.e. $\sigma(x)$ is contained in the set of zeros of P. Now let α be a zero of P. Then there is a $Q \in \mathbb{K}[t]$ with

$$P(t) = (\alpha - t)Q(t)\,.$$

Since $Q(x) \neq 0$ and

$$0 = P(x) = (\alpha 1 - x)Q(x)\,,$$

$\alpha 1 - x$ is a zero-divisor and hence not invertible (Proposition 2.1.2.2). Thus $\alpha \in \sigma(x)$ and $\sigma(x)$ is the set of zeros of P.

Now assume that E is unital and finite-dimensional. Take $x \in E$ and let

$$n := \text{Dim}\, E\,.$$

Then $\{x^k \mid k \in \mathbb{N}_n \cup \{0\}\}$ is linearly dependent. Hence, there is a $P \in \mathbb{K}[t]\backslash\{0\}$ with $P(x) = 0$. By the above,

$$\text{Card}\, \sigma(x) \leq n\,. \qquad \blacksquare$$

Definition 2.1.3.6 (0) (B. Pierce, 1870) *The element x of an algebra is called* ***idempotent*** *if $x^2 = x$.*

Corollary 2.1.3.7 (0) *Let x be an idempotent element of the algebra E. If x is different from 0 and 1, then*

$$\sigma(x) = \{0, 1\}\,.$$

Define

$$P(t) := t^2 - t \in \mathbb{K}[t].$$

Then

$$P(x) = 0, \quad x \neq 0, \quad x - 1 \neq 0$$

and the assertion follows from Corollary 2.1.3.5. ∎

Corollary 2.1.3.8 *Let E be a complex unital algebra. Take $x \in E$ and $P, Q \in \mathbb{C}[t]$ such that $Q(x)$ is invertible, and*

$$R : \sigma(x) \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \frac{P(\alpha)}{Q(\alpha)}.$$

If $\sigma(x) \neq \emptyset$, then

$$\sigma(P(x)Q(x)^{-1}) = R(\sigma(x)).$$

We first remark that

$$0 \notin \sigma(Q(x)) \supset Q(\sigma(x))$$

(Theorem 2.1.3.4 a)), so that R is well defined on $\sigma(x)$. Take $\alpha \in \mathbb{C}$ and let

$$S := P - \alpha Q.$$

The following are equivalent:

$$\alpha \in \sigma(P(x)Q(x)^{-1}),$$

$$P(x)Q(x)^{-1} - \alpha 1 \text{ is not invertible},$$

$$S(x) \text{ is not invertible} \quad \text{(Corollary 2.1.2.6)},$$

$$0 \in \sigma\,(S(x)) = S(\sigma(x)) \quad \text{(Theorem 2.1.3.4 b))},$$

$$S(\beta) = 0, \quad \text{for some} \quad \beta \in \sigma(x),$$

$$R(\beta) = \alpha, \quad \text{for some} \quad \beta \in \sigma(x),$$

$$\alpha \in R(\sigma(x)).$$

∎

Proposition 2.1.3.9 (6) *Let E be a unital algebra and take $x \in E$. For each $\alpha \in \mathbb{K}\backslash\sigma(x)$, put*

$$x_\alpha := (\alpha 1 - x)^{-1}.$$

Then

$$x_\alpha - x_\beta = (\beta - \alpha)x_\alpha x_\beta \quad \textbf{\textit{(the Resolvent Equation)}},$$

$$\{x_\alpha\}^c = \{x\}^c$$

for every $\alpha, \beta \in \mathbb{K}\backslash\sigma(x)$.

Now

$$(\alpha 1 - x)(x_\alpha - x_\beta)(\beta 1 - x) = (\beta 1 - x) - (\alpha 1 - x) = (\beta - \alpha)1$$

and so

$$x_\alpha - x_\beta = (\beta - \alpha)x_\alpha x_\beta.$$

By Proposition 2.1.2.8

$$\{x\}^c = \{\alpha 1 - x\}^c = \{x_\alpha\}^c. \qquad \blacksquare$$

Proposition 2.1.3.10 (0) (Jacobson) *Let E a unital algebra and $x, y \in E$.*

a) $\alpha \in \mathbb{K}\backslash\{0\}$, $\alpha \notin \sigma(xy)$, $z := (\alpha 1 - xy)^{-1} \Rightarrow (\alpha 1 - yx)^{-1} = \frac{1}{\alpha}(1 + yzx)$.

b) $\sigma(xy)\backslash\{0\} = \sigma(yx)\backslash\{0\}$.

c) $r(xy) = r(yx)$.

a) We have

$$1 = z(\alpha 1 - xy) = \alpha z - zxy,$$

$$1 = (\alpha 1 - xy)z = \alpha z - xyz,$$

so that

$$zxy = az - 1, \qquad xyz = \alpha z - 1.$$

Hence

$$(1+yzx)(\alpha 1-yx)=\alpha 1-yx+\alpha yzx-yzxyx=$$

$$=\alpha 1-yx+\alpha yzx-y(\alpha z-1)x=\alpha 1\,,$$

$$(\alpha 1-yx)(1+yzx)=\alpha 1-yx+\alpha yzx-yxyzx=$$

$$=\alpha 1-yx+\alpha yzx-y(\alpha z-1)x=\alpha 1\,,$$

$$(\alpha 1-yx)^{-1}=\frac{1}{\alpha}(1+yzx)\,.$$

b) follows from a).

c) follows from b). ■

Remark. If we denote by u_r and u_ℓ the right and the left shift of ℓ^2, respectively (Example 1.2.2.9), then

$$0\notin\sigma(u_\ell u_r)\,,\quad 0\in\sigma(u_r u_\ell)\,.$$

This example shows the necessity to exclude 0 from the formula in b).

Proposition 2.1.3.11 (0) *If E is an algebra, $x\in E$, and $\alpha\in\mathbb{K}\backslash\{0\}$, then the following are equivalent:*

a) $\alpha\notin\sigma(x)$.

b) *for some $y\in E$*

$$\alpha y-\frac{1}{\alpha}x-xy=0\,,\quad xy=yx\,.$$

Let $\tilde{E}$ be the unital algebra associated to E. Then, given $y\in\tilde{E}$,

$$(\alpha 1-x)(\frac{1}{\alpha}1+y)=1+(\alpha y-\frac{1}{\alpha}x-xy)$$

in $\tilde{E}$.

a $\Rightarrow$ b. Put

$$y:=(\alpha 1-x)^{-1}-\frac{1}{\alpha}1\,.$$

By the above relation,

$$\alpha y-\frac{1}{\alpha}x-xy=0\,,\quad xy=yx\,.$$

Hence

$$y=\frac{1}{\alpha}(\frac{1}{\alpha}x+xy)\in E\,.$$

b $\Rightarrow$ a is obvious. ■

Corollary 2.1.3.12 (0) *Let E, F be algebras and $u : E \to F$ an algebra homomorphism. Then*

$$\sigma_F(ux) \subset \sigma_E(x) \cup \{0\}, \quad r_F(ux) \leq r_E(x)$$

for every $x \in E$. If E is not unital or if E, F, and u are unital, then

$$\sigma_F(ux) \subset \sigma_E(x).$$

Take $\alpha \in \mathbb{K} \backslash (\sigma_E(x) \cup \{0\})$. By Proposition 2.1.3.11 a $\Rightarrow$ b, there is a $y \in E$ with

$$\alpha y - \frac{1}{\alpha} x - xy = 0, \quad xy = yx.$$

Then

$$\alpha uy - \frac{1}{\alpha} ux - (ux)(uy) = 0, \quad (ux)(uy) = (uy)(ux).$$

Hence, by Proposition 2.1.3.11 b $\Rightarrow$ a, $\alpha \notin \sigma_F(ux)$.

The final assertion follows from the fact that under the supplementary hypotheses $0 \notin \sigma_F(ux) \backslash \sigma_E(x)$. ■

Corollary 2.1.3.13 (0) *Let E be an algebra and $u : E \to \mathbb{K}$ an algebra homomorphism. If u is not identically zero, then*

$$ux \in \sigma(x)$$

for every $x \in E$.

If E is unital, then u is also unital (Proposition 2.1.1.12) and the assertion follows from Corollary 2.1.3.12 (and Proposition 2.1.3.2). ■

Corollary 2.1.3.14 (0) *Let F be a subalgebra of an algebra E.*

a) *For $x \in F$,*

$$\sigma_E(x) \subset \sigma_F(x), \quad r_E(x) \leq r_F(x).$$

b) *Assume E is unital, $1 \in F$, and there is a vector subspace G of E such that the map*

$$F \times G \longrightarrow E, \quad (x, y) \longmapsto x + y$$

is bijective and such that $xy, yx \in G$ for every $(x, y) \in F \times G$. Then

$$\sigma_E(x) = \sigma_F(x)$$

for every $x \in F$.

a) follows from Corollary 2.1.3.12.

b) First assume x invertible in E. Let $(y, z) \in F \times G$ such that

$$y + z = x^{-1}.$$

Then

$$xz = 1 - xy \in F \cap G,$$

$$zx = 1 - yx \in F \cap G,$$

so that

$$xz = zx = 0.$$

It follows

$$xy = yx = 1$$

and x is invertible in F.

By a) and the above considerations,

$$\sigma_E(x) = \sigma_F(x). \qquad \blacksquare$$

Corollary 2.1.3.15 (0) *Let E be an algebra and F a subalgebra of E, such that for $(x, y) \in E \times F$, $xy \in F$ whenever $xy = yx$. Then*

$$\sigma_E(x) = \sigma_F(x)$$

for every $x \in F$.

Take $\alpha \in \mathbb{K} \setminus \{0\}$. By Proposition 2.1.3.11,

$$\alpha \notin \sigma_E(x) \iff \exists y \in E,\ \alpha y - \frac{1}{\alpha}x - xy = 0,\ xy = xy,$$

$$\alpha \notin \sigma_F(x) \iff \exists y \in F,\ \alpha y - \frac{1}{\alpha}x - xy = 0,\ xy = xy,$$

On the other hand,

$$xy = yx,\ \alpha y - \frac{1}{\alpha}x - xy = 0 \Longrightarrow y = \frac{1}{\alpha}\left(\frac{1}{\alpha}x + xy\right) \in F$$

for every $y \in E$. Thus

$$\alpha \notin \sigma_E(x) \Longleftrightarrow \alpha \notin \sigma_F(x)\,.$$

Assume that $0 \notin \sigma_E(x)$. Then

$$xx^{-1} = x^{-1}x = 1\,.$$

Hence $1 \in F$ and

$$x^{-1} = 1x^{-1} \in F\,.$$

Thus $0 \notin \sigma_F(x)$. The converse is trivial. ■

Proposition 2.1.3.16 (6) *Let E be a unital algebra. Take $x, y \in E$ with y invertible, and $n \in \mathbb{N}$ such that*

$$xy = yx\,, \quad x^n = 0\,.$$

a) *x is a zero-divisor.*
b) $(y-x)^{-1} = \sum_{k=0}^{n-1} x^k y^{-k-1}$.
c) $\sigma(x) = \{0\}$.

a) Let m be the smallest natural number with $x^m = 0$. The assertion follows from

$$x^{m-1}x = 0\,.$$

b) Use

$$(y-x)\sum_{k=0}^{n-1} x^k y^{-k-1} = \sum_{k=0}^{n-1} x^k y^{-k} - \sum_{k=0}^{n-1} x^{k+1} y^{-k-1} = 1 - x^n y^{-n} = 1\,.$$

c) Since, by a) and Proposition 2.1.2.2, x is not invertible, 0 belongs to $\sigma(x)$. By b), $\alpha 1 - x$ is invertible for every $\sigma \in \mathbb{K}\backslash\{0\}$. Thus

$$\sigma(x) = \{0\}\,.$$ ■

Definition 2.1.3.17 (0) *Let E be an algebra and let $\widetilde{E}$ be the unital algebra associated to E. A maximal proper (left, right) ideal F of E is called* ***regular*** *if there is a maximal proper (left, right) ideal G of $\widetilde{E}$ containing F and different from E.*

If E is unital then every maximal proper (left, right) ideal of E is regular.

Definition 2.1.3.18 (0) *Let E be an algebra. The intersection of all regular maximal proper left ideals of E is called* ***the radical of*** *E (the intersection is defined to be E if there are no regular maximal proper left ideals). E is said to be* ***semi-simple*** *if its radical is $\{0\}$.*

If E is unital then, by Proposition 2.1.1.14 c), there are maximal proper left ideals of E, hence the radical of E is different from E.

Proposition 2.1.3.19 (4) *Let E be an algebra, $\widetilde{E}$ the unital algebra associated to E and R (resp. $\widetilde{R}$) be the radical of E (of $\widetilde{E}$). Then*

$$R = E \cap \widetilde{R}.$$

We may assume E is not unital. Let F be a regular maximal proper left ideal of E. There is a maximal proper left ideal $\widetilde{F}$ of $\widetilde{E}$ containing F and different from E. Then $E \cap \widetilde{F}$ is a proper left ideal of E containing F and so

$$F = E \cap \widetilde{F} \supset E \cap \widetilde{R}.$$

Hence

$$R \supset E \cap \widetilde{R}.$$

Let $\widetilde{F}$ be a maximal proper left ideal of $\widetilde{E}$. Define

$$F := E \cap \widetilde{F}.$$

Then F is a left ideal of E. Assume that F is a proper left ideal of E. Let G be a left ideal of E containing F and different from F. Then $\widetilde{F} + G$ is a left ideal of $\widetilde{E}$ containing $\widetilde{F}$ and different from $\widetilde{F}$. Since $\widetilde{F}$ is maximal,

$$\widetilde{E} = \widetilde{F} + G.$$

In particular, there is a $(y, z) \in \widetilde{F} \times G$ such that

$$1 = y + z.$$

Then

$$x = x(y + z) = xy + xz \in E \cap \widetilde{F} + G \subset G$$

for every $x \in E$. Hence $E = G$ and F is a maximal proper left ideal of E. Since it is obviously regular,

$$R \subset F = E \cap \widetilde{F}, \quad R \subset E \cap \widetilde{R}. \qquad \blacksquare$$

Proposition 2.1.3.20 *Let E be a commutative unital algebra and F the set of nilpotent elements of E. Then F is a proper ideal of E contained in the radical of E.*

Take $x, y \in F$, $\alpha, \beta \in \mathbb{K}$, and $z \in E$. There are $m, n \in \mathbb{N}$ such that

$$x^m = y^n = 0 .$$

Then

$$(\alpha x + \beta y)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} \alpha^k \beta^{m+n-k} x^k y^{m+n-k} = 0 ,$$

$$(xz)^m = x^m z^m = 0 ,$$

and hence $\alpha x + \beta y, xz \in F$ (Corollary 2.1.1.20). Since $1 \notin F$, F is a proper ideal of E.

Take $x \in F$ and let G be a maximal proper ideal of E. If $x \notin G$, then there is a $(y, z) \in E \times G$ with

$$xy + z = 1$$

(Proposition 2.1.1.15). Since xy is nilpotent, z is invertible (Proposition 2.1.3.16 b)) and this is a contradiction (Proposition 2.1.2.9). Hence $x \in G$. It follows that F is contained in the radical of E. ■

Proposition 2.1.3.21 (4) *Let R be the radical of the unital algebra E. Given $x \in E$, the following are equivalent:*

a) $x \in R$.

b) *For each $y \in E$, there is a $z \in E$ with $z(1 - yx) = 1$.*

a $\Rightarrow$ b. Assume that for every $z \in E$,

$$z(1 - yx) \neq 1 .$$

Then, by Proposition 2.1.1.14 c), there is a maximal proper left ideal F of E containing $1 - yx$. Since $R \subset F$ and $yx \in R$, we obtain the contradiction

$$1 = (1 - yx) + yx \in F .$$

b $\Rightarrow$ a. Assume that $x \notin R$. Then there is a maximal proper left ideal F of E with $x \notin F$. Since $\{yx + a \mid y \in E, a \in F\}$ is a left ideal of E containing $F \cup \{x\}$, it contains 1. Hence, there are $y \in E$ and $a \in F$ with

$$a = 1 - yx .$$

By b), there is a $z \in E$ with

$$1 = z(1 - yx) .$$

But then

$$1 = za \in F ,$$

which is a contradiction. ∎

Corollary 2.1.3.22 *The radical of a unital algebra is an ideal.*

Let E be a unital algebra. R its radical, and take $(x, y) \in E \times R$. Assume $yx \notin R$. Then there is a maximal proper left ideal F of E such that $yx \notin F$. Then $\{zyx + a \mid (z, a) \in E \times F\}$ is a left ideal of E containing $F \cup \{yx\}$. Hence there is a $(z, a) \in E \times F$ with

$$zyx + a = 1 .$$

Thus

$$xzyx + xa = x ,$$

$$(1 - xzy)x = x - xzyx = xa \in F .$$

By Proposition 2.1.3.21 a $\Rightarrow$ b, there is a $b \in E$ with

$$b(1 - xzy) = 1 .$$

Hence

$$yx = y1x = yb(1 - xzy)x = ybxa \in F$$

which is a contradiction. Hence $yx \in R$ which proves that R is an ideal of E. ∎

Proposition 2.1.3.23 (4) *Let E be a unital algebra and $\mathfrak{F}$ be the set of left ideals F of E for which $1 - x$ is invertible whenever $x \in F$. Then the radical of E is the greatest element of $\mathfrak{F}$.*

Let R be the radical of E and take $x \in R$. By Proposition 2.1.3.21 a $\Rightarrow$ b, there is a $y \in E$ with

$$y(1-x) = 1 .$$

Thus

$$1 - y = -yx \in R .$$

By Proposition 2.1.3.21 a $\Rightarrow$ b, again, there is a $z \in E$ with

$$1 = z(1-(1-y)) = zy .$$

By Proposition 2.1.2.3, y is invertible and

$$y^{-1} = 1 - x .$$

Hence $1-x$ is invertible and $R \in \mathfrak{F}$.

Take $F \in \mathfrak{F}$ and $x \in F$. Take $y \in E$. Then $yx \in F$ and so $1 - yx$ is invertible. By Proposition 2.1.3.21 b $\Rightarrow$ a, $x \in R$. Hence $F \subset R$ and R is the greatest element of $\mathfrak{F}$. ■

Corollary 2.1.3.24 (4) *Let R be the radical of the algebra E and take $x \in R$. Then*

$$\sigma(x) = \{0\} .$$

By Proposition 2.1.3.19, we may assume E to be unital. By Proposition 2.1.2.9, x is not invertible. Thus $0 \in \sigma(x)$.

Take $\alpha \in \mathbb{K}\backslash\{0\}$. Then $\frac{1}{\alpha}x \in R$, and by Proposition 2.1.3.23, $1 - \frac{1}{\alpha}x$ is invertible. Hence $\alpha 1 - x$ is invertible, $\alpha \notin \sigma(x)$, and

$$\sigma(x) = \{0\} . \qquad ■$$

Proposition 2.1.3.25 (7) *If E is a unital algebra and x is an idempotent element of E then*

$$\{x\}^c = xEx + (1-x)E(1-x) = \{y \in E \mid \exists\, z_1, z_2 \in E ,\ xy = z_1 x ,\ yx = xz_2\} .$$

Let $y, z_1, z_2 \in E$ such that

$$xy = z_1 x , \quad yx = xz_2 .$$

Then

$$y = xyx + xy(1-x) + (1-x)yx + (1-x)y(1-x) =$$

$$= xyx + z_1x(1-x) + (1-x)xz_2 + (1-x)y(1-x) =$$

$$= xyx + (1-x)y(1-x) \in xEx + (1-x)E(1-x).$$

Hence

$$\{y \in E \mid \exists\, z_1, z_2 \in E,\ xy = z_1x,\ yx = xz_2\} \subset$$

$$\subset xEx + (1-x)E(1-x).$$

The inclusions

$$xEx + (1-x)E(1-x) \subset \{x\}^c \subset$$

$$\subset \{y \in E \mid \exists\, z_1, z_2 \in E,\ xy = z_1x,\ yx = xz_2\}$$

are trivial. ■

2.1.4 Standard Examples

Example 2.1.4.1 (0) *Let $(E_t)_{t\in T}$ be a family of algebras. The vector space $\prod_{t\in T} E_t$ is an algebra with respect to pointwise multiplication. It is commutative iff each factor E_t $(t \in T)$ is commutative. It has a unit iff each factor E_t $(t \in T)$ has a unit; then the element $x \in \prod_{t\in T} E_t$ is invertible iff each x_t $(t \in T)$ is invertible and in this case*

$$(x^{-1})_t = (x_t)^{-1}$$

for every $t \in T$. ■

Example 2.1.4.2 (0) *Let T be a set and E an algebra. The vector space E^T is an algebra with respect to pointwise multiplication. E^T is commutative whenever E is commutative. If E is unital, then E^T has a canonical unital algebra structure, with*

$$T \longrightarrow E, \quad t \longmapsto 1$$

as unit. The element $x \in E^T$ is invertible iff $x(t)$ is invertible for each $t \in T$, in which case

$$x^{-1}(t) = x(t)^{-1}$$

for every $t \in T$. ■

Example 2.1.4.3 (0) *Let T be a set. The vector space $\mathbb{K}^T$ is a commutative unital algebra with respect to the pointwise multiplication, the unit being the constant function 1, i.e. e_T. An element x of $\mathbb{K}^T$ is invertible if it does not vanish anywhere, in which case*

$$x^{-1}(t) = \frac{1}{x(t)}$$

for every $t \in T$. $\ell^\infty(T)$ and $c(T)$ are unital subalgebras of $\mathbb{K}^T$. An element x of $\ell^\infty(T)$ (of $c(T)$) is invertible in $\ell^\infty(T)$ (in $c(T)$) iff

$$\inf_{t\in T} |x(t)| > 0.$$

$\ell^p(T)$ and $c_0(T)$ are ideals of $\ell^\infty(T)$ $(p \in [1, \infty[\ )$.

If T is infinite, then:

a) *$\ell^\infty(T)$ is not an ideal of $\mathbb{K}^T$.*

b) $c(T)$ *is not an ideal of* $\ell^\infty(T)$.
c) $\ell^p(T)$ *and* $c_0(T)$ *are proper ideals of* $\ell^\infty(T)$.
d) $c_0(T)$ *is a maximal proper ideal of* $c(T)$.
If $E := \mathbb{K}^T$ $(E := \ell^\infty(T))$, *then*

$$\sigma_E(x) = x(T) \quad (\sigma_E(x) = \overline{x(T)})$$

for every $x \in E$. ■

Example 2.1.4.4 (0) *Let* T *be a topological space. Then* $C(T)$ *is a unital subalgebra of* $\ell^\infty(T)$. *The elements of* $C(T)$ *are invertible in* $C(T)$ *iff they are invertible in* $\ell^\infty(T)$. *If* T *is compact, then the elements of* $C(T)$ *are invertible in* $C(T)$ *iff they are invertible in* $\mathbb{K}^T$. *In particular,*

$$\sigma(x) = x(T)$$

for every $x \in C(T)$. *If* T *is locally compact, then* $C_0(T)$ *is an ideal of* $C(T)$. ■

Proposition 2.1.4.5 *Let* T *be a topological space and* $\mathfrak{F}$ *the set of maximal proper ideals of* $C(T)$. *Define*

$$\mathcal{F}_t := \{x \in C(T) \mid x(t) = 0\}$$

for every $t \in T$. *Then* $\mathcal{F}_t \in \mathfrak{F}$ *for every* $t \in T$ *and if* T *is compact, then the map*

$$T \longrightarrow \mathfrak{F}, \quad t \longmapsto \mathcal{F}_t$$

is bijective.

Take $t \in T$ and set

$$x' : C(T) \longrightarrow \mathbb{K}, \quad x \longmapsto x(t) .$$

Then x' is a homomorphism of unital algebras and

$$\mathcal{F}_t = \operatorname{Ker} x' .$$

By Proposition 2.1.1.12 b), $\mathcal{F}_t \in \mathfrak{F}$.

Now assume that T is compact. By Urysohn's Lemma, the above map is injective. Assume the map is not surjective. Take

$$\mathcal{F} \in \mathfrak{F} \backslash \{\mathcal{F}_t \mid t \in T\} .$$

For each $t \in T$, there is an $x_t \in \mathcal{F} \backslash \mathcal{F}_t$. Then

$$|x_t|^2 = \overline{x}_t x_t \in \mathcal{F}$$

for every $t \in T$. Put

$$U_t := \{|x_t| > 0\}$$

for $t \in T$. Each U_t is an open set containing t and $(U_t)_{t \in T}$ is an open covering of T. Since T is compact, there is a finite subset A of T such that

$$T = \bigcup_{t \in A} U_t .$$

Put

$$x := \sum_{t \in A} |x_t|^2 \in \mathcal{F},$$

and take $s \in T$. There is a $t \in A$ such that $s \in U_t$. Then

$$x(s) \geq |x_t(s)|^2 > 0 .$$

x does not vanish on T, and so it is invertible (Example 2.1.4.4). This is a contradiction (Proposition 2.1.2.9). ■

Example 2.1.4.6 (0) *Let E be a vector space and $\mathcal{F}$ the vector space of linear maps $E \to E$. $\mathcal{F}$ is a unital algebra with respect to the multiplication*

$$\mathcal{F} \times \mathcal{F} \longrightarrow \mathcal{F}, \quad (u, v) \longmapsto v \circ u$$

An element of $\mathcal{F}$ is invertible iff it is bijective.

If E is a normed space, then $\mathcal{L}(E)$ is a unital subalgebra of $\mathcal{F}$. If E is complete, then an element of $\mathcal{L}(E)$ is invertible in $\mathcal{L}(E)$ iff it is bijective, i.e. iff it is invertible in $\mathcal{F}$.

The last assertion follows from the Principle of Inverse Operators. ■

Remark. $\mathcal{F}$ and $\mathcal{L}(E)$ are commutative iff the dimension of E is at most 1. Since $\mathcal{F}$ is a unital algebra, we often write vu instead of $v \circ u \, (u, v \in \mathcal{F})$.

Example 2.1.4.7 *Take $n \in \mathbb{N}$. $\mathcal{L}(\mathbb{K}^n)$ is a unital algebra whose only proper ideal is $\{0\}$. If E is a unital algebra, then every unital algebra homomorphism $u : \mathcal{L}(\mathbb{K}^n) \to E$ is injective. An element of $\mathcal{L}(\mathbb{K}^n)$ is a zero-divisor iff it is not invertible.*

Let $\mathcal{F} \neq \{0\}$ be an ideal of $\mathcal{L}(\mathbb{K}^n)$. Take $u \in \mathcal{F}\backslash\{0\}$ and $x \in \mathbb{K}^n\backslash\operatorname{Ker} u$. Let $(e_i)_{i\in\mathbb{N}_n}$ be the standard algebraic basis of $\mathbb{K}^n$. Take $j \in \mathbb{N}_n$, and $v \in \mathcal{L}(\mathbb{K}^n)$ such that

$$vux = e_j .$$

Define

$$w : \mathbb{K}^n \longrightarrow \mathbb{K}^n , \quad \sum_{i=1}^{n} \alpha_i e_i \longmapsto \alpha_j x ,$$

$$u_j := vuw .$$

Then $u_j \in \mathcal{F}$ and

$$u_j \left(\sum_{i=1}^{n} \alpha_i e_i \right) = \alpha_j e_j .$$

Hence

$$1 = \sum_{j=1}^{n} u_j \in \mathcal{F}$$

and $\mathcal{F} = \mathcal{L}(\mathbb{K}^n)$. Hence $\{0\}$ is the only proper ideal of $\mathcal{L}(\mathbb{K}^n)$. By Proposition 2.1.1.10, every unital algebra homomorphism $u : \mathcal{L}(\mathbb{K}^n) \to E$ is injective.

The final assertion follows from the fact that an element of $\mathcal{L}(\mathbb{K}^n)$ is invertible iff the corresponding determinant does not vanish. ■

Proposition 2.1.4.8 *Let E be a normed space and take $u, v \in \mathcal{L}(E)$ with*

$$uv = vu ,$$

$P \in \mathbb{K}[t]$, and $Q \in \mathbb{K}[s,t]$. Then

$$P(u)' = P(u') , \quad Q(u,v)' = Q(u',v') .$$

An element of $\mathcal{L}(E)$ is idempotent iff it is a projection in E.

The assertion follows immediately from the Corollaries 1.3.4.3, 1.3.4.4, and 1.3.4.5. ■

Proposition 2.1.4.9 (Phillips) *Let E be a Banach space. Then*

$$\sigma(u) = \sigma(u')$$

for every $u \in \mathcal{L}(E)$.

Take $\alpha \in \mathbb{K}$. Then

$$(\alpha 1 - u)' = \alpha 1 - u'$$

(Corollary 1.3.4.3, Corollary 1.3.4.4). By Corollary 1.4.2.5, $\alpha 1 - u$ is invertible iff $(\alpha 1 - u)'$ is invertible. Hence $\alpha \in \sigma(u)$ iff $\alpha \in \sigma(u')$. ■

Example 2.1.4.10 (0) *Let T be a finite set and take $k : T \times T \to \mathbb{K}$. For every $x \in \mathbb{K}^T$, define*

$$\tilde{k}x : T \longrightarrow \mathbb{K}, \quad s \longmapsto \sum_{t \in T} k(s,t)x(t)$$

and

$$u : \mathbb{K}^T \longrightarrow \mathbb{K}^T, \quad y \longmapsto \tilde{k}y.$$

Let $P(r)$ denote the characteristic polynomial of u (with the variable r) i.e. the determinant of the matrix

$$[r\delta(s,t) - k(s,t)]_{s,t \in T}.$$

a) $\alpha \in \sigma(u)$ *iff* $P(\alpha) = 0$ *for every* $\alpha \in \mathbb{K}$.

b) $\operatorname{Card} \sigma(u) \leq \operatorname{Card} T$.

c) *If n is odd or if $\mathbb{K} = \mathbb{C}$, then* $\sigma(u) \neq \{0\}$.

a) $\alpha 1 - u$ is not invertible iff it is not injective, i.e. iff there is an $x \in \mathbb{K}^T \backslash \{0\}$ with

$$ux = \alpha x.$$

The assertion now follows.

b) and c) follow from a). ■

Lemma 2.1.4.11 (0) *Let E be a vector space and $(u_{ij})_{i,j \in \mathbb{N}_2}$ a family of linear maps $E \to E$ such that*

$$u_{ij}u_{kl} = \delta_{jk}u_{il}$$

for every $i, j, k, l \in \mathbb{N}_2$. Then the vector spaces $\operatorname{Im} u_{11}$ and $\operatorname{Im} u_{22}$ are isomorphic.

Now

$$u_{11}u_{12} = u_{12}\,, \quad u_{22}u_{21} = u_{21}\,,$$

$$\operatorname{Im} u_{12} \subset \operatorname{Im} u_{11}\,, \quad \operatorname{Im} u_{21} \subset \operatorname{Im} u_{22}\,.$$

Define

$$v : \operatorname{Im} u_{11} \longrightarrow \operatorname{Im} u_{22}\,, \quad x \longmapsto u_{21}x\,,$$

$$w : \operatorname{Im} u_{22} \longrightarrow \operatorname{Im} u_{11}\,, \quad y \longmapsto u_{12}y\,,$$

Then

$$wvx = u_{12}u_{21}x = u_{11}x = x\,,$$

$$vwy = u_{21}u_{12}y = u_{22}y = y\,,$$

for every $x \in \operatorname{Im} u_{11}$, $y \in \operatorname{Im} u_{22}$. Hence v, w are isomorphisms of vector spaces. ■

Proposition 2.1.4.12 (0) *Let E be a unital algebra, $(x_{ij})_{(i,j)\in I\times I}$ a finite family in E such that*

$$x_{ij}x_{k\ell} = \delta_{jk}x_{i\ell}$$

for $i, j, k, \ell \in I$ and

$$\sum_{i\in I} x_{ii} = 1\,.$$

Let n be a natural number and

$$u : E \longrightarrow \mathcal{L}(\mathbb{K}^n)$$

a unital algebra homomomorphism. Then

$$\frac{n}{\operatorname{Card} I} \in \mathbb{N}\,.$$

Define

$$u_{ij} = ux_{ij}\,, \quad q_i = \operatorname{Dim} \operatorname{Im} u_{ii}$$

for $i, j \in I$. Then

$$u_{ij}u_{kl} = \delta_{jk}u_{il}$$

for $i, j, k, l \in I$ and

$$\sum_{i \in I} u_{ii} = 1 .$$

By Lemma 2.1.4.11, $q_i = q_j$ for every $i, j \in I$. Hence

$$\frac{n}{\operatorname{Card} I} = q_1 \in \mathbb{N} . \qquad \blacksquare$$

Corollary 2.1.4.13 (0) *Let $n, p \in \mathbb{N}$ and let $(e_i)_{i \in \mathbb{N}_n}$ be the standard algebraic basis of $\mathbb{K}^n$. Given $i, j \in \mathbb{N}_n$, let x_{ij} denote the operator on $\mathbb{K}^n$ with*

$$x_{ij}e_k = \delta_{jk}e_i$$

for every $k \in \mathbb{N}_n$.

a) $i, j, k, \ell \in \mathbb{N}_n \Rightarrow x_{ij}x_{k\ell} = \delta_{jk}x_{i\ell}$.

b) $\sum_{i=1}^{n} x_{ii} = 1$.

c) *There is a unital algebra homomorphism*

$$\mathcal{L}(\mathbb{K}^n) \longrightarrow \mathcal{L}(\mathbb{K}^p)$$

iff $\frac{p}{n} \in \mathbb{N}$.

a) For $m \in \mathbb{N}_n$,

$$x_{ij}x_{k\ell}e_m = x_{ij}\delta_{\ell m}e_k = \delta_{\ell m}\delta_{jk}e_i = \delta_{jk}x_{i\ell}e_m$$

so that

$$x_{ij}x_{k\ell} = \delta_{jk}x_{i\ell} .$$

b) For $j \in \mathbb{N}_n$,

$$\left(\sum_{i=1}^{n} x_{ii}\right) e_j = \sum_{i=1}^{n} x_{ii}e_j = \sum_{i=1}^{n} \delta_{ij}e_i = e_j$$

so that

$$\sum_{i=1}^{n} x_{ii} = 1 .$$

c) By a),b), and Proposition 2.1.4.12, if there is a unital algebra homomorphism

$$\mathcal{L}(\mathbb{K}^n) \longrightarrow \mathcal{L}(\mathbb{K}^p)$$

then n dvides p.

The converse is easy to see. ■

Corollary 2.1.4.14 *Let T be an infinite set. Take $p \in [1, \infty]$ and $n \in \mathbb{N}$. Then there is no unital algebra homomomorphism*

$$\mathcal{L}(\ell^p(T)) \longrightarrow \mathcal{L}(\mathbb{K}^n).$$

We may assume there are sets I, S with

$$T = I \times S, \quad \operatorname{Card} I = n + 1.$$

For $x \in \ell^p(T)$ and $i, j \in I$, define

$$x_{ij} : T \longrightarrow \mathbb{K}, \quad (k, s) \longmapsto \begin{cases} x(j, s) & \text{if } k = i \\ 0 & \text{if } k \neq i \end{cases}$$

and

$$u_{ij} : \ell^p(T) \longrightarrow \ell^p(T), \quad y \longmapsto y_{ij}.$$

Then

$$u_{ij} u_{k\ell} = \delta_{jk} u_{i\ell}$$

for every $i, j, k, \ell \in I$ and

$$\sum_{i \in I} u_{ii} = 1.$$

By Proposition 2.1.4.12, there is no unital algebra homomorphism

$$\mathcal{L}(\ell^p(T)) \longrightarrow \mathcal{L}(\mathbb{K}^n). \quad ■$$

Theorem 2.1.4.15 *Let E be a finite-dimensional complex unital algebra. If E is a division algebra, then $E = \mathbb{C}1$.*

Take $x \in E\backslash\{0\}$. Since E is finite-dimensional, $(x^n)_{n\in\mathbb{N}}$ is linearly dependent. Hence there is a $P \in \mathbb{C}[t]\backslash\{0\}$ with

$$P(x) = 0 .$$

There is a finite family $(\alpha_\iota)_{\iota\in I}$ in $\mathbb{C}$ and an $\alpha \in \mathbb{C}\backslash\{0\}$ with

$$P(t) = \alpha \prod_{\iota\in I}(t - \alpha_\iota) .$$

Then

$$\alpha \prod_{\iota\in I}(x - \alpha_\iota 1) = P(x) = 0 .$$

Hence there is a $\iota \in I$ such that $x - \alpha_\iota 1$ is not invertible (Corollary 2.1.2.7). Since E is a division algebra,

$$x - \alpha_\iota 1 = 0 ,$$

$$x = \alpha_\iota 1 \in \mathbb{C}1 .$$

Therefore $E = \mathbb{C}1$. ■

Proposition 2.1.4.16 (0) *Put*

$$I := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} , \qquad J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} , \qquad K := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

and let E denote the real vector subspace of $\mathcal{L}(\mathbb{C}^2)$ generated by $\{1, I, J, K\}$.

a) $I^2 = J^2 = K^2 = -1$
$IJ = -JI = K , \quad JK = -KJ = I , \quad KI = -IK = J .$

b) $1, I, J, K$ *are linearly independent.*

c) $\|\alpha 1 + \beta I + \gamma J + \delta K\|^2$ *and the determinant of* $\alpha 1 + \beta J + \gamma J + \delta K$ *are equal to*

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2$$

and

$$(\alpha 1 + \beta I + \gamma J + \delta K)(\alpha 1 - \beta I - \gamma J - \delta K) = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)1$$

for every $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

d) *E is a real unital subalgebra of the real unital algebra $\mathcal{L}(\mathbb{C}^2)$. It is a division algebra with respect to this structure.*

a) is a simple calculation.

b) Take $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with

$$\alpha 1 + \beta I + \gamma J + \delta K = 0 .$$

Then

$$\begin{bmatrix} \alpha + \beta i & \gamma + \delta i \\ -\gamma + \delta i & \alpha - \beta i \end{bmatrix} = 0 .$$

Hence

$$\alpha = \beta = \gamma = \delta = 0 .$$

c) is a simple calculation. (Use Example 1.2.2.8.)

d) By a), E is a real unital subalgebra of the real unital algebra $\mathcal{L}(\mathbb{C}^2)$. By b) and c) it is a division algebra. ■

Corollary 2.1.4.17 (0) (Hamilton, 1843) *Let $1, i, j, k$ be the unit vectors of $\mathbb{R}^4$. There is a unique multiplication on the vector space $\mathbb{R}^4$ such that 1 is a unit and*

$$i^2 = j^2 = k^2 = -1$$

$$ij = -ji = k , \quad jk = -kj = i , \quad ki = -ik = j .$$

Then

$$\|xy\|_2 = \|x\|_2 \|y\|_2$$

for every $x, y \in \mathbb{R}^4$. $\mathbb{R}^4$ is a division algebra with respect to this multiplication. It is usually denoted by $\mathbb{H}$ and its elements are called **quaternions**.

$\mathbb{H}$ is isomorphic to E in Proposition 2.1.4.16. ■

Remark. The real unital algebra structure of $\mathbb{H}$ cannot be extended to a complex unital algebra structure, as shown by Theorem 2.1.4.15. But

$$\{\alpha + \beta i \mid \alpha, \beta \in \mathbb{R}\}$$

is a unital subalgebra of $\mathbb{H}$, which is isomorphic to the **real** algebra $\mathbb{C}$.

Proposition 2.1.4.18 *Let E be a finite-dimensional real unital algebra and take $x \in E\backslash\mathbb{R}1$. If E is a division algebra then there are $\alpha, \beta \in \mathbb{R}$ with*

$$x^2 + \alpha x + \beta 1 = 0\,, \quad \alpha^2 < 4\beta\,.$$

Put

$$y := \frac{1}{\sqrt{4\beta - \alpha^2}}(\alpha 1 + 2x)\,.$$

Then

$$y^2 = -1\,.$$

Since E is finite-dimensional, $(x^n)_{n\in\mathbb{N}}$ is linearly dependent. Hence, there is a $P \in \mathbb{R}[t]\backslash\{0\}$ with

$$P(x) = 0\,.$$

There are two finite families $(\alpha_\iota)_{\iota\in I}$, $(\beta_\iota)_{\iota\in I}$ in $\mathbb{R}$ and an $\alpha \in \mathbb{R}\backslash\{0\}$ such that

$$P(t) = \alpha \prod_{\iota\in I}(t^2 + \alpha_\iota t + \beta_\iota)\,.$$

Then

$$\alpha \prod_{\iota\in I}(x^2 + \alpha_\iota x + \beta_\iota 1) = P(x) = 0\,.$$

Hence, there is a $\iota \in I$ such that $x^2 + \alpha_\iota x + \beta_\iota 1$ is not invertible (Corollary 2.1.2.7). Since E is a division algebra

$$x^2 + \alpha_\iota x + \beta_\iota 1 = 0\,.$$

If $\alpha_\iota^2 \geq 4\beta_\iota$, then there are $\gamma, \delta \in \mathbb{R}$ with

$$(x + \gamma 1)(x + \delta 1) = x^2 + \alpha_\iota x + \beta_\iota = 0\,.$$

Since E is a division algebra, this implies that $x \in \mathbb{R}1$ (Corollary 2.1.2.6), which contradicts the hypothesis. Hence

$$4\beta_\iota > \alpha_\iota^2\,.$$

Then

$$y^2 = \frac{1}{4\beta - \alpha^2}(\alpha^2 1 + 4\alpha x + 4x^2) = \frac{1}{4\beta - \alpha^2}(\alpha^2 1 + 4\alpha x - 4\alpha x - 4\beta 1) = -1\,. \quad \blacksquare$$

Proposition 2.1.4.19 *Let E be a finite-dimensional real unital algebra. Take $x, y \in E$ linearly independent with*

$$x^2 = y^2 = -1 .$$

If E is a division algebra, then $1, x, y$ are linearly independent and there is an $\alpha \in]-2, 2[$ such that

$$xy + yx = \alpha 1 .$$

By hypothesis, $xy \notin \mathbb{R}1$. So, by Proposition 2.1.4.18, there are $\alpha, \beta \in \mathbb{R}$ with

$$(xy)^2 - \alpha(xy) + \beta 1 = 0 , \quad \alpha^2 < 4\beta .$$

Then

$$\begin{aligned}
0 &= x(xyxy - \alpha xy + \beta 1) = -yxy + \alpha y + \beta x , \\
0 &= y(-yxy + \alpha y + \beta x) = xy - \alpha 1 + \beta yx , \\
0 &= (-yxy + \alpha y + \beta x)y = yx - \alpha 1 + \beta xy , \\
0 &= (xy - \alpha 1 + \beta yx) - \beta(yx - \alpha 1 + \beta xy) = \\
&= xy - \alpha 1 + \beta yx - \beta yx + \alpha\beta 1 - \beta^2 xy = \\
&= \alpha(\beta - 1)1 - (\beta^2 - 1)xy = (\beta - 1)(\alpha 1 - (\beta + 1)xy) , \\
0 &= (\beta - 1)(\alpha 1 - (\beta + 1)xy)y = (\beta - 1)(\alpha y + (\beta + 1)x) .
\end{aligned}$$

Since

$$\beta + 1 > 1$$

and $\{x, y\}$ is linearly independent,

$$\alpha y + (\beta + 1)x \neq 0 .$$

Thus

$$\beta = 1 , \quad \alpha^2 < 4$$

$$xy + yx = \alpha 1 .$$

Take $a, b, c \in \mathbb{R}$ with

$$a1 + bx + cy = 0\,.$$

Then

$$ax - b1 + cxy = 0$$

and

$$ax - b1 + cyx = 0\,.$$

Then

$$2ax - 2b1 + c\alpha 1 = 0\,, \quad 2ax = (2b - c\alpha)1\,, \quad -4a^2 = (2b - c\alpha)^2 \geq 0\,, \quad a = 0\,,$$

$$b = c = 0\,.$$

Hence $1, x, y$ are linearly independent. ■

Proposition 2.1.4.20 *Let E be a finite-dimensional real unital algebra, which is a division algebra, and take $x, y \in E$ linearly independent with*

$$x^2 = y^2 = -1\,.$$

Then there is an injective unital algebra homomorphism $u : \mathbb{H} \to E$ such that

$$x = u\imath\,, \quad y \in u(\mathbb{H})\,.$$

By Proposition 2.1.4.19, there is an $\alpha \in]-2, 2[$ such that

$$xy + yx = \alpha 1\,.$$

Then

$$yxy = y(\alpha 1 - yx) = \alpha y + x\,,$$

$$(xy)^2 = x(\alpha y + x) = \alpha xy - 1\,.$$

Set

$$a := \frac{\alpha}{\sqrt{4 - \alpha^2}}x + \frac{2}{\sqrt{4 - \alpha^2}}y\,,$$

$$b := xa = -\frac{\alpha}{\sqrt{4 - \alpha^2}}1 + \frac{2}{\sqrt{4 - \alpha^2}}xy\,.$$

Then

$$a^2 = \frac{\alpha^2}{4-\alpha^2}x^2 + \frac{2\alpha}{4-\alpha^2}(xy+yx) + \frac{4}{4-\alpha^2}y^2 =$$

$$= -\frac{\alpha^2}{4-\alpha^2}1 + \frac{2\alpha^2}{4-\alpha^2}1 - \frac{4}{4-\alpha^2}1 = \frac{\alpha^2-4}{4-\alpha^2}1 = -1\,,$$

$$b^2 = \frac{\alpha^2}{4-\alpha^2}1 - \frac{4\alpha}{4-\alpha^2}xy + \frac{4}{4-\alpha^2}(xy)^2 =$$

$$= \frac{\alpha^2}{4-\alpha^2}1 - \frac{4\alpha}{4-\alpha^2}xy + \frac{4}{4-\alpha^2}(\alpha xy - 1) =$$

$$= \frac{\alpha^2}{4-\alpha^2}1 - \frac{4\alpha}{4-\alpha^2}xy + \frac{4\alpha}{4-\alpha^2}xy - \frac{4}{4-\alpha^2}1 = \frac{\alpha^2-4}{4-\alpha^2}1 = -1\,,$$

$$ab = -\frac{\alpha^2}{4-\alpha^2}x + \frac{2\alpha}{4-\alpha^2}x^2y - \frac{2\alpha}{4-\alpha^2}y + \frac{4}{4-\alpha^2}yxy =$$

$$= -\frac{\alpha^2}{4-\alpha^2}x - \frac{2\alpha}{4-\alpha^2}y - \frac{2\alpha}{4-\alpha^2}y + \frac{4}{4-\alpha^2}(\alpha y + x) =$$

$$= -\frac{\alpha^2}{4-\alpha^2}x - \frac{4\alpha}{4-\alpha^2}y + \frac{4\alpha}{4-\alpha^2}y + \frac{4}{4-\alpha^2}x = \frac{4-\alpha^2}{4-\alpha^2}x = x\,,$$

$$ba = xa^2 = -x\,,$$

$$ax = a^2b = -b\,,$$

$$bx = xax = -xb = -x^2a = a\,.$$

Define

$$u : \mathbb{H} \longrightarrow E\,, \quad \beta 1 + \gamma i + \delta j + \varepsilon k \longmapsto \beta 1 + \gamma x + \delta a + \varepsilon b\,.$$

By the above, u is a unital algebra homomorphism. By Corollary 2.1.2.10 (and Corollary 2.1.4.17), u is injective. Hence

$$x = ui\,, \quad y \in u(\mathbb{H})\,. \qquad \blacksquare$$

Theorem 2.1.4.21 (Frobenius, 1878) *Every finite-dimensional real unital algebra, which is a divison algebra, is isomorphic to* $\mathbb{R}$, *to* $\mathbb{C}$, *or to* $\mathbb{H}$.

Let F be a finite-dimensional real unital algebra, which is a division algebra. Assume $E \neq \mathbb{R}1$. By Proposition 2.1.4.18, there is an $a \in E$ with

$$a^2 = -1.$$

Hence, if E is two-dimensional, then E is isomorphic to $\mathbb{C}$.

Assume that E is not two-dimensional. By Proposition 2.1.4.18, there is a $b \in E$, with a, b linearly independent and

$$b^2 = -1.$$

By Proposition 2.1.4.20, there is an injective unital algebra homomorphism $u : \mathbb{H} \to E$.

Assume that u is not surjective. Define

$$x := ui, \quad y := uj, \quad z := uk.$$

By Proposition 2.1.4.18, there is a $c \in E \backslash u(\mathbb{H})$ with

$$c^2 = -1.$$

By Proposition 2.1.4.20, there is an injective unital algebra homomorphism $v : \mathbb{H} \to E$ such that

$$x = vi, \quad c \in v(\mathbb{H}).$$

Define

$$Y := vj, \quad Z := vk.$$

Assume that $Y \in u(\mathbb{H})$. Then

$$Z = vk = v(ij) = (vi)(vj) = xY \in u(\mathbb{H}),$$

$$c \in v(\mathbb{H}) \subset u(\mathbb{H}),$$

which is a contradiction. Hence $Y \notin u(\mathbb{H})$. Similarly, $Z \notin u(\mathbb{H})$.

By Proposition 2.1.4.19, there are $\alpha, \beta \in \mathbb{R}$ with

$$yY + Yy = \alpha 1,$$

$$zY + Yz = \beta 1 .$$

Hence

$$\alpha z = (yY + Yy)z = yYz + Yyz = y(\beta 1 - zY) + Yx =$$

$$= \beta y - yzY - Z = \beta y - xY - Z = \beta y - Z - Z ,$$

$$Z = \frac{\beta}{2} y - \frac{\alpha}{2} z \in u(\mathbb{H})$$

which is a contradiction. Hence u is surjective and thus E is isomorphic to $\mathbb{H}$. ■

Proposition 2.1.4.22 *Let S, T be set, Φ the set of bijective maps $S \to T$, and $\mathcal{U}$ the set of algebra isomorphisms $\ell^\infty(T) \to \ell^\infty(S)$. Given $\varphi \in \Phi$, define*

$$u_\varphi : \ell^\infty(T) \longrightarrow \ell^\infty(S) , \quad x \longmapsto x \circ \varphi .$$

Then $u_\varphi \in \mathcal{U}$ and the map

$$\Phi \longrightarrow \mathcal{U} , \quad \varphi \longmapsto u_\varphi$$

is bijective. We may replace ℓ^∞ by c_0 or c in the above enunciation.

It is easy to see that $u_\varphi \in \mathcal{U}$ for every $\varphi \in \Phi$ and that the map

$$\Phi \longrightarrow \mathcal{U} , \quad \varphi \longmapsto u_\varphi$$

is injective. Take $u \in \mathcal{U}$ and $s \in S$. Then there is a $t = \varphi(s) \in T$ such that $u^{-1}(e_s)e_t \neq 0$. Then $\mathbb{K}e_t \subset u^{-1}(\mathbb{K}e_s)$, i.e. $u(\mathbb{K}e_t) \subset \mathbb{K}e_s$. From

$$e_s u(e_t) = u(u^{-1}(e_s)e_t) \neq 0$$

it follows that $\mathbb{K}e_s \subset u(\mathbb{K}e_t)$. Hence $u(\mathbb{K}e_t) = \mathbb{K}e_s$. The map φ is obviously injective. By symmetry, it is bijective.

Take $x \in \ell^\infty(T)$ and $s \in S$. Then

$$x(\varphi(s))e_{\varphi(s)} = xe_{\varphi(s)} ,$$

so

$$(u_\varphi x)e_s = x(\varphi(s))e_s = x(\varphi(s))ue_{\varphi(s)} = u(xe_{\varphi(s)}) =$$

$$= (ux)(ue_{\varphi(s)}) = (ux)e_s .$$

Thus $u_\varphi x = ux$ and $u_\varphi = u$. Hence the map

$$\Phi \longrightarrow \mathcal{U} , \quad \varphi \longmapsto u_\varphi$$

is surjective. ■

Definition 2.1.4.23 (0) *For every vector space E and for every $m, n \in \mathbb{N}$, we denote by $E_{m,n}$ the vector space of $m \times n$-matrices with entries from E, the addition and the scalar multiplication being defined pointwise. If E is an algebra, $m, n, p \in \mathbb{N}$, $a \in E_{m,n}$, $b \in E_{n,p}$, then we denote by ab the matrix of $E_{m,p}$ defined by*

$$(ab)_{i,k} := \sum_{j=1}^{n} a_{ij} b_{jk}$$

for all $i \in \mathbb{N}_m$ and $k \in \mathbb{N}_p$.

Proposition 2.1.4.24 (0) *Let E be an algebra and $m, n, p, q \in \mathbb{N}$.*

a) $a \in E_{m,n}$, $b \in E_{n,p}$, $c \in E_{p,q} \Rightarrow (ab)c = a(bc)$.

b) *$E_{n,n}$ endowed with the bilinear map*

$$E_{n,n} \times E_{n,n} \longrightarrow E_{n,n}, \quad (a, b) \longmapsto ab$$

is an algebra.

c) *If E is unital then $[\delta_{ij} 1]_{i,j \in \mathbb{N}_n}$ is a unit of $E_{n,n}$.* ■

Proposition 2.1.4.25 (0) *Let E be a normed space and $m, n \in \mathbb{N}$. We set*

$$\widetilde{a'} : E_{m,n} \longrightarrow \mathbb{K}, \quad a \longmapsto \sum_{i=1}^{m} \sum_{j=1}^{n} \langle a_{ij}, a'_{ji} \rangle$$

for every $a' \in E'_{n,m}$. If $E_{m,n}$ is endowed with a norm such that

$$\sup_{\substack{i \in \mathbb{N}_m \\ j \in \mathbb{N}_n}} \|a_{ij}\| \leq \|a\| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{ij}\|$$

for every $a \in E_{m,n}$, then $\widetilde{a'} \in (E_{m,n})'$ and

$$\sup_{\substack{i \in \mathbb{N}_m \\ j \in \mathbb{N}_n}} \|a'_{ji}\| \leq \|\widetilde{a'}\| \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \|a'_{ji}\|$$

for every $a' \in E'_{n,m}$ and the map

$$E'_{n,m} \longrightarrow (E_{m,n})', \quad a' \longmapsto \widetilde{a'}$$

is an isomorphism of vector spaces.

We have

$$|\widetilde{a'}(a)| = \left|\sum_{i=1}^{m}\sum_{j=1}^{n}\langle a_{ij}, a'_{ji}\rangle\right| \leq \sum_{i=1}^{m}\sum_{j=1}^{n}|\langle a_{ij}, a'_{ji}\rangle| \leq$$

$$\leq \sum_{i=1}^{m}\sum_{j=1}^{n}\|a_{ij}\|\,\|a'_{ji}\| \leq \|a\|\sum_{i=1}^{m}\sum_{j=1}^{n}\|a'_{ji}\|$$

for every $a \in E_{m,n}$, so $\widetilde{a'} \in (E_{m,n})'$ and

$$\|\widetilde{a'}\| \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\|a'_{ji}\|\,.$$

Let $(p,q) \in \mathbb{N}_m \times \mathbb{N}_n$ such that

$$\|a'_{qp}\| = \sup_{\substack{i\in\mathbb{N}_m\\ j\in\mathbb{N}_n}}\|a'_{ji}\|\,.$$

Further let $\theta \in]0,1[$ and $x \in E^{\#}$ such that

$$\langle x, a'_{qp}\rangle \geq \theta\|a'_{qp}\|\,.$$

We set

$$a := (\delta_{ip}\delta_{qj}x)_{(i,j)\in\mathbb{N}_m\times\mathbb{N}_n}\,.$$

Then $a \in E_{m,n}$,

$$\|a\| = \|x\| \leq 1\,,$$

and

$$\|\widetilde{a'}\| \geq |\widetilde{a'}(a)| = \left|\sum_{i=1}^{m}\sum_{j=1}^{n}\langle a_{ij}, a'_{ji}\rangle\right| =$$

$$= \left|\langle x, a'_{qp}\rangle\right| \geq \theta\|a'_{qp}\|\,.$$

Since θ is arbitrary we get

$$\|\widetilde{a'}\| \geq \|a'_{qp}\| = \sup_{\substack{i\in\mathbb{N}_m\\ j\in\mathbb{N}_n}}\|a'_{ji}\|\,.$$

By Proposition 1.2.1.19 b) (and Remark of Definition 1.1.5.2), the map

$$E'_{n,m} \longrightarrow (E_{m,n})', \quad a' \longmapsto \widetilde{a'}$$

is an isomorphism of vector spaces. ■

Proposition 2.1.4.26 (0) *Let $n \in \mathbb{N}$. If $\mathbb{C}$ (resp. $\mathbb{H}$) is isomorphic to a unital subalgebra of $\mathbb{R}_{n,n}$ (resp. $\mathbb{C}_{n,n}$, resp. $\mathbb{R}_{n,n}$) (Proposition 2.1.4.24 b),c)) then $\frac{n}{2} \in \mathbb{N}$ (resp. $\frac{n}{2} \in \mathbb{N}$, resp. $\frac{n}{4} \in \mathbb{N}$).*

We identify $\mathbb{C}$ (resp. $\mathbb{H}$) with the corresponding unital subalgebra. Then $\mathbb{R}^n$ and $\mathbb{C}^n$ endowed with the maps

$$(x, \xi) \longmapsto x\xi$$

are vector space over $\mathbb{C}$ or over $\mathbb{H}$. If A is an algebraic basis of this vector space then

$$A \cup (iA) \quad (\text{resp. } A \cup (jA), \quad \text{resp. } A \cup (iA) \cup (jA) \cup (kA))$$

is an algebraic basis of the vector space $\mathbb{R}^n$ or $\mathbb{C}^n$. Hence

$$n = 2\text{Card}\, A \quad (\text{resp. } n = 2\text{Card}\, A, \quad \text{resp. } n = 4\text{Card}\, A)$$

so that

$$\frac{n}{2} = \text{Card}\, A \in \mathbb{N} \quad (\text{resp. } \frac{n}{2} = \text{Card}\, A \in \mathbb{N}, \quad \text{resp. } \frac{n}{4} = \text{Card}\, A \in \mathbb{N}). \quad \blacksquare$$

2.1.5 Complexification of Algebras

Complex algebras are easier to handle than real ones. In order to establish some properties of real algebras it is sometimes useful to associate to a real algebra E a canonical complex algebra $\overset{\circ}{E}$ defined on the set $E \times E$ (called the complexification of E) and to deduce the required properties for E using the corresponding properties of $\overset{\circ}{E}$.

Lemma 2.1.5.1 (0) *Let E, F, G, H, I, J be real vector spaces and let $E \times F \to H$, $F \times G \to I$, $H \times G \to J$, $E \times I \to J$ be bilinear maps written in the product form*

$$(a, b) \longmapsto ab .$$

We define the map $E^2 \times F^2 \to H^2$, $F^2 \times G^2 \to I^2$, $H^2 \times G^2 \to J^2$, and $E^2 \times I^2 \to J^2$ by

$$((a, b), (c, d)) \longmapsto (a, b)(c, d) := (ac - bd, ad + bc) .$$

a) *The above defined maps are bilinear.*

b) *If $(ab)c = a(bc)$ for all $(a, b, c) \in E \times F \times G$, then*

$$((a, b)(c, d))(e, f) = (a, b)((c, d)(e, f))$$

for all $((a, b), (c, d), (e, f)) \in E^2 \times F^2 \times G^2$ The same relation holds if we define the map $F^2 \times G^2 \to I^2$, $H^2 \times G^2 \to J^2$ by

$$((a, b), (c, d)) \longmapsto (ac + bd, bc - ad) .$$

c) *E^2 endowed with the map*

$$\mathbb{C} \times E^2 \longrightarrow E^2, \quad ((\alpha, \beta), (a, b)) \longmapsto (\alpha a - \beta b, \alpha b + \beta b) ,$$

*where $\mathbb{C}$ was identified with $\mathbb{R}^2$, is a complex vector space, called **the complexification of** E and denoted by $\overset{\circ}{E}$.*

d) *The map*

$$\overset{\circ}{E} \times \overset{\circ}{F} \longrightarrow \overset{\circ}{G}, \quad ((a, b), (c, d)) \longmapsto (a, b)(c, d)$$

is $\mathbb{C}$-bilinear.

e) *If $(x_\iota)_{\iota \in I}$ is an algebraic basis of E then $((x_\iota, 0))_{\iota \in I}$ is an algebraic basis of $\overset{\circ}{E}$.*

a) is easy to see.

b) We have

$$((a,b)(c,d)(e,f) = (ac-bd, ad+bc)(e,f) =$$

$$= ((ac-bd)e-(ad+bc)f, (ac-bd)f+(ad+bc)e) =$$

$$= (a(ce-df)-b(de+cf), a(cf+de)+b(ce-df)) =$$

$$= (a,b)(ce-df, cf+de) = (a,b)((c,d)(e,f)).$$

The second assertion follows from

$$((a,b)(c,d))(e,f) = (ac-bd, ad+bc)(e,f) =$$

$$= ((ac-bd)e+(ad+bc)f, (ad+bc)e-(ac-bd)f) =$$

$$= (a(ce+df)-b(de-cf), a(de-cf)+b(ce+df)) =$$

$$= (a,b)(ce+df, de-cf) = (a,b)((c,d)(e,f)).$$

c) follows from a) and b) by replacing E and G with $\mathbb{R}$ and F, H, I, J with E.

d) follows from a),b), and c).

e) Take $h \in \mathbb{C}^{(I)}$ with

$$\sum_{\iota \in I} h(\iota)(x_\iota, 0) = 0.$$

Then

$$\Big(\sum_{\iota \in I} \operatorname{re} h(\iota) x_\iota, \quad \sum_{\iota \in I} \operatorname{im} h(\iota) x_\iota\Big) = 0,$$

so that

$$\sum_{\iota \in I} \operatorname{re} h(\iota) x_\iota = \sum_{\iota \in I} \operatorname{im} h(\iota) x_\iota = 0,$$

and thus

$$h(\iota) = 0$$

for every $\iota \in I$. Hence $((x_\iota, 0))_{\iota \in I}$ is linearly independent.

Take $(x, y) \in E^2$. There are $f, g \in \mathbb{R}^{(I)}$ such that

$$x = \sum_{\iota \in I} f(\iota) x_\iota, \quad y = \sum_{\iota \in I} g(\iota) x_\iota .$$

Define

$$h : I \longrightarrow \mathbb{C}, \quad \iota \longmapsto f(\iota) + i g(\iota) .$$

Then

$$\sum_{\iota \in I} h(\iota)(x_\iota, 0) = \Big(\sum_{\iota \in I} f(\iota) x_\iota , \sum_{\iota \in I} g(\iota) x_\iota \Big) = (x, y) .$$

Hence $((x_\iota, 0))_{\iota \in I}$ is an algebraic basis for E^2. ■

Remark. a) We sometimes identify E with the real vector subspace $E \times \{0\}$ of $\overset{\circ}{E}$. By this identification we may write

$$(x, y) = (x, 0) + i(y, 0) = x + iy$$

for every $(x, y) \in \overset{\circ}{E}$.

b) $\mathbb{C}$ is the complexification of $\mathbb{R}$.

Example 2.1.5.2 (0) *Let T be a set and $\overset{\circ}{\widehat{\mathbb{R}^T}}$ the complexification of the vector space $\mathbb{R}^T$. Given $(x, y) \in \overset{\circ}{\widehat{\mathbb{R}^T}}$, define*

$$\widetilde{(x, y)} : T \longrightarrow \mathbb{C}, \quad t \longmapsto x(t) + i y(t) .$$

Then the map

$$\overset{\circ}{\widehat{\mathbb{R}^T}} \longrightarrow \mathbb{C}^T, \quad (x, y) \longmapsto \widetilde{(x, y)}$$

is linear and bijective (we may therefore identify the complexification of $\mathbb{R}^T$ with $\mathbb{C}^T$).

It is obvious that the above map is $\mathbb{R}$–linear and bijective. Take $(x, y) \in \overset{\circ}{\widehat{\mathbb{R}^T}}$. Then

$$\widetilde{i(x, y)}(t) = \widetilde{(-y, x)}(t) = -y(t) + i x(t) =$$

$$= i(x(t) + iy(t)) = i\widetilde{(x,y)}(t) = (i\widetilde{(x,y)})(t)$$

for every $t \in T$, so

$$\widetilde{i(x,y)} = i\widetilde{(x,y)}\,.$$

Hence the above map is $\mathbb{C}$–linear. ■

Lemma 2.1.5.3 (0) *Let E, F be real vector spaces, $L(E,F)$ the real vector space of linear maps of E into F, $\overset{\circ}{E}$, $\overset{\circ}{F}$, and $\overset{\circ}{L}(E,F)$ the complexifications of E, F, and $L(E,F)$, respectively, and $L(\overset{\circ}{E},\overset{\circ}{F})$ the complex vector space of the linear maps of $\overset{\circ}{E}$ into $\overset{\circ}{F}$. For $(u,v) \in \overset{\circ}{L}(E,F)$ define*

$$\widetilde{(u,v)} : \overset{\circ}{E} \longrightarrow \overset{\circ}{F}, \quad (x,y) \longmapsto (ux - vy, uy + vx)\,.$$

Then $\widetilde{(u,v)} \in L(\overset{\circ}{E},\overset{\circ}{F})$ for every $(u,v) \in \overset{\circ}{L}(E,F)$ and the map

$$\overset{\circ}{L}(E,F) \longmapsto L(\overset{\circ}{E},\overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

is linear and bijective. (Thus $\overset{\circ}{L}(E,F)$ and $L(\overset{\circ}{E},\overset{\circ}{F})$ may be identified.) Moreover, if G is another real vector space then (with obvious notation)

$$\widetilde{(c,d)} \circ \widetilde{(a,b)} = \widetilde{(c \circ a - d \circ b,\ c \circ b + d \circ a)}$$

for every $(a,b) \in \overset{\circ}{L}(E,F)$ and $(c,d) \in \overset{\circ}{L}(F,G)$.

The map

$$L(E,F) \times E \longrightarrow F, \quad (u,x) \longmapsto ux$$

is bilinear. By Lemma 2.1.5.1 d), the map

$$\overset{\circ}{L}(E,F) \times \overset{\circ}{E} \longrightarrow \overset{\circ}{F}, \quad ((u,v),(x,y)) \longmapsto \widetilde{(u,v)}((x,y))$$

is bilinear. Hence $\widetilde{(u,v)} \in L(\overset{\circ}{E},\overset{\circ}{F})$ for every $(u,v) \in \overset{\circ}{L}(E,F)$ and the map

$$\overset{\circ}{L}(E,F) \longrightarrow L(\overset{\circ}{E},\overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

is linear. It is obviously injective.

For $w \in L(\overset{\circ}{E},\overset{\circ}{F})$, we define the maps u, v of E into F by

$$(ux, vx) := w((x,0))$$

for all $x \in E$. Then $u, v \in L(E,F)$ and

$$\widetilde{(u,v)}((x,0)) = (ux, vx) = w((x,0))$$

for every $x \in E$. Hence

$$\widetilde{(u,v)} = w$$

and the map

$$\overset{\circ}{L}(E,F) \longrightarrow L(\overset{\circ}{E},\overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

is bijective.

The maps

$$L(F,G) \times F \longrightarrow G, \quad (v,y) \longmapsto vy,$$

$$L(E,F) \times L(F,G) \longrightarrow L(E,G), \quad (u,v) \longmapsto v \circ u,$$

$$L(E,G) \times E \longrightarrow G, \quad (w,x) \longmapsto wx$$

being bilinear, the final assertion follows from Lemma 2.1.5.1 b). ■

Corollary 2.1.5.4 (0) *Let E be a real vector space, $\overset{\circ}{E}$ its complexification, E' (resp. $(\overset{\circ}{E})'$) the algebraic dual of E (resp. of $\overset{\circ}{E}$) and $\overset{\circ}{\widehat{E'}}$ the complexification of E'. Given $(x',y') \in \overset{\circ}{\widehat{E'}}$, define*

$$\widetilde{(x',y')} : \overset{\circ}{E} \longrightarrow \mathbb{C}, \quad (x,y) \longmapsto x'(x) - y'(y) + i(x'(y) + y'(x)).$$

Then $\widetilde{(x',y')} \in (\overset{\circ}{E})'$ for every $(x',y') \in \overset{\circ}{\widehat{E'}}$ and the map

$$\overset{\circ}{\widehat{E'}} \longrightarrow (\overset{\circ}{E})', \quad (x',y') \longmapsto \widetilde{\widetilde{(x',y')}}$$

is linear and bijective. (Thus $(\overset{\circ}{E})'$ and $\overset{\circ}{\widehat{E'}}$ may be identified.) ■

Example 2.1.5.5 *Let $m,n \in \mathbb{N}$. For $a \in \mathbb{K}_{m,n}$, define*

$$\widetilde{a} : \mathbb{K}^n \longrightarrow \mathbb{K}^m, \quad x \longmapsto ax,$$

and identify $\mathbb{K}_{m,n}$ with the vector space $L(\mathbb{K}^n, \mathbb{K}^m)$ of linear maps of $\mathbb{K}^n$ into $\mathbb{K}^m$ via the map

$$\mathbb{K}_{m,n} \longrightarrow L(\mathbb{K}^n, \mathbb{K}^m), \quad a \longmapsto \widetilde{a}.$$

If $\overset{\circ}{L}(\mathbb{R}^n, \mathbb{R}^m)$ denotes the complexification of $L(\mathbb{R}^n, \mathbb{R}^m)$ then the map

$$\overset{\circ}{L}(\mathbb{R}^n, \mathbb{R}^m) \longrightarrow L(\mathbb{C}^n, \mathbb{C}^m), \quad (a,b) \longmapsto a + ib$$

is linear and bijective.

The assertion follows from Example 2.1.5.2 and Lemma 2.1.5.3. ■

Proposition 2.1.5.6 (0) *Let E, F be normed real vector spaces and $\overset{\circ}{E}, \overset{\circ}{F}, \overset{\circ}{\widehat{E'}}, \overset{\circ}{\widehat{F'}}, \overset{\circ}{\mathcal{L}}(E,F)$, and $\overset{\circ}{\mathcal{L}}(F',E')$ be the complexifications of the underlying real vector spaces $E, F, E', F', \mathcal{L}(E,F)$, and $\mathcal{L}(F',E')$, respectively. We assume that $\overset{\circ}{E}$ and $\overset{\circ}{F}$ are endowed with norms such that the canonical bijections*

$$\overset{\circ}{E} \longrightarrow E \times E, \quad \overset{\circ}{F} \longrightarrow F \times F$$

are isomorphisms of real normed spaces. Given $(u,v) \in \overset{\circ}{\mathcal{L}}(E,F)$, define

$$\widetilde{(u,v)} : \overset{\circ}{E} \longrightarrow \overset{\circ}{F}, \quad (x,y) \longmapsto (ux - vy, uy + vx).$$

For $(u,v) \in \overset{\circ}{\mathcal{L}}(F',E')$ put

$$\widetilde{(u,v)} : (\overset{\circ}{F})' \longrightarrow (\overset{\circ}{E})', \quad (x',y') \longmapsto (ux' - vy', uy' + vx').$$

Then $\widetilde{(u,v)} \in \mathcal{L}(\overset{\circ}{E}, \overset{\circ}{F})$ and (with the identification of Corollary 2.1.5.4)

$$\widetilde{(u,v)}' = \widetilde{(u',v')}$$

for every $(u,v) \in \overset{\circ}{\mathcal{L}}(E,F)$ and the map

$$\overset{\circ}{\mathcal{L}}(E,F) \longrightarrow \mathcal{L}(\overset{\circ}{E}, \overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

is an isomorphism of complex vector spaces.

The relation $\widetilde{(u,v)} \in \mathcal{L}(\overset{\circ}{E}, \overset{\circ}{F})$ follows easily from Lemma 2.1.5.3. Take $(x,y) \in \overset{\circ}{E}$ and $(x',y') \in \overset{\circ}{\widehat{F'}}$. Then

$$\langle (x,y), \widetilde{(u',v')}(x',y') \rangle = \langle (x,y), (u'x' - v'y', u'y' + v'x') \rangle =$$

$$= \langle x, u'x' \rangle - \langle x, v'y' \rangle - \langle y, u'y' \rangle - \langle y, v'x' \rangle +$$

$$+ i(\langle x, u'y' \rangle + \langle x, v'x' \rangle + \langle y, u'x' \rangle - \langle y, v'y' \rangle) =$$

$$= \langle ux, x' \rangle - \langle vx, y' \rangle - \langle uy, y' \rangle - \langle vy, x' \rangle +$$

$$+ i(\langle ux, y' \rangle + \langle vx, x' \rangle + \langle uy, x' \rangle - \langle vy, y' \rangle) =$$

$$= \langle (ux - vy, uy + vx), (x',y') \rangle = \langle \widetilde{(u,v)}(x,y), (x',y') \rangle .$$

Hence

$$\widetilde{(u',v')} = \widetilde{(u,v)}' .$$

The last assertion follows from Lemma 2.1.5.3 ■

Proposition 2.1.5.7 (0) *Let E be a real algebra and $\overset{\circ}{E}$ the complexification of the underlying vector space (Lemma 2.1.5.1). Then $\overset{\circ}{E}$ endowed with the bilinear map*

$$\overset{\circ}{E} \times \overset{\circ}{E} \longrightarrow \overset{\circ}{E}, \quad ((a,b),(c,d)) \longmapsto (ac - bd, ad + bc)$$

is a complex algebra, called the ***complexification of*** *E. If $\overset{\circ}{E}$ is unital, E is unital too. If 1 is a unit of E, then $(1,0)$ is a unit of $\overset{\circ}{E}$. If E is commutative, then $\overset{\circ}{E}$ is also commutative. If every left (right) invertible element of $\overset{\circ}{E}$ is invertible, then the same holds for E.*

It is easy to see that the above map is bilinear. By Lemma 2.1.5.1 b), this map is associative.

Assume that $\overset{\circ}{E}$ is unital and let (a,b) be the unit in $\overset{\circ}{E}$. Take $x \in E$. Then

$$(x,0) = (x,0)(a,b) = (xa, xb),$$

$$(x,0) = (a,b)(x,0) = (ax, bx),$$

so that

$$x = xa = ax.$$

Hence a is a unit of E.

The final assertions are trivial. ■

Corollary 2.1.5.8 (0) *Let E be a real vector space, $L(E)$ the unital real algebra of linear maps $E \to E$, (Example 2.1.4.6), $\overset{\circ}{E}$ and $\overset{\circ}{L}(E)$ the complexification of E and $L(E)$, respectively, and $L(\overset{\circ}{E})$ the complex unital algebra of linear maps $\overset{\circ}{E} \to \overset{\circ}{E}$. Given $(u,v) \in \overset{\circ}{L}(E)$, put*

$$\widetilde{(u,v)} : \overset{\circ}{E} \longrightarrow \overset{\circ}{E}, \quad (x,y) \longmapsto (ux - vy, uy + vx).$$

Then the map

$$\overset{\circ}{L}(E) \longrightarrow L(\overset{\circ}{E}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

(Lemma 2.1.5.3) is an isomorphism of unital algebras. Thus the unital algebras $\overset{\circ}{L}(E)$ and $L(\overset{\circ}{E})$ may be identified.

The assertion follows immediately from Lemma 2.1.5.3 and Proposition 2.1.5.7. ■

Example 2.1.5.9 *For $n \in \mathbb{N}$, let $\overset{\circ}{\mathbb{R}}_{n,n}$ be the complexification of $\mathbb{R}_{n,n}$. Then the map*

$$\overset{\circ}{\mathbb{R}}_{n,n} \longrightarrow \mathbb{C}_{n,n}, \quad (a,b) \longmapsto a + ib$$

is an isomorphism of unital algebras. ■

Proposition 2.1.5.10 (0) *Let E be a real algebra and $\overset{\circ}{E}$ its complexification.*

a) *If F is a subalgebra of E, then*

$$\{(x,y) \mid x,y \in F\}$$

is a subalgebra of $\overset{\circ}{E}$.

b) *If F is a (left, right) ideal of E, then*

$$\{(x,y) \mid x,y \in F\}$$

is a (left, right) ideal of $\overset{\circ}{E}$.

c) *Let F be an ideal of E and $p : E \to E/F$, $q : \overset{\circ}{E} \to \overset{\circ}{E}/\overset{\circ}{F}$ the respective quotient maps. Define*

$$r : \overset{\circ}{E} \longrightarrow \overset{\circ}{\widehat{E/F}}, \quad (x,y) \longmapsto (px, py).$$

Then there is a unique $u : \overset{\circ}{\widehat{E/F}} \to \overset{\circ}{E}/\overset{\circ}{F}$ such that $q = u \circ r$ and u is an isomorphism of complex algebras.

The proof is a simple verification. ■

Proposition 2.1.5.11 (0) *Let E, F be real algebras, $\overset{\circ}{E}, \overset{\circ}{F}$ their complexifications, and $u : E \to F$ an algebra homomorphism. Then*

$$\overset{\circ}{u} : \overset{\circ}{E} \longrightarrow \overset{\circ}{F}, \quad (x,y) \longmapsto (ux, uy)$$

is an algebra homomorphism, surjective iff u is surjective. The map $u \mapsto \overset{\circ}{u}$ is injective.

It is obvious that $\overset{\circ}{u}$ is linear. If $(a,b), (c,d) \in \overset{\circ}{E}$, then

$$\overset{\circ}{u}((a,b)(c,d)) = \overset{\circ}{u}((ac - bd, ad + bc)) =$$

$$= ((ua)(uc) - (ub)(ud), (ua)(ud) + (ub)(uc)) =$$

$$= (ua, ub)(uc, ud) = \overset{\circ}{u}((a,b))\overset{\circ}{u}((c,d)),$$

so $\overset{\circ}{u}$ is an algebra homomorphism.

The last assertions are easy to see. ■

Proposition 2.1.5.12 (0) *Let E be a unital real algebra and $\overset{\circ}{E}$ its complexification. Then $x \in E$ is invertible in E iff $(x,0)$ is invertible in $\overset{\circ}{E}$.*

Assume $(x,0)$ invertible in $\overset{\circ}{E}$ and let

$$(y,z) := (x,0)^{-1}.$$

Then

$$(1,0) = (y,z)(x,0) = (yx, zx),$$

$$(1,0) = (x,0)(y,z) = (xy, xz).$$

We get

$$xy = yx = 1,$$

so x is invertible in E. The converse is trivial. ■

Proposition 2.1.5.13 (0) *Let E be a real algebra and F its associated unital algebra. The $\overset{\circ}{F}$ is the unital algebra associated to $\overset{\circ}{E}$.*

The map

$$\overset{\circ}{F} \longrightarrow \mathbb{C} \times \overset{\circ}{E}, \quad ((\alpha,x),(\beta,y)) \longmapsto (\alpha + i\beta, x + iy)$$

is an isomorphism of unital algebras. ■

Corollary 2.1.5.14 (0) *If E is a real algebra and $x \in E$, then*

$$\sigma_E(x) = \mathbb{R} \cap \sigma_{\overset{\circ}{E}}((x,0)),$$

where $\overset{\circ}{E}$ denotes the complexification of E.

By Proposition 2.1.5.13 we may assume that E is unital . Take $\alpha \in \mathbb{R}$. By Proposition 2.1.5.12, $\alpha 1 - x$ is not invertible in E iff $\alpha(1,0) - (x,0)$ is not invertible in $\overset{\circ}{E}$ so

$$\alpha \in \sigma_E(x) \Longleftrightarrow \alpha \in \sigma_{\overset{\circ}{E}}((x,0)).$$ ■

Proposition 2.1.5.15 (4) *Let E be a real algebra and take $x \in E$. Then the following are equivalent:*

a) *x belongs to the radical of E.*

b) $(x,0)$ *belongs to the radical of* $\overset{\circ}{E}$.

In particular, if $\overset{\circ}{E}$ *is semi-simple,* E *is also semi-simple.*

By Proposition 2.1.3.19 and Proposition 2.1.5.13, we may assume E unital.

a $\Rightarrow$ b. Take $(a,b) \in \overset{\circ}{E}$. By Proposition 2.1.3.21 a $\Rightarrow$ b, there are $c,d \in E$ such that

$$c(1-ax) = 1\,, \quad d(1-(a-bxcb)x) = 1\,.$$

Then

$$\begin{aligned}&(d, dbxc)((1,0)-(a,b)(x,0)) = \\ &= (d, dbxc)(1-ax, -bx) = (d(1-ax)+dbxcbx, dbxc(1-ax)-dbx) = \\ &= (d(1-(a-bxcb)x), dbx - dbx) = (1,0)\,.\end{aligned}$$

By Proposition 2.1.3.21 b $\Rightarrow$ a, $(x,0)$ belongs to the radical of $\overset{\circ}{E}$.

b $\Rightarrow$ a. Take $y \in E$. By Proposition 2.1.3.21 a $\Rightarrow$ b, there is an $(a,b) \in \overset{\circ}{E}$ such that

$$(a,b)((1,0)-(y,0)(x,0)) = (1,0)\,.$$

It follows that

$$a(1-yx) = 1\,.$$

By Proposition 2.1.3.21 b $\Rightarrow$ a, x belongs to the radical of E. ■

Proposition 2.1.5.16 *Let* E *(resp.* F*) be a real (resp. complex) algebra,* $\overset{\circ}{E}$ *the complexification of* E *(Proposition 2.1.5.7), and* $\mathcal{F}$ *(resp.* $\mathcal{G}$*) the set of real (resp. complex) algebra homomorphisms of* E *(of* $\overset{\circ}{E}$*) into* F*. For* $u \in \mathcal{F}$*, define*

$$\overset{\circ}{u} : \overset{\circ}{E} \longrightarrow F\,, \quad (x,y) \longmapsto ux + iuy\,.$$

Then $\overset{\circ}{u} \in \mathcal{G}$ *whenever* $u \in \mathcal{F}$ *and the map*

$$\mathcal{F} \longrightarrow \mathcal{G}\,, \quad u \longmapsto \overset{\circ}{u}$$

is bijective. If E, F*, and* u *are unital, then* $\overset{\circ}{u}$ *is also unital.*

Let $u \in \mathcal{F}$. Let $a, b, c, d \in E$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\mathring{u}((a,b)+(c,d)) = \mathring{u}((a+c,b+d)) = u(a+c) + iu(b+d) =$$

$$= ua + uc + iub + iud = \mathring{u}((a,b)) + \mathring{u}((c,d)),$$

$$\mathring{u}(\alpha + i\beta)(a,b)) = \mathring{u}((\alpha a - \beta b, \alpha b + \beta a)) = u(\alpha a - \beta b) + iu(\alpha b + \beta a) =$$

$$= \alpha ua - \beta ub + i\alpha ub + i\beta ua = (\alpha + i\beta)(ua + iub) = (\alpha + i\beta)\mathring{u}((a,b)),$$

$$\mathring{u}((a,b)(c,d)) = \mathring{u}((ac - bd, ad + bc)) = u(ac - bd) + iu(ad + bc) =$$

$$= (ua)(uc) - (ub)(ud) + i(ua)(ud) + i(ub)(uc) = (ua + iub)(uc + iud) =$$

$$= \mathring{u}((a,b))\mathring{u}((c,d)).$$

Hence $\mathring{u} \in \mathcal{G}$.

Given $v \in \mathcal{G}$, define

$$u : E \longrightarrow F, \quad x \longmapsto v((x,0)).$$

Then $u \in \mathcal{F}$ and

$$\mathring{u}((x,0)) = v((x,0))$$

for every $x \in E$. Hence $\mathring{u} = v$ and the map

$$\mathcal{F} \longrightarrow \mathcal{G}, \quad u \longmapsto \mathring{u}$$

is surjective. Since it is obviously injective, it is bijective.

The last assertion is trivial. ■

Corollary 2.1.5.17 *Let E be a real algebra and $\mathring{E}$ its complexification. Let $\mathcal{F}$ and $\mathcal{G}$ denote the sets of algebra homomorphisms of E into $\mathbb{R}$ and of $\mathring{E}$ into $\mathbb{C}$, respectively. Further, given $u \in \mathcal{F}$, let $\mathring{u}$ denote the element of $\mathcal{G}$ defined by*

$$\mathring{E} \longrightarrow \mathbb{C}, \quad (x,y) \longmapsto ux + iuy$$

(Proposition 2.1.5.11) and define

$$\varphi : \mathcal{F} \longrightarrow \mathcal{G}, \quad u \longmapsto \mathring{u}.$$

Then the following are equivalent:

a) φ *is bijective.*

b) φ *is surjective.*

c) *If* u *is an algebra homomorphism of* E *into the real algebra* $\mathbb{C}$, *then* $u(E) \subset \mathbb{R}$.

These conditions are fullfilled if

d) $\sigma_{\overset{\circ}{E}}((x,0)) \subset \mathbb{R}$ *for every* $x \in E$.

a $\Rightarrow$ b is trivial.

b $\Rightarrow$ c . By Proposition 2.1.5.16, the map

$$\overset{\circ}{E} \longrightarrow \mathbb{C}, \quad (x,y) \longmapsto ux + iuy$$

belongs to $\mathcal{G}$. By b), there is a $v \in \mathcal{F}$ such that

$$vx + ivy = ux + iuy$$

for all $x, y \in E$. Thus

$$ux = vx \in \mathbb{R}$$

for every $x \in E$, i.e. $u(E) \subset \mathbb{R}$.

c $\Rightarrow$ a . Take $v \in \mathcal{G}$. By Proposition 2.1.5.16, there is an algebra homomorphism w of E into the real algebra $\mathbb{C}$ such that

$$v(x,y) = wx + iwy$$

for all $x, y \in E$. Using c), define

$$u : E \longrightarrow \mathbb{R}, \quad x \longmapsto wx.$$

Then $u \in \mathcal{F}$ and $\varphi u = v$. Hence φ is surjective. By the last assertion of Proposition 2.1.5.11 φ is bijective.

d $\Rightarrow$ c . Let u be an algebra homomorphism of E into the real algebra $\mathbb{C}$,

$$\overset{\circ}{u} : \overset{\circ}{E} \longrightarrow \mathbb{C}, \quad (x,y) \longmapsto ux + iuy,$$

and take $x \in E$. By Proposition 2.1.5.16, $\overset{\circ}{u}$ is a homomorphism of complex algebras and

$$ux \in \{ux\} = \sigma_{\mathbb{C}}(ux) = \sigma_{\mathbb{C}}(\overset{\circ}{u}(x,0)) \subset \sigma_{\overset{\circ}{E}}((x,0)) \cup \{0\} \subset \mathbb{R}$$

(Proposition 2.1.3.2, Corollary 2.1.3.12), i.e. $u(E) \subset \mathbb{R}$. ∎

Proposition 2.1.5.18 (0) *The following are equivalent for every unital real algebra E.*

a) $x \in E \Rightarrow \sigma_{\overset{\circ}{E}}((x,0)) = \sigma_E(x)$.

b) $x \in E \Rightarrow \sigma_E(x^2) \subset \mathbb{R}_+$.

a $\Rightarrow$ b. Put

$$P(t) := t^2 \in \mathbb{K}[t].$$

By a) and Theorem 2.1.3.4 b),

$$\sigma_E(x^2) = \sigma_{\overset{\circ}{E}}((x^2,0)) = \sigma_{\overset{\circ}{E}}((x,0)^2) = P(\sigma_{\overset{\circ}{E}}(x,0)) = P(\sigma_E(x)) \subset \mathbb{R}_+.$$

b $\Rightarrow$ a. Take $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$. By b),

$$y := (\alpha 1 - x)^2 + \beta^2 1$$

is invertible. From

$$(\alpha 1 - x, \beta 1)((\alpha 1 - x)y^{-1}, -\beta y^{-1}) =$$

$$= ((\alpha 1 - x)^2 y^{-1} + \beta^2 y^{-1}, \beta(\alpha 1 - x)y^{-1} - \beta(\alpha 1 - x)y^{-1}) = (1,0),$$

$$(y^{-1}(\alpha 1 - x), -\beta y^{-1})(\alpha 1 - x, \beta 1) =$$

$$= (y^{-1}(\alpha 1 - x)^2 + \beta^2 y^{-1}, \beta y^{-1}(\alpha 1 - x) - \beta y^{-1}(\alpha 1 - x)) = (1,0)$$

it follows that

$$(\alpha + i\beta)(1,0) - (x,0) = (\alpha 1 - x, \beta 1)$$

is invertible (Proposition 2.1.2.3). Hence

$$\alpha + i\beta \notin \sigma_{\overset{\circ}{E}}((x,0)),$$

$$\sigma_{\overset{\circ}{E}}((x,0)) \subset \mathbb{R},$$

$$\sigma_{\overset{\circ}{E}}((x,0)) = \sigma_E(x)$$

(Corollary 2.1.5.14). ■

Corollary 2.1.5.19 (0) *Let E be a unital real algebra such that*

$$\sigma_{\overset{\circ}{E}}((x,0)) = \sigma_E(x)$$

for every $x \in E$ and let F be an ideal of E. Then

$$\sigma_{\overbrace{\circ}^{E/F}}((X,0)) = \sigma_{E/F}(X)$$

for every $X \in E/F$.

Take $X \in E/F$ and $x \in X$. By Corollary 2.1.3.12 and Proposition 2.1.5.18 a $\Rightarrow$ b,

$$\sigma_{E/F}(X^2) \subset \sigma_E(x^2) \subset \mathbb{R}_+.$$

The assertion follows from Proposition 2.1.5.18 b $\Rightarrow$ a. ■

Exercises

E 2.1.1 A **monoid** is a set T endowed with a map

$$T \times T \longrightarrow T, \quad (s,t) \longmapsto st$$

and an element $1 \in T$ such that

$$(rs)t = r(st), \quad 1t = t1 = t$$

for every $r,s,t \in T$. The monoid T is called **commutative** if

$$st = ts$$

for every $s,t \in T$.

Let T be a monoid. Given $x,y \in \mathbb{K}^{(T)}$, define

$$xy : T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{\substack{r,s \in T \\ rs=t}} x(r)y(s)$$

Show:

a) $(e_t)_{t\in T}$ is a basis of the vector space $\mathbb{K}^{(T)}$ (Proposition 1.1.2.6 a)).

b) The vector space $\mathbb{K}^{(T)}$ is a unital algebra with respect to the multiplication

$$\mathbb{K}^{(T)} \times \mathbb{K}^{(T)} \longrightarrow \mathbb{K}^{(T)}, \quad (x,y) \longmapsto xy$$

c) $e_s e_t = e_{st}$ for every $s,t \in T$.

d) The unital algebra $\mathbb{K}^{(T)}$ is commutative iff T is commutative.

e) The unital algebras $\mathbb{K}[s]$ and $\mathbb{K}[s,t]$ are special cases of the above construction.

E 2.1.2 Let $\sigma(\mathbb{K}[t])$ be the set of unital algebra homomorphisms $\mathbb{K}[t] \to \mathbb{K}$. For $\alpha \in \mathbb{K}$, define

$$f_\alpha : \mathbb{K}[t] \longrightarrow \mathbb{K}, \quad P \longmapsto P(\alpha)$$

Show:

a) $f_\alpha \in \sigma(\mathbb{K}[t])$ for every $\alpha \in \mathbb{K}$.

b) The map

$$\mathbb{K} \longrightarrow \sigma(\mathbb{K}[t]), \quad \alpha \longmapsto f_\alpha$$

is bijective.

E 2.1.3 Let 1 and j be the standard basis vectors of $\mathbb{K}^2$, and for every $\alpha \in \{-1, 0, 1\}$ let E_α be the unital algebra obtained from the vector space $\mathbb{K}^2$ by defining a multiplication such that 1 is the unit and $j^2 = \alpha 1$. Prove the following:

a) If $\mathbb{K} = \mathbb{R}$ then:

- a_1) No two of the algebras E_{-1}, E_0, E_1 are isomorphic.
- a_2) Every two-dimensional real unital algebra is isomorphic to one of these three unital algebras.
- a_3) The group of unital algebra isomorphisms $E_0 \to E_0$ is isomorphic to the multiplicative group $\mathbb{R}\backslash\{0\}$.
- a_4) The group of unital algebra isomorphisms $E_{-1} \to E_{-1}$ ($E_1 \to E_1$) is isomorphic to the multiplicative group $\{-1, 1\}$.
- a_5) E_{-1} is isomorphic to the real unital algebra $\mathbb{C}$.
- a_6) E_1 is isomorphic to the real unital algebra $\mathbb{R}^{\{0,1\}}$ (Example 2.1.4.3).
- a_7) The set of unital algebra homomorphisms $E_\alpha \to \mathbb{R}$ is empty if $\alpha = -1$, has exactly one element if $\alpha = 0$, and has exactly two elements if $\alpha = 1$.

b) If $\mathbb{K} = \mathbb{C}$ then:

- b_1) The unital algebras E_{-1} and E_1 are isomorphic.
- b_2) The unital algebras E_0 and E_1 are not isomorphic.
- b_3) Every two-dimensional complex algebra is isomorphic either to E_0 or to E_1.
- b_4) The group of unital algebra isomorphisms $E_0 \to E_0$ is isomorphic to the multiplicative group $\mathbb{C}\backslash\{0\}$.
- b_5) The group of unital algebra isomorphisms $E_1 \to E_1$ is isomorphic to the multiplicative group $\{-1, 1\}$.
- b_6) E_1 is isomorphic to the complex unital algebra $\mathbb{C}^{\{0,1\}}$ (Example 2.1.4.3).

b$_7$) The set of unital algebra homomorphisms $E_\alpha \to \mathbb{C}$ has exactly one element if $\alpha = 0$ and exactly two elements if $|\alpha| = 1$.

c) Find the zeros of the polynomial $x^2 - 1$ in E_α.

d) Find the spectrum of j in E_α.

e) Find the group of invertible elements of E_α.

f) For which $\alpha \in \{-1, 0, 1\}$ is there a monoid T such that E_α is isomorphic to $\mathbb{K}^{(T)}$ (E 2.1.1).

E 2.1.4 Let K be a subfield of $\mathbb{R}$, such that the dimension of the K-vector space $\mathbb{R}$ is infinite (e.g. $K = \mathbb{Q}$ or K the set of algebraic numbers of $\mathbb{R}$). Show that there are maps

$$\varphi : \mathbb{C} \times \mathbb{R} \longrightarrow \mathbb{R}, \quad \psi : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

such that $\mathbb{R}$ (with the usual addition and with these maps) is a complex unital algebra with

a) $\alpha, \beta \in K \Longrightarrow \varphi(\alpha, \beta) = \psi(\alpha, \beta) = \alpha\beta$;

b) $\mathbb{R}$ is isomorphic to $\mathbb{C}$ as complex unital algebra.

E 2.1.5 Let E a unital algebra, F a maximal proper ideal of E, and $\mathbb{K} \times E$ the unital algebra defined in Proposition 2.1.1.7. Show there is a maximal proper ideal G of $\mathbb{K} \times E$ containing $\{0\} \times F$ other than $\{0\} \times E$.

E 2.1.6 Let E be a unital algebra and take $x, y, z \in E$ with x invertible and $y + z = x$. Prove that

$$yx^{-1}z = zx^{-1}y .$$

E 2.1.7 Let E be a unital algebra, G the set of its invertible elements, and $\mathcal{F}$ the group of isomorphisms of unital algebras $E \to E$. Show:

a) The map

$$G \times G \longrightarrow G, \quad (x, y) \longmapsto xy$$

renders G a group.

b) For every $x \in G$, the map

$$\widetilde{x} : E \longrightarrow E, \quad y \longmapsto x^{-1}yx$$

belongs to $\mathcal{F}$.

c) The map

$$G \longrightarrow \mathcal{F}, \quad x \longmapsto \widetilde{x}$$

is a group homomorphism.

d) $\sigma(x) = \sigma(ux)$ for every $x \in E$ and $u \in \mathcal{F}$.

E 2.1.8 Let E be a unital algebra, $(x_\iota)_{\iota \in I}$ a finite family in E, $(P_\iota)_{\iota \in I}$ a family in $\mathbb{K}[t]$ and let

$$A_\iota := \overset{-1}{P_\iota}(0)$$

for every $\iota \in I$. Assume

1) $\iota \in I \Rightarrow 0 \in A_\iota$, $P_\iota(x_\iota) = 0$, $x_\iota \notin \{\alpha 1 \mid \alpha \in A_\iota\}$.

2) $\iota, \lambda \in I, \iota \neq \lambda \Rightarrow x_\iota x_\lambda = 0$.

Prove that

$$\sigma\Big(\sum_{\iota \in I} x_\iota\Big) \subset \bigcup_{\iota \in I} A_\iota .$$

E 2.1.9 Let E, F be algebras and $u : E \to F$ an algebra homomorphism, which does not vanish identically. Prove the following (generalization of Proposition 2.1.1.12):

a) If E and F are unital and 0 the only zero–divisor in F, then u is unital.

b) If the only ideals of the subalgebras of F are trivial, then $\operatorname{Ker} u$ is a maximal proper ideal of E.

2.2 Normed Algebras

This section lays the foundations for the entire theory. The completeness of Banach algebras allows the construction of the inverses of many elements by means of Neumann series. This has powerful consequences for the spectrum and the resolvent, culminating in a fascinating mathematical theory with the theorems of Riesz, Gelfand, and Gelfand–Mazur as highlights. Applications are numerous including to the algebra of operators on a Banach space, to quotient spaces of such algebras (for example the Calkin algebra), to C^*–algebras, and to harmonic analysis (whose fundamentals appear in this book in summary form).

2.2.1 General Results

Definition 2.2.1.1 (0) (M. Nagumo, 1936) *A* ***normed algebra*** *is an algebra E equipped with a norm such that*

$$\|xy\| \leq \|x\| \, \|y\|$$

for every $x, y \in E$. A ***normed unital algebra*** *is a normed algebra which is unital and for which*

$$\|1\| = 1 \, .$$

A normed (unital) algebra is called ***real*** *or* ***complex*** *when the underlying (unital) algebra is real or complex, respectively. A* ***Banach (unital) algebra*** *is a complete normed (unital) algebra.*

Let E, F be normed (unital) algebras and $u : E \to F$ a (unital) algebra ***isomorphism.*** *u is called an* ***isomorphism of normed (unital) algebras*** *if u and u^{-1} are continuous, i.e. if u is an isomorphism of the underlying normed spaces. u is called an* ***isometry of normed (unital) algebras*** *if*

$$\|ux\| = \|x\|$$

for every $x \in E$, i.e. if u is an isometry of the underlying normed spaces. E and F are called ***isomorphic*** *or* ***isometric*** *if there is an isomorphism or an isometry $E \to F$ of normed (unital) algebras, respectively.*

Let E be a (unital) Banach algebra. Then E can be considered as a (unital) Banach category with a single object.

Proposition 2.2.1.2 (0) *Let E, F be normed unital algebras and $u : E \to F$ a continuous unital algebra homomorphism. Then*

$$\|u\| \geq 1 .$$

Note that

$$1 = \|1\| = \|u1\| \leq \|u\| \, \|1\| = \|u\| . \qquad \blacksquare$$

Proposition 2.2.1.3 (0) *Let E be a normed algebra and let $\mathbb{K} \times E$ the unital algebra defined in Proposition 2.1.1.7. Then*

$$\mathbb{K} \times E \longrightarrow \mathbb{R}_+ , \quad (\alpha, x) \longmapsto |\alpha| + \|x\|$$

is a norm and $\mathbb{K} \times E$ is a normed unital algebra with respect to this norm. It is complete iff E is complete.

By Proposition 1.1.5.1, the above map is a norm and $\mathbb{K} \times E$ is complete in this norm iff E is complete. Given $(\alpha, x), (\beta, y) \in \mathbb{K} \times E$,

$$\|(\alpha, x)(\beta, y)\| = \|(\alpha\beta, \alpha y + \beta x + xy)\| = |\alpha\beta| + \|\alpha y + \beta x + xy\| \leq$$

$$\leq |\alpha| \, |\beta| + |\alpha| \, \|y\| + |\beta| \, \|x\| + \|x\| \, \|y\| =$$

$$= (|\alpha| + \|x\|)(|\beta| + \|y\|) = \|(\alpha, x)\| \, \|(\beta, y)\|$$

and

$$\|(1, 0)\| = 1 .$$

Hence $\mathbb{K} \times E$ is a normed unital algebra with respect to the above norm. $\blacksquare$

Remark. There could be other norms on $\mathbb{K} \times E$ with similar properties.

Definition 2.2.1.4 (0) *Let E be a normed (Banach) algebra and F the unital algebra associated to the underlying algebra of E. F endowed with a norm extending the norm of E (identified with $\{0\} \times E$) and making F to a normed (Banach) unital algebra is called a* ***normed (Banach) unital algebra associated to*** *E (Proposition 2.2.1.3).*

Proposition 2.2.1.5 *Let E be a normed algebra and $\mathbb{K} \times E$ the normed algebra defined in Proposition 2.2.1.3. Define a norm on the vector space $\mathbb{K} \times E'$ by*

$$\mathbb{K} \times E' \longrightarrow \mathbb{R}', \quad (\alpha', x') \longmapsto \sup\{|\alpha'|, \|x'\|\}$$

(Proposition 1.1.5.1) and set

$$\widetilde{(\alpha', x')} : \mathbb{K} \times E \longrightarrow \mathbb{K}, \quad (\alpha, x) \longmapsto \alpha\alpha' + \langle x, x' \rangle$$

for $(\alpha', x') \in \mathbb{K} \times E'$. Then $\widetilde{(\alpha', x')} \in (\mathbb{K} \times E)'$ for every $(\alpha', x') \in \mathbb{K} \times E'$ and the map

$$\mathbb{K} \times E' \longrightarrow (\mathbb{K} \times E)', \quad (\alpha', x') \longmapsto \widetilde{(\alpha', x')}$$

is an isometry of Banach spaces.

The assertion follows from Proposition 1.2.2.13. ■

Proposition 2.2.1.6 (0) *If E is a normed algebra, then the map*

$$E \times E \longrightarrow E, \quad (x, y) \longmapsto xy$$

is continuous and its restriction to $E^\# \times E^\#$ is uniformly continuous.

The assertion follows from Proposition 1.2.9.2 c $\Rightarrow$ a & d. ■

Corollary 2.2.1.7 (0) *If E is a normed (unital) algebra, then A^c is a closed (unital) subalgebra of E for every $A \subset E$.*

By Proposition 2.1.1.17 a), A^c is a (unital) subalgebra of E. Let $x \in A$. Since the map

$$E \longrightarrow E, \quad y \longmapsto xy - yx$$

is continuous (Proposition 2.2.1.6), the set

$$\{y \in E \mid xy = yx\}$$

is closed. Hence

$$A^c = \bigcap_{x \in A} \{y \in E \mid xy = yx\}$$

is also closed. ■

Corollary 2.2.1.8 (0) *If E is a normed algebra and F is a subalgebra (left ideal, right ideal) of E, then $\overline{F}$ is also a subalgebra (left ideal, right ideal) of E .*

$\overline{F}$ is a vector subspace of E (Corollary 1.1.5.4). Define

$$\varphi : E \times E \longrightarrow E, \quad (x,y) \longmapsto xy .$$

Since φ is continuous (Proposition 2.2.1.6), $\overline{\varphi}^{-1}(\overline{F})$ is closed. Now

$$F \times F \subset \overline{\varphi}^{-1}(F) \subset \overline{\varphi}^{-1}(\overline{F})$$

$$\left(E \times F \subset \overline{\varphi}^{-1}(F) \subset \overline{\varphi}^{-1}(\overline{F}), \quad F \times E \subset \overline{\varphi}^{-1}(F) \subset \overline{\varphi}^{-1}(\overline{F})\right),$$

so that

$$\overline{F} \times \overline{F} = \overline{F \times F} \subset \overline{\varphi}^{-1}(\overline{F}) \quad \left(E \times \overline{F} \subset \overline{\varphi}^{-1}(\overline{F}), \quad \overline{F} \times E \subset \overline{\varphi}^{-1}(\overline{F})\right),$$

$$\varphi(\overline{F} \times \overline{F}) \subset \overline{F} \quad \left(\varphi(E \times \overline{F}) \subset \overline{F}, \quad \varphi(\overline{F} \times E) \subset \overline{F}\right),$$

and $\overline{F}$ is thus a subalgebra (left ideal, right ideal) of E . ■

Corollary 2.2.1.9 (0) *Let E be a normed (unital) algebra, A a subset of E, and F the (unital) subalgebra of E generated by A. Then $\overline{F}$ is the smallest closed (unital) subalgebra of E containing A. It is called* ***the closed (unital) subalgebra of E generated by A*** *. If A is countable, then $\overline{F}$ is separable.*

The first assertion follows immediately from Corollary 2.2.1.8.

Suppose that A is countable. If E is unital, we may assume that $1 \in A$. Let B be the set of the elements of E of the form $\prod_{k=1}^{n} x_k$, where $(x_k)_{k \in \mathbb{N}_n}$ is an arbitrary family in A. Then B is countable and F is the vector subspace of E generated by E. Hence F and $\overline{F}$ are separable. ■

Remark. Let E be a normed (unital) algebra and take $x, y \in E$. The closed (unital) subalgebra of E generated by $\{x\}$, (by $\{x,y\}$) is called **the closed (unital) subalgebra of E generated by x (by x and y)**. If E is complete, then we often say **"(unital) Banach subalgebra generated by"** instead of "closed (unital) subalgebra generated by".

Corollary 2.2.1.10 (0) *Let E be a normed (unital) algebra and A a commutative subset of E. The closed (unital) subalgebra of E generated by A is commutative.*

A^{cc} is a commutative closed (unital) subalgebra of E containing A (Proposition 2.1.1.17 e), Corollary 2.2.1.7). ■

Corollary 2.2.1.11 (0) *Let E be a Banach algebra, $(x_\iota)_{\iota\in I}$ a summable family in E and take $x \in E$. Then $(xx_\iota)_{\iota\in I}$ and $(x_\iota x)_{\iota\in I}$ are summable families in E and*

$$\sum_{\iota\in I}(xx_\iota) = x\sum_{\iota\in I}x_\iota\,, \quad \sum_{\iota\in I}(x_\iota x) = \Big(\sum_{\iota\in I}x_\iota\Big)x\,.$$

The assertion follows immediately from Proposition 2.2.1.6. ■

Corollary 2.2.1.12 *Let E be a Banach algebra. Take $x \in E$. Let $\sum_{n=0}^{\infty} t^n x_n$ be a power series in E. Assume there is an $r > 0$ smaller than the radius of convergence of this power series such that*

$$x\left(\sum_{n=0}^{\infty}\alpha^n x_n\right) = \left(\sum_{n=0}^{\infty}\alpha^n x_n\right)x$$

for every $\alpha \in U_r^{\mathbb{K}}(0)$. Then

$$xx_n = x_n x$$

for every $n \in \mathbb{N}\cup\{0\}$.

Take $\alpha \in U_r^{\mathbb{K}}(0)$. Then

$$0 = x\left(\sum_{n=0}^{\infty}\alpha^n x_n\right) - \left(\sum_{n=0}^{\infty}\alpha^n x_n\right)x =$$

$$= \sum_{n=0}^{\infty}\alpha^n x x_n - \sum_{n=0}^{\infty}\alpha^n x_n x = \sum_{n=0}^{\infty}\alpha^n(xx_n - x_n x)$$

(Corollary 2.2.1.11). Thus

$$xx_n - x_n x = 0$$

for every $n \in \mathbb{N}\cup\{0\}$ (Corollary 1.3.3.10). ■

Corollary 2.2.1.13 *Let E be a normed algebra and $\widetilde{E}$ the completion of the underlying normed space. Then there is a unique multiplication on $\widetilde{E}$, rendering $\widetilde{E}$ a Banach algebra such that E is a subalgebra of $\widetilde{E}$. It is called the completion of E. If E possesses a unit 1, then 1 is a unit of $\widetilde{E}$.*

Being uniformly continuous (Proposition 2.2.1.6), the map

$$E^{\#} \times E^{\#} \longrightarrow E, \quad (x,y) \longmapsto xy$$

can be extended continuously to a map

$$\widetilde{E} \times \widetilde{E} \longrightarrow \widetilde{E}, \quad (x,y) \longmapsto xy.$$

All of the assertions follow from the fact that $E \times E$ is dense in $\widetilde{E} \times \widetilde{E}$. ■

Proposition 2.2.1.14 (0) *Let E be a normed (unital) algebra and F a closed (proper) ideal of E. Then E/F is a normed (unital) algebra (Theorem 1.2.4.2 a), Proposition 2.1.1.13 b), d)). If E is complete or commutative, then so is E/F.*

Take $X, Y \in E/F$, $x \in X$ and $y \in Y$. Then $xy \in XY$ and

$$\|XY\| \leq \|xy\| \leq \|x\| \, \|y\| .$$

Since x and y are arbitrary,

$$\|XY\| \leq \|X\| \, \|Y\|.$$

Hence E/F is a normed algebra.

Now let E be a unital algebra and F a proper ideal of E. Let $q : E \to E/F$ be the quotient map. Then

$$\|q1\| \leq \|1\| = 1$$

and

$$\|q1\| = \|(q1)^2\| \leq \|q1\|^2$$

(Proposition 2.1.1.13 d)). Thus

$$\|q1\| = 1$$

and so E/F is a normed unital algebra.

The last assertion follows from Theorem 1.2.4.2 e) and Proposition 2.1.1.13 e). ■

Definition 2.2.1.15 (0) (Peter–Weyl, 1927) *Let E be a normed algebra. An **approximate unit of** E is a filter $\mathfrak{F}$ on E such that $E^{\#}\backslash\{0\} \in \mathfrak{F}$ and*

$$\lim_{y,\mathfrak{F}} xy = \lim_{y,\mathfrak{F}} yx = x$$

*for every $x \in E$. A normed (Banach) algebra possessing an approximate unit will be called **quasiunital normed (Banach) algebra.***

If E is unital, then the neighbourhood filter of the unit of E is an approximate unit since the multiplication is continuous. If $\mathfrak{F}$ and $\mathfrak{G}$ are approximate units of E, then $\mathfrak{F} \cap \mathfrak{G}$ is also an approximate unit of E. For all $p \in [1, \infty[$, the Banach algebra obtained by endowing the Banach space ℓ^p with the pointwise multiplication is not quasiunital.

Proposition 2.2.1.16 (0) *Let E be a Banach algebra. If $\mathfrak{F}$ is an approximate unit of E then*

$$\lim_{x,\mathfrak{F}} \|x\| = 1 .$$

If E has a unit, then $\mathfrak{F}$ converges to this unit.

Take $y \in E$. Then

$$\|y\| = \lim_{x,\mathfrak{F}} \|xy\| \leq \lim_{x,\mathfrak{F}} \inf \|x\| \, \|y\| = \|y\| \lim_{x,\mathfrak{F}} \inf \|x\| \leq$$

$$\leq \|y\| \lim_{x,\mathfrak{F}} \sup \|x\| \leq \|y\|$$

since $E^{\#} \in \mathfrak{F}$, and so

$$\lim_{x,\mathfrak{F}} \|x\| = 1 .$$

If E has a unit then

$$0 = \lim_{x,\mathfrak{F}} \|x1 - 1\| = \lim_{x,\mathfrak{F}} \|x - 1\| .$$

Hence

$$\lim \mathfrak{F} = 1 . \qquad \blacksquare$$

Proposition 2.2.1.17 (0) *Let E be a Banach algebra. For each $x \in E$, define*

$$\widetilde{x} : E \longrightarrow E , \quad y \longmapsto xy .$$

Then $\widetilde{x} \in \mathcal{L}(E)$ *and*

$$\|\widetilde{x}\| \leq \|x\|$$

for every $x \in E$ *and the map*

$$E \longrightarrow \mathcal{L}(E), \quad x \longmapsto \widetilde{x}$$

is a continuous algebra homomorphism. If E *is quasiunital (unital), then*

$$\|\widetilde{x}\| = \|x\|,$$

and

$$\sigma_E(x)\backslash\{0\} \subset \sigma_{\mathcal{L}(E)}(\widetilde{x}) \subset \sigma_E(x) \quad (\sigma_E(x) = \sigma_{\mathcal{L}(E)}(\widetilde{x}))$$

for every $x \in E$. *In particular, if* E *is finite-dimensional then*

$$\operatorname{Card} \sigma_E(x) \leq \operatorname{Dim} E.$$

Since

$$\|\widetilde{x}y\| = \|xy\| \leq \|x\| \, \|y\|,$$

$\widetilde{x} \in \mathcal{L}(E)$ and

$$\|\widetilde{x}\| \leq \|x\|$$

for every $x \in E$. Take $x, y \in E$. Then

$$\widetilde{xy}z = xyz = \widetilde{x}(yz) = \widetilde{x}(\widetilde{y}z) = \widetilde{x}\widetilde{y}z$$

for any $z \in E$. Thus

$$\widetilde{xy} = \widetilde{x}\widetilde{y}.$$

Hence the map

$$E \longrightarrow \mathcal{L}(E), \quad x \longmapsto \widetilde{x}$$

is a continuous algebra homomorphism.

Let $\mathfrak{F}$ be an approximate unit of E and take $x \in E$. Then

$$\|x\| = \lim_{y,\mathfrak{F}} \|xy\| = \lim_{y,\mathfrak{F}} \|\widetilde{x}y\| \leq \|\widetilde{x}\|$$

since $E^{\#} \in \mathfrak{F}$. Hence, by the above result,

$$\|\widetilde{x}\| = \|x\|\,.$$

By Corollary 2.1.3.12,

$$\sigma_{\mathcal{L}(E)}(\widetilde{x}) \subset \sigma_E(x)\,.$$

Take $\alpha \in \mathbb{K}\backslash\left(\sigma_{\mathcal{L}(E)}(\widetilde{x}) \cup \{0\}\right)$. Then $\alpha 1 - \widetilde{x}$ is surjective. Hence there is a $y \in E$ with

$$(\alpha 1 - \widetilde{x})y = \frac{1}{\alpha}x\,.$$

It now follows successively that

$$\alpha y - xy = \frac{1}{\alpha}x, \quad (\alpha 1 - \widetilde{x})\widetilde{y} = \alpha\widetilde{y} - \widetilde{x}\widetilde{y} = \frac{1}{\alpha}\widetilde{x},$$

$$\widetilde{y} = (\alpha 1 - \widetilde{x})^{-1}\frac{1}{\alpha}\widetilde{x} = \frac{1}{\alpha}\widetilde{x}(\alpha 1 - \widetilde{x})^{-1}, \quad \widetilde{y}(\alpha 1 - \widetilde{x}) = \frac{1}{\alpha}\widetilde{x}.$$

$$\widetilde{x}\widetilde{y} = \widetilde{y}\widetilde{x},$$

and

$$xy = yx\,.$$

Hence $\alpha \notin \sigma_E(x)$ by Proposition 2.1.3.11 b $\Rightarrow$ a. Thus

$$\sigma_E(x)\backslash\{0\} \subset \sigma_{\mathcal{L}(E)}(\widetilde{x})\,.$$

Assume that E is unital. Assume that $\widetilde{x}$ is invertible. Then there is a $y \in E$ with

$$xy = \widetilde{x}y = 1_E\,.$$

Thus

$$\widetilde{x}\widetilde{y} = 1_{\mathcal{L}(E)}, \quad \widetilde{y} = \widetilde{x}^{-1}, \quad \widetilde{y}\widetilde{x} = 1_{\mathcal{L}(E)},$$

and

$$yx = 1_E\,.$$

Hence x is invertible. Thus

$$\sigma_E(x) = \sigma_{\mathcal{L}(E)}(\widetilde{x})\,. \qquad \blacksquare$$

Remark. If

$$\|\widetilde{x}\| = \|x\|$$

for every $x \in E$ and if we identify E with $\{\widetilde{x} \mid x \in E\}$, then the closed subalgebra of $\mathcal{L}(E)$ generated by $\{1_E\} \cup E$ is an associated unital Banach algebra of E.

Proposition 2.2.1.18 (0) *Let E be a real Banach space. Put*

$$\|(x,y)\| := \sup_{\theta\in\mathbb{R}} \|(\cos\theta)x + (\sin\theta)y\|$$

for every $(x,y) \in \overset{\circ}{E}$, where $\overset{\circ}{E}$ is the complexification of E (Proposition 2.1.5.7).

a) $(x,y) \in \overset{\circ}{E} \Rightarrow \sup\{\|x\|, \|y\|\} \leq \|(x,y)\| \leq \|x\| + \|y\|$.

b) *The map*

$$\overset{\circ}{E} \longrightarrow \mathbb{R}_+, \quad (x,y) \longmapsto \|(x,y)\|$$

is a complete norm.

a) is easy to see.

b) Take $(x_1,y_1), (x_2,y_2) \in \overset{\circ}{E}$. For $\theta \in \mathbb{R}$,

$$\|\cos\theta(x_1+x_2) + \sin\theta(y_1+y_2)\| \leq$$

$$\leq \|(\cos\theta)x_1 + (\sin\theta)y_1\| + \|(\cos\theta)x_2 + (\sin\theta)y_2\| \leq$$

$$\leq \|(x_1,y_1)\| + \|(x_2,y_2)\|.$$

Hence

$$\|(x_1,y_1) + (x_2,y_2)\| \leq \|(x_1,y_1)\| + \|(x_2,y_2)\|.$$

Take $(x,y) \in \overset{\circ}{E}$ and $re^{i\varphi} \in \mathbb{C}$. Then

$$re^{i\varphi}(x,y) = ((r\cos\varphi)x - (r\sin\varphi)y, \quad (r\sin\varphi)x + (r\cos\varphi)y).$$

For $\theta \in \mathbb{R}$,

$$\|\cos\theta((r\cos\varphi)x - (r\sin\varphi)y) + \sin\theta((r\sin\varphi)x + (r\cos\varphi)y)\| =$$

$$= r\|(\cos\theta\cos\varphi + \sin\theta\sin\varphi)x + (\sin\theta\cos\varphi - \cos\theta\sin\varphi)y\| =$$

$$= r\|(\cos(\theta-\varphi))x + (\sin(\theta-\varphi))y\|.$$

Thus

$$\|re^{i\varphi}(x,y)\| = r\|(x,y)\|.$$

By the above, the map

$$\overset{\circ}{E} \longrightarrow \mathbb{R}_+, \quad (x,y) \longmapsto \|(x,y)\|$$

is a seminorm. By a), it is a complete norm. ■

Proposition 2.2.1.19 (0) (Goodearl, 1982) *If E is a real Banach algebra then there is a norm on $\overset{\circ}{E}$ making it into a complex Banach algebra and such that*

$$\|(x,0)\| = \|x\|$$

for every $x \in E$. $\overset{\circ}{E}$ endowed with such a norm is called a ***complexification*** *of E.*

Put

$$\|(x,y)\|_0 := \sup_{\theta \in \mathbb{R}} \|(\cos\theta)x + (\sin\theta)y\|$$

for every $(x,y) \in \overset{\circ}{E}$. By Proposition 2.2.1.18 b), the map

$$\overset{\circ}{E} \longrightarrow \mathbb{R}_+, \quad (x,y) \longmapsto \|(x,y)\|_0$$

is a complete norm. We set

$$\|(x,y)\| := \sup\{\|(a,b)(x,y)\|_0 \mid (a,b) \in \overset{\circ}{E},\ \|(a,b)\|_0 = 1\}$$

for all $(x,y) \in \overset{\circ}{E}$.

Step 1 $\|(\cdot,\cdot)\|$ is a complete norm

Take $(x_1,y_1), (x_2,y_2) \in \overset{\circ}{E}$. For $(a,b) \in \overset{\circ}{E}$, $\|(a,b)\|_0 = 1$,

$$\|(a,b)((x_1,y_1)+(x_2,y_2))\|_0 \leq \|(a,b)(x_1,y_1)\|_0 + \|(a,b)(x_2,y_2)\|_0 \leq$$

$$\leq \|(x_1,y_1)\| + \|(x_2,y_2)\|,$$

so that

$$\|(x_1,y_1)+(x_2,y_2)\| \leq \|(x_1,y_1)\| + \|(x_2,y_2)\|.$$

Take $(x,y) \in \overset{\circ}{E}$ and $re^{i\varphi} \in \mathbb{C}$. For $(a,b) \in \overset{\circ}{E}$, $\|(a,b)\|_0 = 1$,

$$\|e^{i\varphi}(a,b)\|_0 = \|(a,b)\|_0 = 1$$

(Proposition 2.2.1.18 b)). It follows that

$$\|re^{i\varphi}(x,y)\| = \sup\{\|(a,b)(re^{i\varphi}(x,y)\|_0 \mid (a,b) \in \overset{\circ}{E},\ \|(a,b)\|_0 = 1\} =$$

$$= r \sup\{\|(e^{i\varphi}(a,b))(x,y)\|_0 \mid (a,b) \in \overset{\circ}{E},\ \|(a,b)\|_0 = 1\} = r\|(x,y)\|\,.$$

Moreover, by Proposition 2.2.1.18 a),

$$\|(a,b)(x,y)\|_0 = \|(ax - by, ay + bx)\|_0 \leq$$

$$\leq \|ax - by\| + \|ay + bx\| \leq \|ax\| + \|by\| + \|ay\| + \|bx\| \leq$$

$$\leq \|a\|\,\|x\| + \|b\|\,\|y\| + \|a\|\,\|y\| + \|b\|\,\|x\| =$$

$$= (\|a\| + \|b\|)(\|x\| + \|y\|) \leq 4\|(x,y)\|_0\,,$$

so that

$$\|(x,y)\| \leq 4\|(x,y)\|_0\,.$$

By Proposition 2.2.1.18 a), we may assume

$$\|a\| \geq \frac{1}{2}\,.$$

It follows that

$$\|(x,y)\| \geq \|(a,0)(x,y)\|_0 =$$

$$= \|(ax,ay)\|_0 \geq \sup\{\|ax\|, \|ay\|\} \geq \frac{1}{2}\sup\{\|x\|, \|y\|\}\,.$$

By the above and by Proposition 2.2.1.18 b), the map

$$\overset{\circ}{E} \longrightarrow \mathbb{R}_+\,, \quad (x,y) \longmapsto \|(x,y)\|$$

is a complete norm.

Step 2 $\overset{\circ}{E}$ is a Banach algebra

Take $(x_{,1}\,, y_1)\,,\ (x_2, y_2) \in \overset{\circ}{E}$. For $(a,b) \in \overset{\circ}{E}$, $\|(a,b)\|_0 = 1$,

$$\|(a,b)((x_1,y_1)(x_2,y_2))\|_0 = \|((a,b)(x_1,y_1))(x_2,y_2)\|_0 \leq$$

$$\leq \|(a,b)(x_1,y_1)\|_0\,\|(x_2,y_2)\| \leq \|(x_1,y_1)\|\,\|(x_2,y_2)\|\,,$$

so that

$$\|(x_1, y_1)(x_2, y_2)\| \leq \|(x_1, y_1)\| \, \|(x_2, y_2)\| .$$

Step 3 $\quad x \in E \Rightarrow \|(x, 0)\| = \|x\|$

Take $(a, b) \in \overset{\circ}{E}$, $\quad \|(a, b)\|_0 = 1$. For $\theta \in \mathbb{R}$,

$$\|((\cos \theta)a + (\sin \theta)b)x\| \leq \|(\cos \theta)a + (\sin \theta)b\| \, \|x\| \leq \|(a, b)\|_0 \|x\| = \|x\| .$$

It follows that

$$\|(a, b)(x, 0)\|_0 = \|(ax, bx)\|_0 = \sup_{\theta \in \mathbb{R}} \|(\cos \theta)ax + (\sin \theta)bx\| \leq \|x\| ,$$

so that

$$\|(x, 0)\| \leq \|x\| .$$

In order to prove the reverse inequality, we may assume E unital (Proposition 2.2.1.3). Then

$$\|(1, 0)\|_0 = \|1\| = 1 ,$$

so that

$$\|(x, 0)\| \geq \|(1, 0)(x, 0)\|_0 = \|(x, 0)\|_0 = \|x\|$$

(Proposition 2.2.1.18 a)). ■

2.2.2 The Standard Examples

Example 2.2.2.1 (0) *Let T be a set. $\ell^\infty(T)$ is a commutative unital Banach algebra with respect to pointwise multiplication. $c(T)$ is a closed unital subalgebra of $\ell^\infty(T)$. $c_0(T)$ is a closed ideal of $\ell^\infty(T)$. If T is a topological space, then $C(T)$ is a closed unital subalgebra of $\ell^\infty(T)$. If T is locally compact, then $C_0(T)$ is a closed ideal of $C(T)$ and the Banch algebra $C_0(T)$ is quasiunital.* ■

Example 2.2.2.2 (0) *If $(T, \mathfrak{R}, \mu)$ is a measure space, then $L^\infty(\mu)$ is a commutative unital Banach algebra. For every $x \in L^\infty(\mu)$,*

$$\sigma(x) = \{\alpha \in \mathbb{K} \mid \varepsilon > 0 \Longrightarrow \overline{x}^{-1}(U_\varepsilon^{\mathbb{K}}(\alpha)) \text{ is not a } \mu\text{-null set}.\}$$

If there is an $\varepsilon > 0$ such that $\overline{x}^{-1}(U_\varepsilon^{\mathbb{K}}(0))$ is a μ–null set, then

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \frac{1}{x(t)}$$

is the inverse of x in $L^\infty(\mu)$. If x is invertible then $\overline{x}^{-1}(U_\varepsilon^{\mathbb{K}}(0))$ is a μ–null set, where

$$\varepsilon := \frac{1}{\|x^{-1}\|}\,.$$ ■

Example 2.2.2.3 (0) *Let $(E_t)_{t\in T}$ be a family of normed (unital) algebras and*

$$E := \{x \in \prod_{t\in T} E_t \mid (\|x(t)\|)_{t\in T} \in \ell^\infty(T)\}\,.$$

Then E is a (unital) subalgebra of the (unital) algebra $\prod_{t\in T} E_t$ (Example 2.1.4.1), E endowed with the norm

$$E \longrightarrow \mathbb{R}_+, \quad x \longmapsto \sup_{t\in T} \|x(t)\|$$

(Proposition 1.1.2.7 a)) is a normed unital algebra, and

$$\{x \in E \mid (\|x(t)\|)_{t\in T} \in c_0(T)\}$$

is a closed ideal of E. If all E_t $(t \in T)$ are complete then E is complete too. If T is a topological space and all E_t $(t \in T)$ are equal then

$$\{x \in E \mid x \text{ is continuous}\}$$

is a (unital) closed subalgebra of E.

The last two assertions follow from Proposition 1.1.2.7 b) and Corollary 1.1.2.8. ■

Example 2.2.2.4 (0) *If E is a normed (Banach) space, then $\mathcal{L}(E)$ is a normed (Banach) semi-simple algebra, unital if $E \neq \{0\}$.*

By Example 2.1.4.6, Corollary 1.2.1.5, and Corollary 1.2.1.11, $\mathcal{L}(E)$ is a normed (Banach) algebra.

We now show that $\mathcal{L}(E)$ is semi-simple. Take $x \in E\backslash\{0\}$ and put

$$\mathcal{F} := \{u \in \mathcal{L}(E) \mid ux = 0\}.$$

$\mathcal{F}$ is a left ideal of $\mathcal{L}(E)$. Let $\mathcal{G}$ be a left ideal of $\mathcal{L}(E)$ strictly containing $\mathcal{F}$. Put

$$F := \{ux \mid u \in \mathcal{G}\}.$$

Then F is a non-zero vector subspace of E such that

$$u(F) \subset F$$

for every $u \in \mathcal{L}(E)$. Take $y \in E\backslash F$. Then there is a $z \in F$ and a $u \in \mathcal{L}(E)$ such that $y = uz$. Thus $y \in F$, which is a contradiction. Hence $F = E$. Therefore there is a $v \in \mathcal{G}$ such that

$$vx = x.$$

If $u \in \mathcal{L}(E)$, then

$$(uv - u)x = ux - ux = 0,$$

so that

$$uv - u \in \mathcal{F}.$$

$$u = uv - (uv - u) \in \mathcal{G},$$

$$\mathcal{G} = \mathcal{L}(E).$$

Hence $\mathcal{F}$ is a maximal proper left ideal. It follows immediately that $\{0\}$ is the radical of $\mathcal{L}(E)$. ■

Example 2.2.2.5 (1) *Let T be a set. Take $p \in [1,\infty] \cup \{0\}$ and let q be the conjugate exponent of p.*

a) *The Banach space $\ell^{p,q}(T,T)$ endowed with the multiplication*

$$\ell^{p,q}(T,T) \times \ell^{p,q}(T,T) \longrightarrow \ell^{p,q}(T,T), \quad (h,k) \longmapsto h \circ k$$

(Proposition 1.2.3.5) is a Banach algebra and the map

$$\ell^{p,q}(T,T) \longrightarrow \mathcal{L}(\ell^p(T)), \quad k \longmapsto \overset{\cap}{k}$$

(Proposition 1.2.3.4 b)) is an injective continuous algebra homomorphism.

b) *If $p = \infty$ then the above algebra and the above map are unital.*

c) *If $k \in \ell^{2,2}(T,T)$ then $\sigma(k) = \sigma(\overset{\cap}{k})$.*

a) and b) follow immediately from Proposition 1.2.3.5, Proposition 1.2.3.4 b), and Example 2.2.2.4.

c) By a), the map

$$u : \ell^{2,2}(T,T) \longrightarrow \mathcal{L}(\ell^2(T)), \quad k \longmapsto \overset{\cap}{k}$$

is an injective algebra homomorphism. $\operatorname{Im} u$ is thus a subalgebra of $\mathcal{L}(\ell^2(T))$ so we have that

$$\sigma(k) = \sigma_{\operatorname{Im} u}(\overset{\cap}{k}).$$

By Proposition 1.2.3.4 g),

$$(v,w) \in (\operatorname{Im} u) \times \mathcal{L}(\ell^2(T)),\ vw = wv \Longrightarrow vw \in \operatorname{Im} u.$$

Hence, by Corollary 2.1.3.15,

$$\sigma_{\operatorname{Im} u}(\overset{\cap}{k}) = \sigma(\overset{\cap}{k}).$$

Thus

$$\sigma(k) = \sigma(\overset{\cap}{k}). \qquad \blacksquare$$

Example 2.2.2.6 *If Λ is a (unital) Banach category, then $\Lambda(E)$ is a (unital) Banach algebra for each object E of Λ.* $\blacksquare$

Theorem 2.2.2.7 (4) *Let T be a group and let 1 be its neutral element. Given $x, y \in \ell^1(T)$, define*

$$x * y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{s \in T} x(ts^{-1})y(s)$$

(the convolution of x with y).

a) *$x * y$ is in $\ell^1(T)$ whenever $x, y \in \ell^1(T)$ and $\ell^1(T)$, equipped with the multiplication*

$$\ell^1(T) \times \ell^1(T) \longrightarrow \ell^1(T), \quad (x, y) \longmapsto x * y,$$

is a unital Banach algebra with e_1 as unit.

b) *The map*

$$t \longmapsto e_t$$

is a group homomorphism of T into the multiplicative group of invertible elements of $\ell^1(T)$.

c) *T is commutative iff $\ell^1(T)$ is commutative.*

d) *Define*

$$s \sim t :\Longleftrightarrow \exists r \in T, r^{-1}sr = t$$

for all $s, t \in T$. Then:

d_1) *$\sim$ is an equivalence relation on T; its equivalence classes are called* ***conjugacy classes of*** *T.*

d_2) *$s, t \in T \Rightarrow st \sim ts$.*

d_3) *An element of $\ell^1(T)$ belongs to $\ell^1(T)^c$ iff it is constant on the conjugacy classes of T.*

d_4) *If T is the permutation group of a finite set then two elements of T are equivalent with respect to $\sim$ iff they have the same cycle structure (i.e. the complete factorizations into disjoint cycles have the same number of n-cycles for each n).*

d_5) *If Q is a conjugacy class of T then $\{t^{-1} | t \in Q\}$ is a conjugacy class of T and so it is either equal to Q or disjoint from Q.*

e) *If S is a subgroupf of T then $\ell^1(S)$ may be canonically identified with a unital subalgebra of $\ell^1(T)$ and*

$$\sigma_{\ell^1(S)}(x) = \sigma_{\ell^1(T)}(x)$$

for every $x \in \ell^1(S)$.

a) Take $x, y \in \ell^1(T)$. Since $\ell^1(T) \subset \ell^\infty(T)$, $x * y$ is well-defined. Now

$$\sum_{t\in T} |(x*y)(t)| \le \sum_{t\in T}\sum_{s\in T} |x(ts^{-1})|\,|(y(s)| =$$

$$= \sum_{s\in T} |y(s)| \sum_{t\in T} |x(ts^{-1})| = \sum_{s\in T} |y(s)| \|x\|_1 = \|x\|_1\|y\|_1\,,$$

and so $x * y \in \ell^1(T)$ with

$$\|x*y\|_1 \le \|x\|_1\|y\|_1\,.$$

Take $x, y, z \in \ell^1(T)$ and $\alpha, \beta \in \mathbb{K}$. Then

$$(\alpha x + \beta y) * z = \alpha x * z + \beta y * z\,, \quad z * (\alpha x + \beta y) = \alpha z * x + \beta z * y\,.$$

Take $t \in T$. Since

$$((x*y)*z)(t) = \sum_{s\in T} (x*y)(ts^{-1})z(s) = \sum_{s\in T}\Big(\sum_{r\in T} x(ts^{-1}r^{-1})y(r)\Big)z(s) =$$

$$= \sum_{s\in T}\Big(\sum_{q\in T} x(tq^{-1})y(qs^{-1})\Big)z(s) = \sum_{q\in T} x(tq^{-1})\Big(\sum_{s\in T} y(qs^{-1})z(s)\Big) =$$

$$= \sum_{q\in T} x(tq^{-1})(y*z)(q) = (x*(y*z))(t)\,,$$

it follows that

$$(x*y)*z = x*(y*z)\,.$$

Finally,

$$(e_1 * x)(t) = \sum_{s\in T} e_1(ts^{-1})x(s) = x(t)\,,$$

$$(x * e_1)(t) = \sum_{s \in T} x(ts^{-1})e_1(s) = x(t),$$

for every $t \in T$. Thus

$$e_1 * x = x * e_1 = x.$$

Since $\|e_1\|_1 = 1$, $\ell^1(T)$ is a unital Banach algebra with e_1 as unit.

b) Take $s, t \in T$. Then

$$(e_s * e_t)(r) = \sum_{q \in T} e_s(rq^{-1})e_t(q) = e_s(rt^{-1}) = e_{st}(r)$$

for every $r \in T$. Thus

$$e_s * e_t = e_{st}$$

and the map

$$t \longmapsto e_t$$

is a group homomorphism of T to the multiplicative group of invertible elements of T.

c) Suppose that T is commutative and take $x, y \in \ell^1(T)$. Then

$$\begin{aligned}(x * y)(t) &= \sum_{s \in T} x(ts^{-1})y(s) = \sum_{s \in T} x(ts)y(s^{-1}) = \\ &= \sum_{r \in T} x(r)y(r^{-1}t) = \sum_{r \in T} y(tr^{-1})x(r) = (y * x)(t)\end{aligned}$$

for every $t \in T$. Thus

$$x * y = y * x$$

and $\ell^1(T)$ is commutative.

If $\ell^1(T)$ is commutative, then

$$e_{st} = e_s * e_t = e_t * e_s = e_{ts}.$$

Hence

$$st = ts$$

for $s, t \in T$ and T is thus commutative.

d_1) and d_2) are easy to see.

d_3) Take $x \in \ell^1(T)$. Then for $s, t \in T$,

$$(x * e_t)(s) = \sum_{r \in T} x(sr^{-1})e_t(r) = x(st^{-1}),$$

$$(e_t * x)(s) = \sum_{r \in T} e_t(sr^{-1})x(r) = x(t^{-1}s).$$

Hence if x is constant on the equivalence classes of $\sim$ then, by d_2),

$$x * e_t = e_t * x$$

for every $t \in T$. It follows that $x \in \ell^1(T)^c$.

Now assume $x \in \ell^1(T)^c$ and let $s, t \in T$, $s \sim t$. Then there is an $r \in T$ such that

$$r^{-1}sr = t.$$

By the above,

$$x(s) = x(srr^{-1}) = (x * e_r)(sr) =$$

$$= (e_r * x)(sr) = x(r^{-1}sr) = x(t).$$

Thus x is constant on the equivalence classes of $\sim$.

d_4) If r is a transposition and s is a cycle then it is easy to see that $r^{-1}sr$ is a cycle of the same length. Hence s and $r^{-1}sr$ have the same cycle structure for every $s \in T$. Since every permutation is a product of transpositions, it follows that $s \sim t$ implies that s and t have the same cycle structure.

Assume now that s and t have the same cycle structure. Place the complete factorization of s over that of t, so that cycles of the same lenght correspond and let r be the permutation sending the bottom to the top. Then

$$r^{-1}sr = t.$$

d_5) follows from

$$(s^{-1}ts)^{-1} = s^{-1}t^{-1}s$$

for all $(s, t) \in T \times Q$.

e) For $y \in \ell^1(S)$ $(y \in \ell^1(T\backslash S))$ put

$$\tilde{y} : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} y(t) & \text{if } t \in S \quad (\text{ if } t \in T\backslash S) \\ 0 & \text{if } T\backslash S \quad (\text{ if } t \in S) \end{cases}$$

and identify $\ell^1(S)$ and $\ell^1(T\backslash S)$ with vector subspaces of $\ell^1(T)$ using the maps $y \mapsto \tilde{y}$. Then the map

$$\ell^1(S) \times \ell^1(T\backslash S) \longrightarrow \ell^1(T), \quad (y,z) \longmapsto y+z$$

is bijective.

Let $y,z \in \ell^1(S)$ and $t \in T\backslash S$. Then

$$(y*z)(t) = \sum_{s\in T} y(ts^{-1})z(s) = \sum_{s\in S} y(ts^{-1})z(s) = 0,$$

so that $y*z \in \ell^1(S)$. Hence, by the above identification, $\ell^1(S)$ is a unital subalgebra of $\ell^1(T)$.

Let $(y,z) \in \ell^1(S) \times \ell^1(T\backslash S)$ and $t \in S$. Thus

$$(y*z)(t) = \sum_{s\in T} y(ts^{-1})z(s) = \sum_{s\in T\backslash S} y(ts^{-1})z(s) = 0,$$

$$(z*y)(t) = \sum_{s\in T} z(ts^{-1})y(s) = \sum_{s\in S} z(ts^{-1})y(s) = 0,$$

so that $y*z$, $z*y \in \ell^1(T\backslash S)$.

By Corollary 2.1.3.14 b),

$$\sigma_{\ell^1(T)}(x) = \sigma_{\ell^1(S)}(x). \qquad \blacksquare$$

Example 2.2.2.8 *The following assertions hold for every $x \in \ell^1(\mathbb{Z}_2)$ (Theorem 2.2.2.7).*

a) *x is invertible iff*

$$x(0)^2 \neq x(1)^2.$$

b) *The spectrum of x is*

$$\{x(0) \pm x(1)\}.$$

a) Given $y \in \ell^1(\mathbb{Z}_2)$,

$$x*y = (x(0)y(0) + x(1)y(1))e_0 + (x(0)y(1) + x(1)y(0))e_1.$$

Hence x is invertible iff it is possible to solve the system of equations

$$\begin{cases} x(0)y(0) + x(1)y(1) = 1 \\ x(1)y(0) + x(0)y(1) = 0 \end{cases}$$

with $y(0)$ and $y(1)$ as unknown. This occurs iff

$$x(0)^2 \neq x(1)^2 .$$

b) By a), an $\alpha \in \mathbb{K}$ belongs to the spectrum of x iff

$$(x(0) - \alpha)^2 = x(1)^2$$

i.e. iff

$$\alpha = x(0) \pm x(1) . \qquad \blacksquare$$

Example 2.2.2.9 *Let βT be the Stone-Čech compactification of the group T with respect to the discrete topology. Take $s_0, t_0 \in \beta T$. Let $\ell^1(T)$ be the unital Banach algebra defined in Theorem 2.2.2.7. Identify $\ell^1(T)'$ canonically with $\ell^\infty(T)$ (Example 1.2.2.3 d)) and $\ell^\infty(T)$ with $C(\beta T)$ - by identifying each element of $\ell^\infty(T)$ with its continuous extension to βT. Finally, identify $\ell^\infty(T)'$ with the Banach space of Radon measures on βT (Example 1.2.2.11). If $\delta_{s_0}, \delta_{t_0}$ denote the Dirac measures on βT at s_0 and t_0 respectively, then, (using the notation of Definition 1.5.2.10),*

$$\langle \delta_{s_0} \dashv \delta_{t_0}, x' \rangle = \lim_{T \ni t \to t_0} \lim_{T \ni s \to s_0} x'(st) ,$$

$$\langle \delta_{s_0} \vdash \delta_{t_0}, x' \rangle = \lim_{T \ni s \to s_0} \lim_{T \ni t \to t_0} x'(st) ,$$

for every $x' \in \ell^\infty(T)$.

Now

$$\langle \delta_{s_0} \dashv \delta_{t_0}, x' \rangle = \langle \delta_{t_0}, x' \delta_{s_0} \rangle = \lim_{T \ni t \to t_0} \langle \delta_t, x' \delta_{s_0} \rangle =$$

$$= \lim_{T \ni t \to t_0} \langle \delta_{s_0}, \delta_t x' \rangle = \lim_{T \ni t \to t_0} \lim_{T \ni s \to s_0} \langle \delta_s, \delta_t x' \rangle =$$

$$= \lim_{T \ni t \to t_0} \lim_{T \ni s \to s_0} \langle \delta_s \delta_t, x' \rangle = \lim_{T \ni t \to t_0} \lim_{T \ni s \to s_0} x'(st) ,$$

$$\langle \delta_{s_0} \vdash \delta_{t_0}, x' \rangle = \langle \delta_{s_0}, \delta_{t_0} x' \rangle = \lim_{T \ni s \to s_0} \langle \delta_s, \delta_{t_0} x' \rangle =$$

$$= \lim_{T \ni s \to s_0} \langle \delta_{t_0}, x' \delta_s \rangle = \lim_{T \ni s \to s_0} \lim_{T \ni t \to t_0} \langle \delta_t, x' \delta_s \rangle =$$

$$= \lim_{T \ni s \to s_0} \lim_{T \ni t \to t_0} \langle \delta_s \delta_t, x' \rangle = \lim_{T \ni s \to s_0} \lim_{T \ni t \to t_0} x'(st)\,,$$

where δ_t denotes the Dirac measure on βT at $t \in T$. ■

Remark. Take $T := \mathbb{Z}$. Let $\mathfrak{F}$ be a free ultrafilter on $\mathbb{Z}$ with $\mathbb{N} \in \mathfrak{F}$ and put

$$s_0 := \lim_{n,\mathfrak{F}} n\,, \quad t_0 := \lim_{n,\mathfrak{F}} (-n)\,.$$

Then

$$\mathrm{Supp}(\delta_{s_0} \dashv \delta_{t_0}) \subset \overline{\mathbb{N}}\,, \quad \mathrm{Supp}(\delta_{s_0} \vdash \delta_{t_0}) \subset \beta\mathbb{Z} \backslash \overline{\mathbb{N}}\,.$$

Thus

$$\delta_{s_0} \dashv \delta_{t_0} \neq \delta_{s_0} \vdash \delta_{t_0}\,.$$

Theorem 2.2.2.7 admits the following generalization.

Theorem 2.2.2.10 (4) *Let T be a locally compact group and for every $t \in T$ let δ_t be the Dirac measure at $t \in T$.*

a) *For every $\mu, \nu \in \mathcal{M}_b(T)$ there is a unique element of $M_b(T)$, denoted by $\mu * \nu$ (**the convolution of μ with ν**), such that*

$$\int x d(\mu * \nu) = \int x(st) d(\mu \otimes \nu)(s,t)$$

for every $x \in \mathcal{C}_0(T)$.

b) *The multiplication*

$$\mathcal{M}_b(T) \times \mathcal{M}_b(T) \longrightarrow \mathcal{M}_b(T)\,, \quad (\mu, \nu) \longmapsto \mu * \nu$$

renders $\mathcal{M}_b(T)$ a unital Banach algebra with δ_1 as unit.

c) *$\mathcal{M}_b(T)$ is commutative iff T is commutative.*

d) *The map*

$$u : \ell^1(T) \longrightarrow \mathcal{M}_b(T)\,, \quad x \longmapsto \sum_{t \in T} x(t) \delta_t$$

is a unital algebra homomorphism preserving the norms (Theorem 2.2.2.7). In particular

$$\delta_{st} = \delta_s * \delta_t$$

for every $s, t \in T$.

Let $x \in \ell^1(T)$. *If* $\mu \in \mathcal{M}_b(T)$ *is atomless then* $(ux) * \mu$ *and* $\mu * ux$ *are atomless. In particular,*

$$\sigma_{\ell^1(T)}(x) = \sigma_{\mathcal{M}_b(T)}(ux).$$

e) *If* λ *is a left invariant Haar measure on* T, *then* $\{x \cdot \lambda \mid x \in L^1(\mu)\}$ *is a closed subalgebra of* $\mathcal{M}_b(T)$ *and*

$$\|x \cdot \lambda\| = \|x\|_1, \quad (x \cdot \lambda) * (y \cdot \lambda) = (x * y) \cdot \lambda$$

for every $x, y \in L^1(\lambda)$, *where*

$$x * y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \int x(s) y(s^{-1}t) d\lambda(s)$$

(the convolution of x ***with*** y***).*** $L^1(\lambda)$ *has a unit iff* T *is discrecte.*

f) *The function*

$$T \times T \longrightarrow \mathbb{K}, \quad (s, t) \longmapsto x(st)$$

is in $\mathcal{L}^1(\mu \otimes \nu)$ *and*

$$\int x(st) d(\mu \otimes \nu)(s, t) = \int x d(\mu * \nu).$$

for every $\mu, \nu \in \mathcal{M}_b(T)$ *and* $x \in \mathcal{L}^1(\mu * \nu)$.

g) *The map*

$$\mathcal{M}_b(T) \longrightarrow \mathbb{K}, \quad \mu \longmapsto \mu(T)$$

is a unital algebra homomorphism.

h) $\mu(T) \in \sigma(\mu)$ *for every* $\mu \in \mathcal{M}_b(T)$.

i) $\int x d\lambda \in \sigma_{L^1(T)}(x)$ *for every* $x \in L^1(T)$.

a) and b). The map

$$C_0(T) \longrightarrow \mathbb{K}, \quad x \longmapsto \int x(st) d(\mu \otimes \nu)(s, t)$$

is linear and

$$\left|\int x(st)d(\mu\otimes\nu)(s,t)\right| \leq \|x\|_\infty\|\mu\otimes\nu\| \leq \|x\|_\infty\|\mu\|\,\|\nu\|$$

for every $x \in \mathcal{C}_0(T)$. Hence there is a unique element $\mu * \nu$ of $M_b(T)$, such that

$$\int x d(\mu * \nu) = \int x(st)d(\mu\otimes\nu)(s,t)$$

for every $x \in \mathcal{C}_0(T)$. Moreover,

$$\|\mu * \nu\| \leq \|\mu\|\,\|\nu\|\,.$$

It is obvious that the map

$$\mathcal{M}_b(T) \times \mathcal{M}_b(T) \longrightarrow \mathcal{M}_b(T)\,, \quad (\mu,\nu) \longmapsto \mu * \nu$$

is bilinear. Take $\lambda, \mu, \nu \in \mathcal{M}_b(T)$. Then

$$\int x d((\lambda * \mu) * \nu) = \int x(st)d((\lambda * \mu)\otimes\nu)(s,t) =$$

$$= \int \Big(\int x(st)d((\lambda * \mu)(s)\Big)d\nu(t) = \int \Big(\int x(rst)d(\lambda\otimes\mu)(r,s)\Big)d\nu(t) =$$

$$= \int x(rst)d(\lambda\otimes\mu\otimes\nu)(r,s,t) = \int x d(\lambda * (\mu * \nu))$$

for every $x \in \mathcal{C}_0(T)$. Thus

$$(\lambda * \mu) * \nu = \lambda * (\mu * \nu)\,.$$

It is easy to see that δ_1 is the unit of $\mathcal{M}_b(T)$.

c) is trivial.

d) The map

$$u : \ell^1(T) \longrightarrow \mathcal{M}_b(T)\,, \quad x \longmapsto \sum_{t\in T} x(t)\delta_t$$

is obviously linear and preserves the norms. Take $s, t \in T$. Then

$$\int f d(\delta_s * \delta_t) = \int f(s't')d(\delta_s\otimes\delta_t)(s',t') = f(st)$$

for every $f \in \mathcal{C}_0(T)$, so that

$$\delta_s * \delta_t = \delta_{st}\,.$$

By linearity,

$$u(x * y) = (ux) * (uy)$$

for every $x, y \in \mathbb{K}^{(T)}$. By continuity,

$$u(x * y) = (ux) * (uy)$$

for every $x, y \in \ell^1(T)$. Hence u is a unital algebra homomorphism.

Now we prove the last assertion. $\delta_t * \mu$ and $\mu * \delta_t$ are obviously atomless. Hence $(ux) * \mu$ and $\mu * ux$ are also atomless. By Corollary 2.1.3.14 b),

$$\sigma_{\ell^1(T)}(x) = \sigma_{\mathcal{M}_b(T)}(ux).$$

e) Given $f \in C_0(T)$,

$$\int f d((x \cdot \lambda) * (y \cdot \lambda)) = \int f(st) d((x \cdot \lambda) \otimes (y \cdot \lambda))(s,t)) =$$

$$= \int \Big(\int f(st) y(t) d\lambda(t) \Big) x(s) d\lambda(s) =$$

$$= \int \Big(\int f(t) y(s^{-1}t) d\lambda(t) \Big) x(s) d\lambda(s) =$$

$$= \int f(t) \Big(\int x(s) y(s^{-1}t) d\lambda(s) \Big) d\lambda(t) = \int f(x * y) d\lambda$$

and so

$$(x \cdot \lambda) * (y \cdot \lambda) = (x * y) \cdot \lambda .$$

If T is discrete, then $L^1(\lambda) = \mathcal{M}_b(T)$, so that $L^1(\lambda)$ has a unit. Assume that $L^1(\lambda)$ has a unit x. Let K be a compact set of T containing $1 \in T$. By approximating e_K with a downward directed family of real functions in $C_0(T)$, we obtain

$$1 = e_K(1) = (x * e_K)(1) = \int x(s) e_K(s^{-1}) d\lambda(s) = \int\limits_{K^{-1}} x \, d\lambda ,$$

and this implies that

$$x = 0 \qquad \lambda\text{-a.e.}$$

on $T\backslash\{1\}$ and that

$$\lambda(\{1\}) > 0\,.$$

Hence $\{1\}$ is an open set of T and T is discrete.

f) The assertion holds if x is the characteristic function of a compact set of T, since in this case x may be approximated by a downward directed family of real functions in $\mathcal{C}_0(T)$. The Lebesgue Convergence Theorem enables us to extend the assertion to arbitrary x.

g) follows from f).

h) and i) follow from g) and Corollary 2.1.3.13. ■

Lemma 2.2.2.11 (4) *Let T be a locally compact group and take $x \in \mathcal{C}_0(T)$. Given $t \in T$, define*

$$x_t : T \longrightarrow \mathbb{K}\,, \quad s \longmapsto x(st) \quad (\textit{resp. } x(ts))\,.$$

Then

$$\lim_{t\to 1} \|x_t - x\|_\infty = 0\,.$$

Given $t \in T$, let s_t be a point of T such that

$$\|x_t - x\|_\infty = |x_t(s_t) - x(s_t)|\,.$$

Let $\mathfrak{F}$ be an ultrafilter on T converging to 1. If

$$a := \lim_{t,\mathfrak{F}} s_t \in T\,,$$

then

$$\lim_{t,\mathfrak{F}} s_t t = a\,.$$

Hence

$$\lim_{t,\mathfrak{F}} \|x_t - x\|_\infty = \lim_{t,\mathfrak{F}} |x(s_t t) - x(s_t)| = |x(a) - x(a)| = 0\,.$$

If $\lim\limits_{t,\mathfrak{F}} s_t$ is the Alexandroff point of T, then $\lim\limits_{t,\mathfrak{F}} s_t t$ is also the the Alexandroff point of T and so

$$\lim_{t,\mathfrak{F}} \|x_t - x\|_\infty = \lim_{t,\mathfrak{F}} |x(s_t t) - x(s_t)| = |0 - 0| = 0\,.$$

Since $\mathfrak{F}$ is arbitrary,

$$\lim_{t\to 1} \|x_t - x\|_\infty = 0\,.$$ ■

Lemma 2.2.2.12 (4) *Let T be a locally compact group, λ a left invariant Haar measure on T, and take $p \in [1, \infty[$. For $x \in L^p(\lambda)$ and $t \in T$, define*

$$x_t : T \longrightarrow \mathbb{K}, \quad s \longmapsto x(ts) \quad (\text{resp. } x(st)).$$

Then, for every compact set $\mathcal{F}$ of $L^p(\lambda)$,

$$\lim_{t \to 1} \|x_t - x\|_p = 0$$

uniformly in $x \in \mathcal{F}$.

Take $\varepsilon > 0$. There is a finite set $\mathcal{G}$ of continuous functions with compact support on T such that

$$\mathcal{F} \subset \bigcup_{y \in \mathcal{G}} U_{\frac{\varepsilon}{4}}(y).$$

Let K be a compact neighbourhood of 1 in T containing the carriers of the $y \in \mathcal{G}$. By Lemma 2.2.2.11, there is a neighbourhood V of 1 in T such that

$$\|y_t - y\|_\infty < \frac{\varepsilon}{1 + 3\lambda(KK)^{\frac{1}{p}}}, \quad |\Delta(t)^{\frac{1}{p}} - 1| < \frac{1}{3}$$

for every $y \in \mathcal{G}$ and $t \in V$, where Δ is the module of G. Then

$$\|y_t - y\|_p = \Big(\int |y_t - y|^p d\lambda\Big)^{\frac{1}{p}} \leq \Big(\int_{KK} \|y_t - y\|_\infty^p d\lambda\Big)^{\frac{1}{p}} \leq$$

$$\leq \frac{\varepsilon}{1 + 3\lambda(KK)^{\frac{1}{p}}} \lambda(KK)^{\frac{1}{p}} < \frac{\varepsilon}{3}$$

for every $y \in \mathcal{G}$ and $t \in V \cap K$.

Take $x \in \mathcal{F}$. There is a $y \in \mathcal{G}$ with

$$\|x - y\|_p < \frac{\varepsilon}{4}.$$

Then

$$\|x_t - y_t\|_p = \|x - y\|_p < \frac{\varepsilon}{3},$$

$$(\text{resp. } \|x_t - y_t\|_p \leq \Delta(t)^{\frac{1}{p}} \|x - y\|_p \leq (1 + \frac{1}{3})\frac{\varepsilon}{4} = \frac{\varepsilon}{3})$$

$$\|x_t - x\|_p \leq \|x_t - y_t\|_p + \|y_t - y\| + \|y - x\|_p < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

for every $t \in V \cap K$. Hence

$$\lim_{t \to 1} \|x_t - x\|_p = 0$$

uniformly in $x \in \mathcal{F}$. ∎

proposition 2.2.2.13 (4) *Let T be a locally compact group, λ a left invariant Haar measure on T, and $\mathfrak{V}$ the neighbourhood filter of 1 in T. Given $V \in \mathfrak{V}$, let $\mathcal{U}_V$ denote the set of positive real functions x of $L^1(\lambda) \cap L^\infty(\lambda)$ such that*

$$\operatorname{Supp} x \subset V,$$

$$\int x(s)d\lambda(s) = 1 \quad (\textit{resp.} \int x(s^{-1})d\lambda(s) = 1).$$

Let $\mathfrak{F}$ denote the filter on $L^1(\lambda)$ generated by the filter base $\{\mathcal{U}_V \mid V \in \mathfrak{V}\}$. Then

$$\lim_{x,\mathfrak{F}} \|x * y - y\|_1 = 0 \quad (\textit{resp.} \quad \lim_{x,\mathfrak{F}} \|y * x - y\|_1 = 0)$$

for every $y \in L^1(\lambda)$. In particular, $L^1(\lambda)$ has an approximate unit. If $x \in L^1(\lambda)$ and

$$\widetilde{x} : L^1(\lambda) \longrightarrow L^1(\lambda), \quad y \longmapsto x * y,$$

then

$$\|\widetilde{x}\| = \|x\|, \quad \sigma(x)\backslash\{0\} \subset \sigma(\widetilde{x}) \subset \sigma(x).$$

Take $\varepsilon > 0$. For $t \in T$, put

$$y_t : T \longrightarrow \mathbb{K}, \quad s \longmapsto y(ts) \quad (\text{resp. } y(st)).$$

By Lemma 2.2.2.12, there is a symmetric neighbourhood V of 1 such that

$$\|y_t - y\|_1 < \varepsilon$$

for every $t \in V$. Then, for every $x \in \mathcal{U}_V$,

$$\|x * y - y\|_1 = \int |x * y - y| d\lambda =$$

$$= \int \left| \int x(t)y(t^{-1}s)d\lambda(t) - y(s)\int x(t)d\lambda(t) \right| d\lambda(s) \leq$$

$$\leq \int \Big(\int x(t) \mid y(t^{-1}s) - y(s) \mid d\lambda(t) \Big) d\lambda(s) =$$

$$= \int \Big(\int |y_{t^{-1}}(s) - y(s)| d\lambda(s) \Big) x(t) d\lambda(t) =$$

$$= \int \|y_{t^{-1}} - y\|_1 x(t) d\lambda(t) < \varepsilon$$

$$\text{(resp. } \|y * x - y\|_1 = \int |y * x - y| d\lambda =$$

$$= \int \left| \int y(t) x(t^{-1}s) d\lambda(t) - y(s) \right| d\lambda(s) =$$

$$= \int \left| \int y(st) x(t^{-1}) d\lambda(t) - \int y(s) x(t^{-1}) d\lambda(t) \right| d\lambda(s) \leq$$

$$\leq \int \Big(\int |y_t(s) - y(s)| x(t^{-1}) d\lambda(t) \Big) d\lambda(s) =$$

$$= \int \Big(\int |y_t(s) - y(s)| d\lambda(s) \Big) x(t^{-1}) d\lambda(t) =$$

$$= \int \|y_t - y\|_1 x(t^{-1}) dt < \varepsilon) .$$

Hence

$$\lim_{x,\mathfrak{F}} \|x * y - y\| = 0 \quad \text{(resp. } \lim_{x,\mathfrak{F}} \|y * x - y\| = 0)$$

and $L^1(\lambda)$ has an approximate unit. The last assertion now follows from Proposition 2.2.1.17. ■

Proposition 2.2.2.14 (4) *Let λ be a left invariant Haar measure on the locally compact group T and take $p \in [1, \infty]$. Assume that $p \neq \infty$ or T is compact and define*

$$A(\mu, x) := \{t \in T \mid T \longrightarrow \mathbb{K}, \quad s \longmapsto x(s^{-1}t) \text{ is not } \mu\text{-integrable}\},$$

$$\mu * x : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} \int x(s^{-1}t) d\mu(s) & \text{if } t \notin A(\mu, x) \\ 0 & \text{if } t \in A(\mu, x) \end{cases}$$

for every $\mu \in \mathcal{M}_b(T)$ and every Borel function $x : T \to \mathbb{K}$.

a) *$A(\mu,x)$ is a λ-null set, $\mu * x$ is a Borel function of $\mathcal{L}^p(\lambda)$,*

$$\|\mu * x\|_p \leq \|\mu\|\,\|x\|_p,$$

and the function

$$T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto x(s^{-1}t)$$

is in $L^p(\mu\otimes\lambda)$ for every $\mu \in \mathcal{M}_b(T)$ and every Borel function $x \in \mathcal{L}^p(\lambda)$. For $\mu \in \mathcal{M}_b(T)$, put

$$\widetilde{\mu} : L^p(\lambda) \longrightarrow L^p(\lambda), \quad x \longmapsto \mu * x.$$

b) *If T is compact, then $\mu * x \in C(T)$ for every $\mu \in \mathcal{M}_b(T)$ and every $x \in C(T)$.*

c) *$\widetilde{\mu} \in \mathcal{L}(L^p(\mu))$ and $\|\widetilde{\mu}\| \leq \|\mu\|$ for every $\mu \in \mathcal{M}_b(T)$.*

d) *$(\mu,x) \in \mathcal{M}_b(T) \times L^1(\lambda) \Rightarrow (\mu * x)\cdot\lambda = \mu * (x\cdot\lambda)$.*

e) *The map*

$$M_b(T) \longrightarrow \mathcal{L}(L^p(\lambda)), \quad \mu \longmapsto \widetilde{\mu}$$

is an injective unital continuous algebra homomorphism (Theorem 2.2.2.10 b)).

f) *For every $x \in L^1(T)$ and every Borel function $y \in \mathcal{L}^p(T)$,*

$$((x\cdot\lambda) * y)(s) = \int x(st)y(t)d\lambda(t)$$

for λ-almost all $s \in T$.

g) *If T is compact, then $\{x\cdot\lambda \mid x \in L^p(\lambda)\}$ and $\{x\cdot\lambda \mid x \in C(T)\}$ are ideals of $M_b(T)$.*

a) Case 1 $p \neq \infty$

For $t \in T$, define

$$x_t : T \longrightarrow \mathbb{K}, \quad s \longmapsto x(s^{-1}t).$$

Then, by Hölder's Inequality,

$$\int^* |x(s^{-1}t)|d|\mu|(s) = \int^* |x_t|d|\mu| \leq \Big(\int^* |x_t|^p d|\mu|\Big)^{\frac{1}{p}} \Big(\int d|\mu|\Big)^{\frac{p-1}{p}} =$$

$$= \|\mu\|^{\frac{p-1}{p}} \Big(\int^* |x_t|^p d|\mu|\Big)^{\frac{1}{p}}$$

for every $t \in T$, so that

$$\int^* \Big(\int^* |x(s^{-1}t)|d|\mu|(s)\Big)^p d\lambda(t) \leq$$

$$\leq \|\mu\|^{p-1} \int^* \Big(\int^* |x_t|^p d|\mu|\Big) d\lambda(t) =$$

$$= \|\mu\|^{p-1} \int^* \Big(\int^* |x(s^{-1}t)|^p d\lambda(t)\Big) d|\mu|(s) =$$

$$= \|\mu\|^{p-1} \|x\|_p^p \|\mu\| = \|\mu\|^p \|x\|_p^p$$

by Tonelli's Theorem. Hence, $A(\mu, x)$ is a λ-null set, $\mu * x \in \mathcal{L}^p(\lambda)$,

$$\|\mu * x\|_p \leq \|\mu\| \, \|x\|_p,$$

and the function

$$T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto x(s^{-1}t)$$

is in $\mathcal{L}^p(\mu \otimes \lambda)$.

Case 2 $p = \infty$, T compact, and $x = 0$ λ-a.e.

By Tonelli's Theorem,

$$\int^* \Big(\int^* |x(s^{-1}t)|(d|\mu|(s)\Big) d\lambda(t) = \int^* \Big(\int^* |x(s^{-1}t)|d\lambda(t)\Big) d|\mu|(s) = 0,$$

so that

$$\mu * x = 0 \qquad \lambda\text{-a.e.}$$

Case 3 $p = \infty$, and T is compact

We may assume by Case 2, that $|x| \leq \|x\|_\infty$. Then, obviously

$$|\mu * x| \leq \|\mu\| \, \|x\|_\infty.$$

b) follows from Integration Theory.

c) follows from a).

d) Now

$$((\mu * x)\cdot\lambda)(K) = \int_K (\mu * x)d\lambda = \int \Big(\int x(s^{-1}t)d\mu(s)\Big)e_K(t)d\lambda(t) =$$

$$= \int \Big(\int e_K(t)x(s^{-1}t)d\lambda(t)\Big)d\mu(s) =$$

$$= \int \Big(\int e_K(st)x(t)d\lambda(t)\Big)d\mu(s) = \int e_K(st)d(\mu \otimes (x\cdot\lambda))(s,t) = (\mu * (x\cdot\lambda))(K)$$

for every compact set K of T (by a)).

e) The map is obviously linear and continuous. Take $\mu, \nu \in \mathcal{M}_b(T)$. Then, by d), for $x \in L^1(\lambda) \cap L^p(\lambda)$,

$$(\widetilde{\mu * \nu}x)\cdot\lambda = ((\mu * \nu) * x)\cdot\lambda = (\mu * \nu) * (x\cdot\lambda) =$$

$$= \mu * (\nu * (x\cdot\lambda)) = \mu * ((\nu * x)\cdot\lambda) = \mu * ((\widetilde{\nu}x)\cdot\lambda) =$$

$$= (\mu * (\widetilde{\nu}x))\cdot\lambda = (\widetilde{\mu}\widetilde{\nu}x)\cdot\lambda,$$

so that

$$\widetilde{\mu * \nu}x = \widetilde{\mu}\widetilde{\nu}x.$$

Since $L^1(\lambda) \cap L^p(\lambda)$ is dense in $L^p(\lambda)$,

$$\widetilde{\mu * \nu} = \widetilde{\mu} * \widetilde{\nu}.$$

For $x \in L^p(\lambda)$ and $t \in T$,

$$(\delta_1 * x)(t) = \int x(s^{-1}t)d\delta_1(s) = x(t),$$

so that

$$\widetilde{\delta}_1 x = x$$

for every $x \in L^p(\lambda)$. Hence the map

$$\mathcal{M}_b(T) \longrightarrow \mathcal{L}(L^p(\lambda)), \quad \mu \longmapsto \widetilde{\mu}$$

is a unital homomorphism of Banach algebras.

Take $\mu \in \mathcal{M}_b(T)$ with $\tilde{\mu} = 0$ and let K be a compact set of T. Then

$$0 = \mu * e_{K^{-1}}(1) = \int e_{K^{-1}}(s^{-1}) d\mu(s) = \mu(K).$$

Since K is arbitrary, $\mu = 0$. Hence the map

$$\mathcal{M}_b(T) \longrightarrow \mathcal{L}(L^p(\lambda)), \quad \mu \longmapsto \tilde{\mu}$$

is injective.

f) By a), $A(x \cdot \lambda, y)$ is a λ-null set and

$$((x \cdot \lambda) * y)(s) = \int y(s^{-1}t) x(t) d\lambda(t) = \int y(t) x(st) d\lambda(t)$$

for every $s \in T \backslash A(x \cdot \lambda, y)$.

g) If T is compact, then λ is bounded. Hence $L^p(\lambda) \subset L^1(\lambda)$. Take $x \in L^p(\lambda)$ $(x \in \mathcal{C}(T))$ and $\mu \in \mathcal{M}_b(T)$. Then

$$\mu * (x \cdot \lambda) = (\mu * x) \cdot \lambda$$

by d) and $\mu * x \in L^p(\lambda)$ by a) $(\mu * x \in \mathcal{C}(T)$ by b)). Since λ is also right invariant, $\{x \cdot \lambda \mid x \in L^p(\lambda)\}$ and $\{x \cdot \lambda \mid x \in \mathcal{C}(T)\}$ are ideals of $\mathcal{M}_b(T)$. ■

Proposition 2.2.2.15 (4) *Let T be a locally compact group, λ a left invariant Haar measure on T, and $\mathfrak{V}$ the neighbourhood filter of 1 in T. Given $V \in \mathfrak{V}$, let $\mathcal{U}_V$ denote the set of real positive functions x of $\mathcal{L}^1(\lambda) \cap L^\infty(\lambda)$, such that*

$$\operatorname{Supp} x \subset V, \quad \int x \, d\lambda = 1$$

and let $\mathcal{F}$ be the filter on $L^1(\lambda)$ generated by the filter base $\{\mathcal{U}_V \mid V \in \mathfrak{V}\}$. Then

$$\lim_{x,\mathfrak{F}} \langle y, (\mu * x) \cdot \lambda - \mu \rangle = 0, \quad \lim_{x,\mathfrak{F}} \|\mu * x\|_1 = \|\mu\|$$

for every $\mu \in \mathcal{M}_b(T)$ and $y \in \mathcal{C}_0(T)$.

Take $\varepsilon > 0$. For $t \in T$, put

$$y_t : T \longrightarrow \mathbb{K}, \quad s \longmapsto y(st).$$

By Lemma 2.2.2.11, there is a $V \in \mathfrak{V}$, such that

$$\|y_t - y\|_\infty < \varepsilon$$

for every $t \in V$. Then

$$|\langle y, (\mu * x)\cdot\lambda - \mu\rangle| = \left|\int y(\mu * x)d\lambda - \int y\,d\mu\right| =$$

$$= \left|\int \left(\int x(s^{-1}t)d\mu(s)\right) y(t)d\lambda(t) - \int y\,d\mu\right| =$$

$$= \left|\int \left(\int x(s^{-1}t)y(t)d\lambda(t)\right) d\mu(s) - \int y\,d\mu\right| =$$

$$= \left|\int \left(\int x(t)y(st)d\lambda(t)\right) d\mu(s) - \int \left(\int y(s)x(t)d\lambda(t)\right) d\mu(s)\right| =$$

$$= \left|\int \left(\int (y(st) - y(s))x(t)d\lambda(t)\right) d\mu(s)\right| \leq$$

$$\leq \int \left(\int |y_t(s) - y(s)|x(t)d\lambda(t)\right) d|\mu|(s) \leq$$

$$\leq \int \left(\int \|y_t - y\|_\infty x(t)d\lambda(t)\right) d|\mu|(s) \leq \varepsilon\|\mu\|$$

for every $x \in \mathcal{U}_V$. Hence

$$\lim_{x,\mathfrak{F}} \langle y, (\mu * x)\cdot\lambda - \mu\rangle = 0\,.$$

Thus

$$|\langle y, \mu\rangle| = \lim_{x,\mathfrak{F}} |\langle y, (\mu * x)\cdot\lambda\rangle| \leq \lim_{x,\mathfrak{F}} \inf \|(\mu * x)\cdot\lambda\| = \lim_{x,\mathfrak{F}} \inf \|\mu * x\|_1$$

for every $y \in C_0(T)^\#$ (Proposition 2.2.2.14 d)) and

$$\|\mu\| = \sup\{|\langle y, \mu\rangle| \mid y \in C_0(T)^\#\} \leq \lim_{x,\mathfrak{F}} \inf \|\mu * x\|_1 \leq$$

$$\leq \lim_{x,\mathfrak{F}} \sup \|\mu * x\|_1 \leq \lim_{x,\mathfrak{F}} \sup \|\mu\|\,\|x\|_1 = \|\mu\|\,,$$

$$\lim_{x,\mathfrak{F}} \|\mu * x\|_1 = \|\mu\|\,.$$

■

Corollary 2.2.2.16 (4) *Let λ be a left invariant measure on the compact group T. Using the notation from Proposition 2.2.2.14 a) b), the map*

$$\mathcal{M}_b(T) \longrightarrow \mathcal{L}(\mathcal{C}(T)), \quad \mu \longmapsto \tilde{\mu}$$

is injective.

Take $\mu \in \mathcal{M}_b(T)$ with $\tilde{\mu} = 0$. Let $\mathfrak{F}$ be the filter introduced in Proposition 2.2.2.15 and $\mathfrak{G}$ its trace on $\mathcal{C}(T)$. Then, by Proposition 2.2.2.15,

$$\int y \, d\mu = \lim_{x,\mathfrak{G}} \langle y, (\mu * x) \cdot \lambda \rangle = 0$$

for every $y \in \mathcal{C}(T)$. Thus $\mu = 0$. ■

Proposition 2.2.2.17 *Let T be a locally compact group. Take $\mu \in \mathcal{M}_b(T)$ and $t \in T$. Then*

$$|\mu * \delta_t|(A) = |\mu|(At^{-1})$$

for every Borel set A of T.

Let $\mathcal{B}$ be the set of bounded Borel functions on T with compact support. Then

$$\int x d(\mu * \delta_t) = \int x(rs) d(\mu \otimes \delta_t)(r, s) =$$

$$= \int \left(\int x(rs) d\delta_t(s) \right) d\mu(r) = \int x(rt) d\mu(r)$$

for every $x \in \mathcal{C}_0(T)$ so that, by Lebesgue's Convergence Theorem,

$$\int x d(\mu * \delta_t) = \int x(rt) d\mu(r)$$

for every $x \in \mathcal{B}$. Hence

$$|\mu * \delta_t|(A) = \sup \left\{ \left| \int x d(\mu * \delta_t) \right| \;\middle|\; x \in \mathcal{B}, \, |x| \leq e_A \right\} =$$

$$= \sup \left\{ \left| \int x(rt) d\mu(r) \right| \;\middle|\; x \in \mathcal{B}, \, |x| \leq e_A \right\} =$$

$$= \sup \left\{ \left| \int y \, d\mu \right| \;\middle|\; y \in \mathcal{B}, \, |y| \leq e_{At^{-1}} \right\} = |\mu|(At^{-1}) \, . \quad ■$$

Proposition 2.2.2.18 *Let λ be a left invariant Haar measure on the locally compact group T, and $(\mu_\iota)_{\iota\in I}$ a finite family in $\mathcal{M}_b(T)\backslash\{0\}$. Then there is a compact set K of T such that*

$$\|\mu_\iota - \mu_\kappa * \delta_t\| > \inf_{\iota\in I} \|\mu_\iota\|,$$

for every $\iota, \kappa \in I$, $t \in T\backslash K$.

Put

$$\alpha := \inf_{\iota\in I} \|\mu_\iota\| > 0.$$

There is a compact set L of T with $1 \in L$ and

$$\sup_{\iota\in I} |\mu_\iota|(T\backslash L) < \frac{\alpha}{4}.$$

Now define

$$K := L^{-1}L := \{s^{-1}t \mid s, t \in L\}.$$

Then

$$|\mu_\iota|(L) = |\mu_\iota|(T) - |\mu_\iota|(T\backslash L) > \alpha - \frac{\alpha}{4} = \frac{3\alpha}{4}$$

for every $\iota \in I$ and

$$L \cap (Lt^{-1}) = \emptyset$$

for every $t \in T\backslash K$. By Proposition 2.2.2.17,

$$|\mu_\kappa * \delta_t|(L) = |\mu_\kappa|(Lt^{-1}) \leq |\mu_\kappa|(T\backslash L) < \frac{\alpha}{4},$$

$$|\mu_\kappa * \delta_t|(T\backslash L) = |\mu_\kappa * \delta_t|(T) - |\mu_\kappa * \delta_t|(L) >$$

$$> |\mu_\kappa|(Tt^{-1}) - \frac{\alpha}{4} \geq \alpha - \frac{\alpha}{4} = \frac{3\alpha}{4},$$

so that

$$\|\mu_\iota - \mu_\kappa * \delta_t\| = |\mu_\iota - \mu_\kappa * \delta_t|(L) + |\mu_\iota - \mu_\kappa * \delta_t|(T\backslash L) \geq$$

$$\geq |\mu_\iota|(L) - |\mu_\kappa * \delta_t|(L) + |\mu_\kappa * \delta_t|(T\backslash L) - |\mu_\iota|(T\backslash L) >$$

$$> \frac{3\alpha}{4} - \frac{\alpha}{4} + \frac{3\alpha}{4} - \frac{\alpha}{4} = \alpha.$$

■

Corollary 2.2.2.19 *Let T be a locally compact group and take*

$$\mu \in \mathcal{M}_b(T)\backslash\{0\}.$$

If T is not compact, then there is a sequence $(t_n)_{n\in\mathbb{N}}$ in T such that

$$\|\mu * \delta_{t_m} - \mu * \delta_{t_n}\| > \|\mu\|$$

for distinct $m, n \in \mathbb{N}$.

We construct the sequence inductively, starting with $t_1 = 1$. Take $n \in \mathbb{N}\backslash\{1\}$ and assume that the sequence has already been constructed for all $m \in \mathbb{N}_{n-1}$. Then

$$\|\mu * \delta_t\| = \|\mu\|$$

for every $t \in T$ (Proposition 2.2.2.17). By Proposition 2.2.2.18, there is a $t_n \in T$ such that

$$\|\mu * \delta_{t_m} - \mu * \delta_{t_n}\| > \|\mu\|$$

for every $m \in \mathbb{N}_{n-1}$. ■

Lemma 2.2.2.20 *Let λ be a right invariant measure on the locally compact group T. Take $p \in [1, \infty[$ and let $(x_\iota)_{\iota\in I}$ be a finite family in $L^p(\lambda)\backslash\{0\}$. For $x \in L^p(\lambda)$ and $t \in T$, define*

$$x^t : T \longrightarrow \mathbb{K}, \quad s \longmapsto x(st).$$

Then there is a compact set K of T such that

$$\|x_\iota - x_\kappa^t\|_p > \inf_{\iota\in I} \|x_\iota\|_p$$

for every $\iota, \kappa \in I$ and $t \in T\backslash K$.

Define

$$\alpha := \inf_{\iota\in I} \|x_\iota\|_p^p > 0.$$

Take $\beta \in]1, \infty[$ such that

$$\left(\frac{\beta-1}{\beta}\right)^{\frac{1}{p}} - \left(\frac{1}{\beta}\right)^{\frac{1}{p}} = \left(\frac{1}{2}\right)^{\frac{1}{p}}.$$

There is a compact set L of T such that

$$\int_{T\setminus L} |x_\iota|^p d\lambda < \frac{\alpha}{\beta}$$

for every $\iota \in I$. Define

$$K := LL^{-1} := \{st^{-1} \mid s, t \in L\}.$$

Then

$$\int_L |x_\iota|^p d\lambda = \int_T |x_\iota|^p d\lambda - \int_{T\setminus L} |x_\iota|^p d\lambda > \alpha - \frac{\alpha}{\beta} = \frac{\beta - 1}{\beta}\alpha$$

for every $\iota \in I$ and

$$L \cap (Lt) = \emptyset$$

for every $t \in T\setminus K$.

Take $\iota, \kappa \in I$ and $t \in T\setminus K$. Then

$$\int_L |x_\kappa^t|^p d\lambda = \int e_L(s)|x_\kappa(st)|^p d\lambda(s) =$$

$$= \int e_L(st^{-1})|x_\kappa(s)|^p d\lambda(s) = \int_{Lt} |x_\kappa(s)|^p d\lambda(s) \leq \int_{T\setminus L} |x_\kappa|^p d\lambda < \frac{\alpha}{\beta},$$

$$\int_{T\setminus L} |x_\kappa^t|^p d\lambda = \int_T |x_\kappa^t|^p d\lambda - \int_L |x_\kappa^t|^p d\lambda > \alpha - \frac{\alpha}{\beta} = \frac{\beta - 1}{\beta}\alpha,$$

$$\left(\int_L |x_\iota - x_\kappa^t|^p d\lambda\right)^{\frac{1}{p}} \geq \left(\int_L |x_\iota|^p d\lambda\right)^{\frac{1}{p}} - \left(\int_L |x_\kappa^t|^p d\lambda\right)^{\frac{1}{p}} >$$

$$> \left(\frac{\beta - 1}{\beta}\right)^{\frac{1}{p}} \alpha^{\frac{1}{p}} - \left(\frac{1}{\beta}\right)^{\frac{1}{p}} \alpha^{\frac{1}{p}} = \left(\frac{\alpha}{2}\right)^{\frac{1}{p}},$$

$$\left(\int_{T\setminus L} |x_\iota - x_\kappa^t|^p d\lambda\right)^{\frac{1}{p}} \geq \left(\int_{T\setminus L} |x_\kappa^t|^p d\lambda\right)^{\frac{1}{p}} - \left(\int_{T\setminus L} |x_\iota|^p d\lambda\right)^{\frac{1}{p}} >$$

$$> \left(\frac{\beta - 1}{\beta}\right)^{\frac{1}{p}} \alpha^{\frac{1}{p}} - \left(\frac{1}{\beta}\right)^{\frac{1}{p}} \alpha^{\frac{1}{p}} = \left(\frac{\alpha}{2}\right)^{\frac{1}{p}},$$

$$\|x_\iota - x_\kappa^t\|_p^p = \int_L |x_\iota - x_\kappa^t|^p d\lambda + \int_{T\setminus L} |x_\iota - x_\kappa^t|^p d\lambda > \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha. \qquad \blacksquare$$

Corollary 2.2.2.21 *Let λ be Haar measure on the unimodular locally compact group T. Take $p \in [1, \infty[$, $\mu \in \mathcal{M}_b(T)$, and $x \in L^p(\lambda)$, such that*

$$\mu * x \neq 0$$

(Proposition 2.2.2.14 a)). We set

$$y^t : T \longrightarrow \mathbb{K}, \quad s \longmapsto y(st)$$

for every $y \in L^p(\lambda)$ and $t \in T$. If T is not compact, then there is a sequence $(t_n)_{n\in\mathbb{N}}$ in T such that

$$\|\mu * x^{t_m} - \mu * x^{t_n}\|_p > \|\mu * x\|_p$$

for all distinct $m, n \in \mathbb{N}$.

Take $t \in T$. Then

$$(\mu * x^t)(s) = \int x^t(r^{-1}s)d\mu(r) = \int x(r^{-1}st)d\mu(r) = (\mu * x)(st) = (\mu * x)^t(s)$$

for every $s \in T$. Thus

$$\mu * x^t = (\mu * x)^t, \quad \|\mu * x^t\|_p = \|(\mu * x)^t\|_p = \|\mu * x\|_p.$$

We construct the sequence inductively, starting with $t_1 := 1$. Let $n \in \mathbb{N}\backslash\{1\}$ and assume the sequence has been constructed for all $m \in \mathbb{N}_{n-1}$. By the above considerations and by Lemma 2.2.2.20, there is a $t_n \in T$ such that

$$\|\mu * x^{t_m} - \mu * x^{t_n}\|_p = \|(\mu * x)^{t_m} - (\mu * x)^{t_n}\|_p > \|\mu * x\|_p . \qquad \blacksquare$$

Example 2.2.2.22 (0) *Let $(T, \mathfrak{T}, \mu)$ be a measure space and take $p \in [1, \infty]$. Given $x \in L^\infty(\mu)$, define*

$$\widetilde{x} : L^p(\mu) \longrightarrow L^p(\mu), \quad y \longmapsto xy.$$

a) *Given $x \in L^\infty(\mu)$, $\widetilde{x} \in \mathcal{L}(L^p(\mu))$, $\|\widetilde{x}\| = \|x\|_\infty$. $\widetilde{x}$ is called a* ***multiplication operator on*** *$L^p(\mu)$.*

b) *The map*

$$L^\infty(\mu) \longrightarrow \mathcal{L}(L^p(\mu)), \quad x \longmapsto \widetilde{x}$$

is a continuous unital algebra homomorphism (Examples 2.2.2.2 and 2.2.2.4).

c) *For each* $x \in L^\infty(\mu)$,

$$\sigma(\widetilde{x}) = \{\alpha \in \mathbb{K} \mid \varepsilon > 0 \Longrightarrow \overline{x}^1(U_\varepsilon^{\mathbb{K}}(\alpha)) \text{ is not a } \mu\text{-null set}\}.$$

d) *Let* μ *be a* σ*-finite measure or a Radon measure on a locally compact space and put*

$$\widetilde{L^\infty(\mu)} := \{\widetilde{x} \mid x \in L^\infty(\mu)\}.$$

Then

$$\left(\widetilde{L^\infty(\mu)}\right)^c := \widetilde{L^\infty(\mu)}.$$

e) *Let* μ *be a positive Radon measure on the locally compact space* T. *Take* $p < \infty$ *and put*

$$\mathcal{K}(T) := \{x \in C(T) \mid \operatorname{Supp} x \text{ is compact}\},$$

$$\widetilde{\mathcal{K}(T)} := \{\widetilde{x} \mid x \in \mathcal{K}(T)\},$$

$$\widetilde{L^\infty(\mu)} := \{\widetilde{x} \mid x \in L^\infty(\mu)\}.$$

Then

$$\left(\widetilde{\mathcal{K}(T)}\right)^c = \left(\widetilde{L^\infty(\mu)}\right)^c = \widetilde{L^\infty(\mu)}.$$

Moreover if $\mathfrak{F}$ *denotes the upper section filter of*

$$\{x \in \mathcal{K}(T) \mid 0 \leq x < e_T\}.$$

Then

$$\lim_{x,\mathfrak{F}} \widetilde{x}y = y$$

for every $y \in L^p(\mu)$.

a) $\widetilde{x}$ is obviously linear and

$$\|\widetilde{x}y\|_p = \left(\int |x|^p |y|^p d\mu\right)^{\frac{1}{p}} \leq \|x\|_\infty \|y\|_p$$

for all $y \in L^p(\mu)$ if $p \neq \infty$. This inequality also holds for $p = \infty$. Hence $\widetilde{x} \in \mathcal{L}(L^p(\mu))$ and

$$\|\tilde{x}\| \leq \|x\|_\infty .$$

In order to prove the reverse inequality, we may assume that $\|x\| \neq 0$. Take $\alpha \in [0, \|x\|_\infty[$. There is a μ–integrable set A with

$$A \subset \{|x| > \alpha\}, \qquad \int e_A d\mu > 0 .$$

We have

$$\|\tilde{x}\| \, \|e_A\|_p \geq \|\tilde{x} e_A\|_p = \Big(\int_A |x|^p d\mu \Big)^{\frac{1}{p}} \geq \alpha \|e_A\|_p$$

if $p \neq \infty$. This inequality also holds for $p = \infty$ and so

$$\|\tilde{x}\| \geq \alpha .$$

Since α is arbitrary, we get

$$\|\tilde{x}\| \geq \|x\|_\infty .$$

b) is easy to see.

c) Take $\alpha \in \sigma(\tilde{x})$ and $\varepsilon > 0$. If

$$\overset{-1}{x}(U_\varepsilon^{\mathbb{K}}(\alpha))$$

is a μ–null set then $\alpha e_T - x$ is invertible in $L^\infty(\mu)$. By b), $\alpha 1 - \tilde{x}$ is invertible in $\mathcal{L}(L^p(\mu))$, which is a contradicition.

Now take $\alpha \in \mathbb{K} \backslash \sigma(\tilde{x})$ and put

$$u := (\alpha 1 - \tilde{x})^{-1} .$$

Take $\varepsilon \in]0, \|u\|[$ and let A be a μ–integrable set contained in $\overset{-1}{x}(U_\varepsilon^{\mathbb{K}}(\alpha))$. Then

$$\|(\alpha 1 - \tilde{x}) e_A\|_p = \Big(\int_A |\alpha e_T - x|^p d\mu \Big)^{\frac{1}{p}} \leq \varepsilon \|e_A\|_p$$

if $p \neq \infty$ and the same inequality holds for $p = \infty$. Since

$$\|e_A\|_p = \|u(\alpha 1 - \tilde{x}) e_A\|_p \leq \|u\| \, \|(\alpha 1 - \tilde{x}) e_A\|_p \leq \|u\| \varepsilon \|e_A\|_p ,$$

we see that $e_A = 0$. A being arbitrary, it follows that $\overset{-1}{x}(U_\varepsilon^{\mathbb{K}}(\alpha))$ is a μ–null set.

d) Take $u \in \Big(\widetilde{L^\infty(\mu)}\Big)^c$. First suppose that $T \in \mathfrak{T}$. Put

$$x := u e_T \in L^p(\mu) .$$

Then

$$xy = \widetilde{y}x = \widetilde{y}ue_T = u\widetilde{y}e_T = uy$$

for every $y \in L^\infty(\mu)$. Take $\alpha > 0$ such that

$$A := \{|x| > \alpha\}$$

is not a μ-null set. Define

$$y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} \frac{\overline{x}(t)}{|x(t)|} & \text{if } t \in A \\ 0 & \text{if } t \in T \backslash A \,. \end{cases}$$

Then $y \in L^\infty(\mu)$ so that

$$|x|e_A = xy = uy \,.$$

Thus

$$\alpha \|e_A\|_p \leq \| \, |x|e_A\|_p = \|uy\|_p \leq \|u\| \, \|y\|_p = \|u\| \, \|e_A\|_p \,, \qquad \alpha \leq \|u\| \,.$$

Hence $x \in L^\infty(\mu)$. Since

$$\widetilde{x}y = xy = uy$$

for every $y \in L^\infty(\mu)$ and since $L^\infty(\mu)$ is dense in $L^p(\mu)$, we get that

$$u = \widetilde{x} \in \widetilde{L^\infty(\mu)} \,.$$

If μ is either a σ-finite measure or a Radon measure on a locally compact space, then there is a set $\mathfrak{A}$ of pairwise disjoint sets of $\mathfrak{T}$ and such that for every $A \in \mathfrak{A}$ the set

$$A \backslash \bigcup_{B \in \mathfrak{T}_A} B$$

is a μ-null set, where

$$\mathfrak{T}_A := \{B \in \mathfrak{T} \mid \mu(A \cap B) \neq 0\} \,.$$

Given $A \in \mathfrak{A}$, define

$$x_A := ue_A$$

and

$$x : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} x_A(t) & \text{if } t \in A \text{ for an } A \in \mathfrak{A} \\ 0 & \text{if } t \in T \setminus \bigcup_{A \in \mathfrak{A}} A . \end{cases}$$

By the above considerations, $x \in \mathcal{L}^\infty(\mu)$ and

$$u = \tilde{x} \in \widetilde{L^\infty(\mu)} .$$

Hence

$$\left(\widetilde{L^\infty(\mu)}\right)^c \subset \widetilde{L^\infty(\mu)} .$$

The reverse inclusion is trivial since $\widetilde{L^\infty(\mu)}$ is commutative.

e) Take $u \in \left(\widetilde{\mathcal{K}(T)}\right)^c$. We set

$$\mathcal{F} := \{x \in L^\infty(\mu) \mid \tilde{x}u = u\tilde{x}\} .$$

Let x be a lower semicontinuous function in $\mathcal{L}^\infty(\mu)_+$,

$$\mathcal{G} := \{y \in \mathcal{K}(T)_+ | y \leq x\} ,$$

and $\mathfrak{F}$ the upper section filter of $\mathcal{G}$. Then

$$\tilde{x}uz = xuz = \lim_{y,\mathfrak{F}} yuz = \lim_{y,\mathfrak{F}} \tilde{y}uz = \lim_{y,\mathfrak{F}} u\tilde{y}z =$$

$$= \lim_{y,\mathfrak{F}} u(yz) = u(xz) = u\tilde{x}z$$

for every $z \in L^p(\mu)$, where the limits are taken in $L^p(\mu)$. Thus

$$\tilde{x}u = u\tilde{x} ,$$

$$x \in \mathcal{F} .$$

Let $(x_n)_{n \in \mathbb{N}}$ be a monotone sequence in $\mathcal{F}$ converging pointwise to an $x \in L^\infty(\mu)$. Then

$$\tilde{x}uy = xuy = \lim_{n \to \infty} x_n uy = \lim_{n \to \infty} \tilde{x}_n uy = \lim_{n \to \infty} u\tilde{x}_n y =$$

$$= \lim_{n \to \infty} u(x_n y) = u(xy) = u\tilde{x}y$$

for every $y \in L^p(\mu)$, where the limits are taken in $L^p(\mu)$. Thus

$$\tilde{x}u = u\tilde{x} ,$$

$$x \in \mathcal{F}.$$

By the above considerations,

$$\mathcal{F} = L^\infty(\mu).$$

Hence, by d),

$$u \in \left(\widetilde{L^\infty(\mu)}\right)^c = \widetilde{L^\infty(\mu)},$$

$$\left(\widetilde{\mathcal{K}(T)}\right)^c = \left(\widetilde{L^\infty(\mu)}\right)^c = \widetilde{L^\infty(\mu)}.$$

The final assertion is easy to see. ■

Example 2.2.2.23 *Let E be a Banach space with topological cardinality $\aleph > \aleph_0$ and $\aleph', \aleph''$ cardinal numbers such that:*

1) $\aleph' < \aleph$.

2) $\aleph'' \leq \aleph$.

3) *The sum of any countable family of cardinal numbers strictly smaller than $\aleph''$ is strictly smaller than $\aleph''$.*

Denote by $\mathcal{F}$ the set of $u \in \mathcal{L}(E)$ for which the topological cardinality of $\operatorname{Im} u$ is at most equal to $\aleph'$ (is strictly smaller than $\aleph''$). Then $\mathcal{F}$ is a closed proper ideal of $\mathcal{L}(E)$ (Example 2.2.2.4).

Take $u, v \in \mathcal{F}$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\operatorname{Im}(\alpha u + \beta v) \subset \operatorname{Im} u + \operatorname{Im} v$$

so that $\alpha u + \beta v \in \mathcal{F}$. Hence $\mathcal{F}$ is a vector subspace of $\mathcal{L}(E)$.

Take $u \in \mathcal{L}(E)$ and $v \in \mathcal{F}$. Then

$$\operatorname{Im}(uv) = u(\operatorname{Im} v), \quad \operatorname{Im}(vu) \subset \operatorname{Im} v,$$

so that $uv, vu \in \mathcal{F}$. Hence $\mathcal{F}$ is an ideal of $\mathcal{L}(E)$ and it is obviously proper.

Let $u \in \overline{\mathcal{F}}$ and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ converging to u. Then

$$\operatorname{Im} u \subset \overline{\bigcup_{n\in\mathbb{N}} \operatorname{Im} u_n}$$

so that $u \in \mathcal{F}$ and $\mathcal{F}$ is closed. ■

2.2.3 The Exponential Function and the Neumann Series

Proposition 2.2.3.1 (0) *Let E be an algebra and f a positive real function on E such that*

$$f(xy) \leq f(x)f(y)$$

for $x, y \in E$. Then, given $x \in E$,

$$f(x^n) \leq f(x)^n$$

for every $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} f(x^n)^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} f(x^n)^{\frac{1}{n}} \leq f(x).$$

The first inequality follows by complete induction.

Take $k \in \mathbb{N}$. For every $n \in \mathbb{N}$, there are uniquely determined

$$p(n), q(n) \in \mathbb{N} \cup \{0\}$$

such that

$$n = kp(n) + q(n), \quad q(n) < k.$$

Clearly,

$$\lim_{n\to\infty} \frac{q(n)}{n} = 0, \quad \lim_{n\to\infty} \frac{p(n)}{n} = \frac{1}{k}.$$

Then

$$f(x^n) = f(x^{kp(n)+q(n)}) \leq f(x^k)^{p(n)} f(x)^{q(n)},$$

$$f(x^n)^{\frac{1}{n}} \leq f(x^k)^{\frac{p(n)}{n}} f(x)^{\frac{q(n)}{n}}$$

for every $n \in \mathbb{N}$ and therefore

$$\limsup_{n\to\infty} f(x^n)^{\frac{1}{n}} \leq f(x^k)^{\frac{1}{k}}.$$

Since k is arbitrary, we see that

$$\limsup_{n\to\infty} f(x^n)^{\frac{1}{n}} \leq \inf_{n\in\mathbb{N}} f(x^n)^{\frac{1}{n}} \leq \liminf_{n\to\infty} f(x^n)^{\frac{1}{n}}.$$

Hence

$$\lim_{n\to\infty} f(x^n)^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} f(x^n)^{\frac{1}{n}} \leq f(x). \qquad \blacksquare$$

Proposition 2.2.3.2 (0) *Let E be a unital Banach algebra and $\sum_{n=0}^{\infty} \alpha_n t^n$ a power series in $\mathbb{K}$ whose radius of convergence is r. Given $x \in E$ with*

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} < r\,,$$

the family $(\alpha_n x^n)_{n\in\mathbb{N}\cup\{0\}}$ is absolutely summable and the map

$$U_r^E(0) \longrightarrow E\,, \quad x \longmapsto \sum_{n=0}^{\infty} \alpha_n x^n$$

is continuous.

Now

$$\limsup_{n\to\infty} \|\alpha_n x^n\|^{\frac{1}{n}} = \limsup_{n\to\infty} |\alpha_n|^{\frac{1}{n}} \|x^n\|^{\frac{1}{n}} =$$

$$= \left(\limsup_{n\to\infty} |\alpha^n|^{\frac{1}{n}}\right) \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \frac{1}{r} \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} < 1\,.$$

By the Cauchy Root Test, the family $(\alpha_n x^n)_{n\in\mathbb{N}\cup\{0\}}$ is absolutely summable.

Take $x \in U_r^E(0)$. By Proposition 2.2.3.1,

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\| < r\,.$$

Take $\varepsilon > 0$. There is a $p \in \mathbb{N}$, such that

$$\sum_{n=p+1}^{\infty} |\alpha_n| \left(\frac{r + \|x\|}{2}\right)^n < \frac{\varepsilon}{3}\,.$$

Since

$$\lim_{y\to x} \sum_{n=0}^{p} \alpha_n y^n = \sum_{n=0}^{p} \alpha_n x^n$$

(Proposition 2.2.1.6), there is a $\delta \in \left]0, \frac{r-\|x\|}{2}\right[$ such that

$$\left\| \sum_{n=0}^{p} \alpha_n y^n - \sum_{n=0}^{p} \alpha_n x^n \right\| < \frac{\varepsilon}{3}$$

whenever $y \in U_\delta^E(x)$. Then

$$\|y\| \leq \|y - x\| + \|x\| < \frac{r - \|x\|}{2} + \|x\| = \frac{r + \|x\|}{2}$$

and so

$$\left\| \sum_{n=0}^{\infty} \alpha_n y^n - \sum_{n=0}^{\infty} \alpha_n x^n \right\| \leq \left\| \sum_{n=0}^{p} \alpha_n y^n - \sum_{n=0}^{p} \alpha_n x^n \right\| +$$

$$+2 \sum_{n=p+1}^{\infty} |\alpha_n| \left(\frac{r + \|x\|}{2} \right)^n < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

for every $y \in U_\delta^E(x)$. Hence the map

$$U_r^E(0) \longrightarrow E, \quad x \longmapsto \sum_{n=0}^{\infty} \alpha_n x^n$$

is continuous. ■

Proposition 2.2.3.3 (0) *Let E be a unital Banach algebra,*

$$\sum_{n=0}^{\infty} \alpha_n t^n, \sum_{n=0}^{\infty} \beta_n t^n$$

power series in $\mathbb{K}$ with radii of convergence r and r', respectively, and $\sum_{n=0}^{\infty} \gamma_n t^n$ the product of the two power series. Then

$$\left(\sum_{n=0}^{\infty} \alpha_n x^n \right) \left(\sum_{n=0}^{\infty} \beta_n x^n \right) = \sum_{n=0}^{\infty} \gamma_n x^n$$

for every $x \in E$, so that

$$\lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} < \inf\{r, r'\}.$$

By Proposition 2.2.3.2, the above series are well defined. By Proposition 1.2.9.6 (and Proposition 2.2.1.6),

$$\left(\sum_{n=0}^{\infty} \alpha_n x^n \right) \left(\sum_{n=0}^{\infty} \beta_n x^n \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \alpha_k \beta_{n-k} \right) x^n = \sum_{n=0}^{\infty} \gamma_n x^n. \quad ■$$

Corollary 2.2.3.4 *Let E be a unital Banach algebra. Take $x \in E$. Let $\sum_{n=0}^{\infty} \alpha_n t^n$, $\sum_{n=0}^{\infty} \beta_n t^n$ be power series in $\mathbb{K}$ with radii of convergence r and r', respectively, and $\sum_{n=0}^{\infty} \gamma_n t^n$ the power series in 0 corresponding to the analytic function*

$$t \longmapsto \sum_{n=0}^{\infty} \beta_n \left(\sum_{k=0}^{\infty} \alpha_k t^k \right)^n.$$

If $\|x\| < r$ *and*

$$\left\| \sum_{n=0}^{\infty} \alpha_n x^n \right\| < r',$$

then

$$\sum_{n=0}^{\infty} \beta_n \left(\sum_{k=0}^{\infty} \alpha_k x^k \right)^n = \sum_{n=0}^{\infty} \gamma_n x^n .$$

By Proposition 2.2.3.2, the above series are well defined. By Proposition 2.2.3.3 and Proposition 1.1.6.11, the above relation holds whenever $\{n \in \mathbb{N} \mid \beta_n \neq 0\}$ is finite. The general result now follows by passing to the limit. ■

Definition 2.2.3.5 (0) *Let* E *be a unital Banach algebra and take* $x \in E$. *Define (Proposition 2.2.3.2)*

$$e^x := \sum_{n=0}^{\infty} \frac{1}{n!} x^n \qquad \text{(E.N. Laguerre, 1867)}.$$

For every $\alpha \in \mathbb{K}$ *with*

$$\lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} < \alpha$$

define (Proposition 2.2.3.2)

$$\frac{1}{\alpha - x} := \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} x^n \qquad \text{(C. Neumann, 1870)}.$$

The map

$$E \longrightarrow E, \quad x \longmapsto e^x$$

is called ***the exponential function (on*** E ***)***.

Remark. Let F be the closed unital subalgebra of E generated by x. Then F is commutative (Corollary 2.2.1.10) and

$$e^x, \frac{1}{\alpha - x} \in F .$$

Proposition 2.2.3.6 *Let E be a unital Banach algebra and take $x, y \in E$. If*

$$ye^{\alpha x} = e^{\alpha x}y$$

for every $\alpha \in \mathbb{K}$, then

$$xy = yx\,.$$

Given $\alpha \in \mathbb{K}$,

$$y\left(\sum_{n=0}^{\infty} \frac{\alpha^n}{n!}x^n\right) = \left(\sum_{n=0}^{\infty} \frac{\alpha^n}{n!}x^n\right)y\,,$$

so that

$$yx = yx$$

by Corollary 2.2.1.12. ■

Proposition 2.2.3.7 (0) *Let E be a unital Banach algebra and take $x, y \in E$ with*

$$xy = yx\,.$$

a) $e^{x+y} = e^x e^y$.

b) $(e^x)^{-1} = e^{-x}$.

c) $e^x \neq 0$.

a) Given $n \in \mathbb{N} \cup \{0\}$, define

$$z_n := \sum_{k=0}^{n} \frac{1}{k!(n-k)!}\,x^k y^{n-k}\,.$$

Then

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!}\,x^k y^{n-k} = n!z_n\,,$$

$$z_n = \frac{1}{n!}(x+y)^n$$

for every $n \in \mathbb{N} \cup \{0\}$. It follows that

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n = \sum_{n=0}^{\infty} z_n = \left(\sum_{n=0}^{\infty} \frac{1}{n!}x^n\right)\left(\sum_{n=0}^{\infty} \frac{1}{n!}y^n\right) = e^x e^y$$

(Proposition 1.2.9.6 and Proposition 2.2.1.6).

b) By a),

$$e^x e^{-x} = e^{x-x} = 1$$

and so

$$(e^x)^{-1} = e^{-x}.$$

c) follows from b). ■

Remark. e^{x+y} is not always $e^x e^y$. For example, let $E := \mathcal{L}(\mathbb{K}^2)$ and take

$$x := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad y := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then

$$x^2 = y^2 = 0,$$

$$xy = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad yx = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and so

$$(x+y)^{2n} = 1, \quad (x+y)^{2n-1} = x+y$$

for every $n \in \mathbb{N}$. Hence

$$e^x = 1 + x, \quad e^y = 1 + y$$

$$e^x e^y = 1 + x + y + xy,$$

$$e^{x+y} = \sum_{n=0}^{\infty} \frac{1}{(2n)!}1 + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(x+y) = (ch\,1)1 + (sh\,1)(x+y) \neq e^x e^y.$$

Corollary 2.2.3.8 (Le Page, 1967) *Let E be a unital complex Banach algebra. If there is an $\alpha \in \mathbb{R}_+$ such that*

$$\|xy\| \leq \alpha\|yx\|$$

for every $x, y \in E$, then E is commutative.

Take $x, y \in E$ and consider

$$f : \mathbb{C} \longrightarrow E, \quad \beta \longmapsto e^{\beta x} y e^{-\beta x}.$$

f is analytic (Propositions 1.3.10.3 and 1.3.10.10) and

$$\|f(\beta)\| = \|e^{\beta x} y e^{-\beta x}\| \leq \alpha \|y e^{-\beta x} e^{\beta x}\| = \alpha \|y\|$$

for every $\beta \in \mathbb{C}$ (Proposition 2.2.3.7 b)). By Liouville's Theorem, f is constant. Hence

$$y = f(0) = f(\beta) = e^{\beta x} y e^{-\beta x}, \qquad y e^{\beta x} = e^{\beta x} y$$

for every $\beta \in \mathbb{C}$ (Proposition 2.2.3.7 b)), and so

$$yx = xy$$

(Proposition 2.2.3.6). ∎

Remark. $\mathbb{H}$ (Corollary 2.1.4.17) is a counterexample to the above Corollary for the case $\mathbb{K} = \mathbb{R}$.

Proposition 2.2.3.9 *Let E be a unital Banach algebra. Given $x \in U_1^E(0)$, put*

$$\log(1 - x) := -\sum_{n=1}^{\infty} \frac{1}{n} x^n$$

(Proposition 2.2.3.2). Then

$$e^{\log(1-x)} = 1 - x$$

for every $x \in U_1^E(0)$ and

$$\log e^x := \log(1 - (1 - e^x)) = x$$

for every $x \in E$, so that $e^x \in U_1^E(1)$.

Now

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(-\sum_{k=1}^{\infty} \frac{1}{k} \alpha^k \right)^n = e^{\log(1-\alpha)} = 1 - \alpha$$

for every $\alpha \in U_1^{\mathbb{K}}(0)$, and

$$-\sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \sum_{k=1}^{\infty} \frac{1}{k!} \alpha^k \right)^n = \log(1 - (1 - e^{\alpha})) = \log e^{\alpha} = \alpha$$

for every $\alpha \in \mathbb{K}$. Thus $|1 - e^{\alpha}| < 1$. The assertions now follow from Corollary 2.2.3.4. ∎

Definition 2.2.3.10 (0) *Given* $\alpha \in \mathbb{C}$ *and* $n \in \mathbb{N}$, *define*

$$\binom{\alpha}{0} := 1, \quad \binom{\alpha}{n} := \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}.$$

We have

$$\binom{p}{n} = 0$$

for every $n, p \in \mathbb{N}$, with $n > p$.

Lemma 2.2.3.11 (0) (Raabe's Ratio Test) *A series* $\sum_{n=0}^{\infty} \alpha_n$ *in* $\mathbb{K}$ *is absolutely summable if*

$$\liminf_{n\to\infty} n\left(\left|\frac{\alpha_n}{\alpha_{n+1}}\right| - 1\right) > 1, \qquad \left(\left|\frac{\alpha_n}{\alpha_{n+1}}\right| := \infty \quad \text{if } \alpha_{n+1} = 0\right).$$

Choose

$$\alpha \in \left]1, \liminf_{n\to\infty} n\left(\left|\frac{\alpha_n}{\alpha_{n+1}}\right| - 1\right)\right[.$$

Since

$$\lim_{\substack{\beta\to 0\\ \beta>0}} \frac{(1+\beta)^\alpha - 1}{\beta} = \alpha,$$

there is an $n_0 \in \mathbb{N}$ such that

$$n\left(\left|\frac{\alpha_n}{\alpha_{n+1}}\right| - 1\right) > \frac{(1+\frac{1}{n})^\alpha - 1}{\frac{1}{n}}$$

for every $n \in \mathbb{N}$, $n \geq n_0$. Thus

$$\left|\frac{\alpha_n}{\alpha_{n+1}}\right| \geq \left(\frac{n+1}{n}\right)^\alpha,$$

$$|\alpha_{n+1}| \leq \left(\frac{n}{n+1}\right)^\alpha |\alpha_n|,$$

$$|\alpha_{n+1}| \leq \left(\frac{n_0}{n+1}\right)^\alpha |\alpha_{n_0}|$$

for all $n \in \mathbb{N}$, $n \geq n_0$, so that

$$\sum_{n=0}^{\infty} |\alpha_n| < \infty. \qquad \blacksquare$$

Lemma 2.2.3.12 (0) (Binomial Theorem) *If* $\alpha \in \mathbb{R}_+$, *then the power series*

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} t^n$$

converges absolutely and uniformly on $[-1,1]$ *to* $(1+t)^\alpha$.

We may assume $\alpha > 0$. We have for every $n \in \mathbb{N}$, $n > \alpha$,

$$n\left(\left|\frac{\binom{\alpha}{n}}{\binom{\alpha}{n+1}}\right| - 1\right) = n\left(\left|\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}\,\frac{(n+1)!}{\alpha(\alpha-1)\dots(\alpha-n)}\right| - 1\right) =$$

$$= n\left(\frac{n+1}{n-\alpha} - 1\right) = n\frac{n+1-n+\alpha}{n-\alpha} = (1+\alpha)\frac{n}{n-\alpha},$$

so

$$\lim_{n\to\infty} n\left(\left|\frac{\binom{\alpha}{n}}{\binom{\alpha}{n+1}}\right| - 1\right) = 1 + \alpha > 1.$$

By Lemma 2.2.3.11

$$\sum_{n=0}^{\infty} \left|\binom{\alpha}{n}\right| < \infty.$$

Hence the power series

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} t^n$$

converges absolutes and uniformly on $[-1,1]$ to an $x \in C([-1,1])$. x is differentiable on $]-1,1[$ and for every $t \in]-1,1[$ we have

$$(1+t)x'(t) = (1+t)\sum_{n=0}^{\infty} n\binom{\alpha}{n} t^{n-1} = \sum_{k=1}^{\infty} k\binom{\alpha}{k} t^{k-1} + \sum_{n=0}^{\infty} n\binom{\alpha}{n} t^n =$$

$$= \sum_{n=0}^{\infty} (n+1)\binom{\alpha}{n+1} t^n + \sum_{n=0}^{\infty} n\binom{\alpha}{n} t^n =$$

$$= \sum_{n=0}^{\infty} \left((n+1)\frac{\alpha(\alpha-1)\dots(\alpha-n)}{(n+1)!} + n\frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}\right) t^n =$$

$$= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}(\alpha-n+n)t^n = \alpha \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n = \alpha x(t) .$$

If we define

$$y :]-1,1[\longrightarrow \mathbb{R}, \quad t \longmapsto \frac{x(t)}{(1+t)^\alpha},$$

then

$$y'(t) = \frac{(1+t)^\alpha x'(t) - \alpha(1+t)^{\alpha-1}x(t)}{(1+t)^{2\alpha}} = \frac{(1+t)x'(t) - \alpha x(t)}{(1+t)^{\alpha+1}} = 0 .$$

Hence

$$\frac{x(t)}{(1+t)^\alpha} = y(t) = y(0) = x(0) = 1 ,$$

$$x(t) = (1+t)^\alpha .$$

By continuity, the above relation holds for every $t \in [-1,1]$. ■

Proposition 2.2.3.13 (0) *Let E be a unital Banach algebra, $\alpha \in \mathbb{R}_+$, and $x \in E$ such that $\|x\| \leq 1$ or*

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} < 1 .$$

a) *The family $\left(\binom{\alpha}{n} x^n\right)_{n\in\mathbb{N}\cup\{0\}}$ is absolutely summable; we put*

$$[1+x]^\alpha := \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n .$$

b) *The series*

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$

converges uniformly on $E^\#$, so that the map

$$E^\# \longrightarrow E, \qquad x \longmapsto [1+x]^\alpha$$

is continuous.

c) *For every $k \in \mathbb{N}$,*

$$\left([1+x]^{\frac{1}{k}}\right)^k = 1 + x .$$

a & b. Since

$$\left\|\binom{\alpha}{n}x^n\right\| \leq \left|\binom{\alpha}{n}\right|$$

for every $n \in \mathbb{N}$ and $x \in E^{\#}$, it follows from Lemma 2.2.2.12 that the series

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}x^n$$

converges absolutey and uniformly on $E^{\#}$. Moreover, the radius of convergence of the power series

$$\sum_{n=0}^{\infty}\binom{\alpha}{n}t^n$$

in $\mathbb{K}$ is at least 1. By Proposition 2.2.3.2, the family $\left(\binom{\alpha}{n}x^n\right)_{n\in\mathbb{N}}$ is absolutely summable whenever

$$\lim_{n\to\infty}\|x^n\|^{\frac{1}{n}} < 1.$$

c) By Proposition 2.2.3.3 and Lemma 2.2.3.12

$$\left([1+x]^{\frac{1}{k}}\right)^k = 1 + x$$

for all $x \in E$ with

$$\lim_{n\to\infty}\|x^n\|^{\frac{1}{n}} < 1.$$

In particular, the above relation holds for every $x \in E$ with $\|x\| < 1$ (Proposition 2.2.3.1). By continuity, the relation holds for every $x \in E^{\#}$. ■

2.2.4 Invertible Elements of Unital Banach Algebras

Proposition 2.2.4.1 (0) *Let E be a unital Banach algebra. Take $x \in E$ and $\alpha \in \mathbb{K}$ such that*

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} < |\alpha| .$$

Then

$$(\alpha 1 - x)^{-1} = \frac{1}{\alpha - x},$$

$$r(x) \leq \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\| .$$

Now

$$(\alpha 1 - x)\frac{1}{\alpha - x} = (\alpha 1 - x)\sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} x^n = \alpha \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} x^n - x \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} x^n =$$

$$= \sum_{n=0}^{\infty} \frac{1}{\alpha^n} x^n - \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} x^{n+1} = 1 + \sum_{n=1}^{\infty} \frac{1}{\alpha^n} x^n - \sum_{n=0}^{\infty} \frac{1}{\alpha^n} x^n = 1$$

(Corollary 2.2.1.11), so that

$$(\alpha 1 - x)^{-1} = \frac{1}{\alpha - x} .$$

It follows that

$$r(x) \leq \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} .$$

The inequality

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\|$$

was proved in Proposition 2.2.3.1. ■

Corollary 2.2.4.2 (0) *Let E be a Banach algebra and take $x \in E$. If $r(x) = \|x\|$ then*

$$r(x)^n = r(x^n) = \|x^n\| = \|x\|^n$$

for every $n \in \mathbb{N}$.

We may assume that E is unital (Proposition 2.2.1.3). By Theorem 2.1.3.4 a),

$$r(x)^n \leq r(x^n)$$

and by Proposition 2.2.4.1,

$$r(x^n) \leq \|x^n\| .$$

Thus $r(x) = \|x\|$ implies that

$$\|x\|^n = r(x)^n \leq r(x^n) \leq \|x^n\| \leq \|x\|^n$$

(Proposition 2.2.3.1). Hence

$$r(x)^n = r(x^n) = \|x^n\| = \|x\|^n . \qquad \blacksquare$$

Corollary 2.2.4.3 (Le Page, 1967) *Let E be a complex unital Banach algebra. If there is an $\alpha \in \mathbb{R}_+$ such that*

$$\|x\| \leq \alpha r(x)$$

for all $x \in E$, then E is commutative.

Given $x, y \in E$,

$$\|xy\| \leq \alpha r(xy) = \alpha r(yx) \leq \alpha \|yx\|$$

(Proposition 2.1.3.10 c), Proposition 2.2.4.1). By Corollary 2.2.3.8, E is commutative. $\blacksquare$

Corollary 2.2.4.4 (0) *Let E be a unital Banach algebra and take $x, y \in E$. If x is invertible and*

$$\|x - y\| < \frac{1}{\|x^{-1}\|} ,$$

then y is also invertible,

$$\|1 - x^{-1}y\| < 1 ,$$

and

$$y^{-1} = \frac{1}{1 - (1 - x^{-1}y)} x^{-1} = \sum_{n=0}^{\infty} (1 - x^{-1}y)^n x^{-1} .$$

Now

$$\|1 - x^{-1}y\| = \|x^{-1}(x-y)\| \leq \|x^{-1}\|\,\|x-y\| < 1\,.$$

Since

$$x^{-1}y = 1 - (1 - x^{-1}y)\,,$$

we deduce that

$$(x^{-1}y)^{-1} = \frac{1}{1-(1-x^{-1}y)}$$

(Proposition 2.2.4.1). From

$$y = x(x^{-1}y)$$

it follows further that

$$y^{-1} = (x^{-1}y)^{-1}x^{-1} = \frac{1}{1-(1-x^{-1}y)}x^{-1} = \sum_{n=0}^{\infty}(1-x^{-1}y)^n x^{-1}$$

(Corollary 2.1.2.6, Corollary 2.2.1.11). ■

Corollary 2.2.4.5 (0) *Let E be a Banach algebra. Take $x \in E$. Then $\sigma(x)$ is compact.*

We may assume that E is unital (Proposition 2.2.1.3). Choose $\alpha \in \mathbb{K}\backslash\sigma(x)$ and put

$$\beta := \frac{1}{\|(\alpha 1 - x)^{-1}\|}\,.$$

Take $\gamma \in U_\beta^{\mathbb{K}}(\alpha)$. Since $\alpha 1 - x$ is invertible and

$$\|(\gamma 1 - x) - (\alpha 1 - x)\| = |\gamma - \alpha| < \beta\,,$$

it follows that $\gamma 1 - x$ is also invertible (Corollary 2.2.4.4). Hence

$$U_\beta^{\mathbb{K}}(\alpha) \subset \mathbb{K}\backslash\sigma(x)\,,$$

$\mathbb{K}\backslash\sigma(x)$ is open and $\sigma(x)$ is closed. Being bounded (Proposition 2.2.4.1), $\sigma(x)$ is compact. ■

Corollary 2.2.4.6 *Let E be a unital Banach algebra and x an element of E. Let*

$$f : \mathbb{K}\backslash\sigma(x) \longrightarrow E$$

be the resolvent of x. Then f is differentiable and

$$f' = -f^2\,.$$

Take $\alpha \in \mathbb{K}\backslash\sigma(x)$. Then

$$\frac{1}{\beta-\alpha}(f(\beta)-f(\alpha)) = -f(\alpha)f(\beta)$$

for every $\beta \in \mathbb{K}\backslash\sigma(x)$, $\beta \neq \alpha$ (Proposition 2.1.3.9). The assertion follows. ■

Corollary 2.2.4.7 (0) *Let E be a unital Banach algebra and x an invertible element of E with*

$$\|x\| = \|x^{-1}\| = 1\,.$$

Then

$$\sigma(x) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,.$$

By Proposition 2.2.4.1,

$$r(x) \leq \|x\| = 1\,,$$

so that

$$\sigma(x) \subset \{\alpha \in \mathbb{K} \mid |\alpha| \leq 1\}\,.$$

Take $\alpha \in \mathbb{K}$ with $|\alpha| < 1$. Then

$$x - \alpha 1 \in U_1^E(x) = U^E_{\frac{1}{\|x^{-1}\|}}(x)\,.$$

By Corollary 2.2.4.4, $x - \alpha 1$ is invertible, i.e. $\alpha \notin \sigma(x)$. Hence

$$\sigma(x) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,.$$ ■

Corollary 2.2.4.8 (4) *Let E be a Banach space and $u : E \to E$ an isometry. Then*

$$\sigma(u) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,.$$

Now

$$\|u\| = \|u^{-1}\| = 1$$

and the assertion follows from Corollary 2.2.4.7 (and Example 2.2.2.4). ■

Corollary 2.2.4.9 *Let E be a Banach space. Take $u \in \mathcal{L}(E)$ and $\delta > 0$, such that*

$$\|ux\| \geq \delta\|x\|$$

for every $x \in E$. Then either

$$U_\delta^{\mathbb{K}}(0) \subset \sigma(u)$$

or

$$U_\delta^{\mathbb{K}}(0) \cap \sigma(u) = \emptyset .$$

Assume there is an

$$\alpha \in U_\delta^{\mathbb{K}}(0) \backslash \sigma(u) .$$

Then

$$\|\alpha x - ux\| \geq \|ux\| - \|\alpha x\| \geq \delta\|x\| - |\alpha|\,\|x\| = (\delta - |\alpha|)\|x\|$$

for every $x \in E$. By Proposition 1.2.1.18 b),

$$\|(\alpha 1 - u)^{-1}\| \leq \frac{1}{\delta - |\alpha|} .$$

Take $\beta \in U_{\delta-|\alpha|}^{\mathbb{K}}(\alpha)$. Then

$$\|(\beta 1 - u) \cdot (\alpha 1 - u)\| = |\beta - \alpha| < \delta - |\alpha| \leq \frac{1}{\|(\alpha 1 - u)^{-1}\|} .$$

By Corollary 2.2.4.4, $\beta 1 - u$ is invertible, i.e. $\beta \notin \sigma(u)$. Hence

$$U_{\delta-|\alpha|}^{\mathbb{K}}(\alpha) \cap \sigma(u) = \emptyset .$$

It follows that

$$U_\delta^{\mathbb{K}}(0) \cap \sigma(u) = \emptyset . \qquad \blacksquare$$

Corollary 2.2.4.10 (0) *Let E be a unital Banach algebra. Take $x \in E$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence of invertible elements of E, converging to x. If $(x_n^{-1})_{n\in\mathbb{N}}$ is bounded, then x is invertible.*

Put

$$\alpha := \sup_{n\in\mathbb{N}} \|x_n^{-1}\| .$$

There is an $n \in \mathbb{N}$ with

$$\|x_n - x\| < \frac{1}{\alpha} .$$

By Corollary 2.2.4.4, x is invertible. $\blacksquare$

Example 2.2.4.11 (4) *Let T be a group. Take $s \in T$ and $p \in [1,\infty]$. Let E be one of the following Banach spaces: $\ell^p(T)$, $c_0(T)$, or $c(T)$. Given $x \in E$, set*

$$\tilde{x} : T \longrightarrow \mathbb{K}, \quad t \longmapsto x(st)$$

and define

$$u : E \longrightarrow E, \quad y \longmapsto \tilde{y}.$$

a) *If $s^n \neq 1$ for every $n \in \mathbb{N}$, then*

$$\sigma(u) = \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}.$$

b) *If n is the least natural number such that $s^n = 1$, then*

$$\sigma(u) = \{\alpha \in \mathbb{K} \mid \alpha^n = 1\}.$$

u is an isometry, so that

$$\sigma(u) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}$$

(Corollary 2.2.4.8). Take $\alpha \in \mathbb{K}$ with $|\alpha| = 1$, and $x, y \in E$ such that

$$\alpha x - ux = y.$$

Then

$$x(s^{k+1}) = \alpha x(s^k) - y(s^k)$$

for every $k \in \mathbb{Z}$. Hence, by complete induction,

$$x(s^k) = \alpha^k x(1) - \sum_{j=0}^{k-1} \alpha^j y(s^{k-1-j})$$

for every $k \in \mathbb{N}$.

a) Case 1 $E = \ell^\infty(T)$

Define

$$y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} \alpha^k & \text{if } t = s^k \text{ for a } k \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$x(s^k) = \alpha^k x(1) - \sum_{j=0}^{k-1} \alpha^j \alpha^{k-1-j} = \alpha^k x(1) - k\alpha^{k-1}$$

for every $k \in \mathbb{N}$ which is impossible since x is bounded. Hence $\alpha 1 - u$ is not surjective and so $\alpha \in \sigma(u)$.

Case 2 $E = c(T)$ and $\alpha = 1$

Put $y := e_T$. Then

$$x(s^k) = x(1) - k$$

for every $k \in \mathbb{N}$, which is impossible since x is bounded. Hence $\alpha 1 - u$ is not surjective and therefore $\alpha \in \sigma(u)$.

Case 3 The other cases

Put $y := e_1$. Then

$$x(s^{-\kappa}) = \alpha^{-k} x(1)$$

for every $k \in \mathbb{N}$ by complete induction, which is impossible since $x \in E$. Hence $\alpha 1 - u$ is not surjective and so $\alpha \in \sigma(u)$.

b) Put $y := e_1$ and assume that $\alpha^n = 1$. Then

$$x(1) = x(s^n) = \alpha^n x(1) - \sum_{j=0}^{n-1} \alpha^j e_1(s^{n-1-j}) = x(1) - \alpha^{n-1},$$

which is a contradiction. Hence $\alpha 1 - u$ is not surjective and therefore $\alpha \in \sigma(u)$. Thus

$$\{\alpha \in \mathbb{K} \mid \alpha^n = 1\} \subset \sigma(u).$$

Now let $\alpha \in \mathbb{K}$ with $\alpha^n \neq 1$. Given $y \in E$, define

$$x : T \longrightarrow \mathbb{K}, \quad t \longmapsto \frac{1}{\alpha^n - 1} \sum_{j=0}^{n-1} \alpha^j y(s^{n-1-j}t).$$

It is easy to see that $x \in E$. Then

$$((\alpha 1 - u)x)(t) = \alpha x(t) - x(st) =$$

$$= \frac{\alpha}{\alpha^n - 1} \sum_{j=0}^{n-1} \alpha^j y(s^{n-1-j}t) - \frac{1}{\alpha^n - 1} \sum_{j=0}^{n-1} \alpha^j y(s^{n-j}t) =$$

$$= \frac{1}{\alpha^n - 1} \sum_{j=1}^{n} \alpha^j y(s^{n-j}t) - \frac{1}{\alpha^n - 1} \sum_{j=0}^{n-1} \alpha^j y(s^{n-j}t) =$$

$$= \frac{1}{\alpha^n - 1} \alpha^n y(t) - \frac{1}{\alpha^n - 1} y(s^n t) = y(t)$$

for every $t \in T$, so that

$$(\alpha 1 - u)x = y\,.$$

Hence $\alpha 1 - u$ is surjective.

Take $z \in E$ with

$$(\alpha 1 - u)z = 0\,.$$

For $t \in T$,

$$z(st) = \alpha z(t)\,,$$

so that

$$z(t) = z(s^n t) = \alpha^n z(t)$$

by complete induction. Hence

$$z(t) = 0\,,$$

$$z = 0\,,$$

and $\alpha 1 - u$ is injective. By the Principle of Inverse Operators, $\alpha 1 - u$ is invertible. Thus $\alpha \notin \sigma(u)$ and

$$\alpha(u) = \{\alpha \in \mathbb{K} \mid \alpha^n = 1\}\,. \qquad \blacksquare$$

Corollary 2.2.4.12 (4) *Let T be a group. Take $t \in T$ and let $\ell^1(T)$ be the Banach algebra defined in Theorem 2.2.2.7 a). If $t^n \neq 1$ for every $n \in \mathbb{N}$, then*

$$\sigma(e_t) = \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,.$$

If n is the least natural number such that $t^n = 1$, then

$$\sigma(e_t) = \{\alpha \in \mathbb{K} \mid \alpha^n = 1\}\,.$$

Given $x \in \ell^1(T)$, define

$$\widetilde{x} : \ell^1(T) \longrightarrow \ell^1(T), \quad y \longmapsto x * y .$$

By Proposition 2.2.1.17,

$$\sigma(e_t) = \sigma(\widetilde{e}_t) .$$

For $x \in \ell^1(T)$ and $s \in T$,

$$(\widetilde{e}_t x)(s) = (e_t * x)(s) = \sum_{r \in T} e_t(sr^{-1})x(r) = x(t^{-1}s) .$$

The assertions now follow from Example 2.2.4.11. ■

Corollary 2.2.4.13 (4) *Let T be a locally compact group, $M_b(T)$ the unital Banach algebra defined in Theorem 2.2.2.10 b), and δ_t the Dirac measure at the point $t \in T$. If $t^n \neq 1$ for every $n \in \mathbb{N}$, then*

$$\sigma(\delta_t) = \{\alpha \in \mathbb{K} \mid |\alpha| = 1\} .$$

If n is the least natural number such that $t^n = 1$, then

$$\sigma(\delta_t) = \{\alpha \in \mathbb{K} \mid \alpha^n = 1\} .$$

The assertion follows immediately from Theorem 2.2.2.10 d) and Corollary 2.2.4.12. ■

Theorem 2.2.4.14 (0) *Let E be a unital Banach algebra. Then the set U of invertible elements of E is open and the map*

$$U \longrightarrow U, \quad x \longmapsto x^{-1}$$

is a homeomorphism. In particular, U is a topological group with respect to the multiplication.

Take $x \in U$. By Corollary 2.2.4.4

$$U^E_{\frac{1}{\|x^{-1}\|}}(x) \subset U .$$

Hence U is open. Since

$$\lim_{y \to x}(1 - x^{-1}y) = 1 - 1 = 0 ,$$

it follows that

$$\lim_{y\to x} \frac{1}{1-(1-x^{-1}y)} = 1$$

(Proposition 2.2.3.2) and so

$$\lim_{y\to x} y^{-1} = \lim_{y\to x} \frac{1}{1-(1-x^{-1}y)} x^{-1} = x^{-1}$$

(Corollary 2.2.4.4). ∎

Remark. a) Let

$$T := \{\alpha \in \mathbb{C} \mid |\alpha| \leq 1\}.$$

It is easy to see that

$$T \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \alpha$$

is an interior point of the set of non-invertible elements of $\mathcal{C}(T)$. Hence the set of invertible elements of a unital Banach algebra is not necessarily dense.

b) If $x \in U$ then

$$\lim_{\substack{y\to x \\ y\in U\setminus\{x\}}} \frac{\|y^{-1}-x^{-1}-x^{-1}(y-x)x^{-1}\|}{\|y-x\|} = 0.$$

Indeed, let $x, y \in U$ such that

$$\|x-y\| < \frac{1}{\|x^{-1}\|}.$$

Then

$$1-x^{-1}y = x^{-1}(x-y), \quad \|1-x^{-1}y\| \leq \|x^{-1}\|\,\|x-y\|$$

so, by Corollary 2.2.4.4,

$$\|y^{-1}-x^{-1}-x^{-1}(y-x)x^{-1}\| = \left\|\sum_{n=2}^{\infty}(1-x^{-1}y)^n x^{-1}\right\| \leq$$

$$\leq \sum_{n=2}^{\infty} \|x^{-1}\|^{n+1}\|x-y\|^n = \frac{\|x^{-1}\|^3\|x-y\|^2}{1-\|x^{-1}\|\,\|x-y\|}$$

and the assertion follows.

Corollary 2.2.4.15 (0) *Let E, F be Banach spaces. The set $\mathcal{U}$ of isomorphisms $E \to F$ is an open set of $\mathcal{L}(E,F)$.*

Take $v \in \mathcal{U}$. The map

$$w : \mathcal{L}(E,F) \longrightarrow \mathcal{L}(E), \quad u \longmapsto v^{-1} \circ u$$

is linear and continuous. If $\mathcal{V}$ is the set of invertible elements of $\mathcal{L}(E)$, then

$$\mathcal{U} = \overset{-1}{w}(\mathcal{V}).$$

Since $\mathcal{V}$ is open (Theorem 2.2.4.14), $\mathcal{U}$ also is open. ■

Corollary 2.2.4.16 (0) *Let E be a unital Banach algebra. The closure of a proper (left, right) ideal of E is a proper (left, right) ideal of E. The maximal proper (left, right) ideals of E are closed.*

Let F be a (left, right) proper ideal of E. By Corollary 2.2.1.8, $\overline{F}$ is a (left, right) ideal of E . Let U be the set of invertible elements of E. Then

$$F \cap U = \emptyset$$

(Proposition 2.1.2.9). Since U is open (Theorem 2.2.4.14),

$$\overline{F} \cap U = \emptyset,$$

and hence $\overline{F}$ is a proper (left, right) ideal of E.

Let G be a maximal proper (left, right) ideal of E. By the above considerations, $\overline{G}$ is a proper (left, right) ideal of E. Hence

$$G = \overline{G},$$

i.e., G is closed. ■

Corollary 2.2.4.17 *The regular maximal proper (left, right) ideals and the radical of a Banach algebra are closed.*

Let E be a Banach algebra and F a regular maximal proper (left, right) ideal. Let $\tilde{E}$ be a unital Banach algebra associated to E (Definition 2.2.1.4) and let $G \neq E$ be a maximal proper (left, right) ideal of $\tilde{E}$ containing F. Since F is maximal,

$$F = E \cap G.$$

By Corollary 2.2.4.16, G is closed, so that F is also closed.

It follows that the radical of E is closed. ■

Proposition 2.2.4.18 *Let E be a unital Banach algebra, U the topological group of invertible elements of E (Theorem 2.2.4.14), and V the connected component of U containing 1. Then V is a clopen normal subgroup of U and*

$$V = \left\{ \prod_{\iota \in I} e^{x_\iota} \;\middle|\; (x_\iota)_{\iota \in I} \text{ finite family in } E \right\} .$$

If E is commutative, then

$$V = \{e^x \mid x \in E\}$$

and no element of $(U/V)\backslash\{1\}$ has finite order if $\mathbb{K} = \mathbb{C}$.

Take $x \in V$ and $y \in U$. Then $x^{-1}V$, $y^{-1}Vy$ are connected sets of U (Proposition 2.2.1.6), containing 1. Hence

$$x^{-1}V \subset V, \quad y^{-1}Vy \subset V .$$

Thus V is a normal subgroup of U.

Since U is obviously locally connected, V is open. Being a subgroup of U, it is a closed set of U.

By Proposition 2.2.3.7 b),

$$\{e^x \mid x \in E\} \subset U .$$

Since E is connected and the map

$$E \longrightarrow E, \quad x \longmapsto e^x$$

is continuous (Proposition 2.2.3.2), $\{e^x \mid x \in E\}$ is connected. Hence

$$\{e^x \mid x \in E\} \subset V .$$

Thus

$$W := \left\{ \prod_{\iota \in I} e^{x_\iota} \mid (x_\iota)_{\iota \in I} \text{ finite family in } E \right\} \subset V .$$

By Proposition 2.2.3.9,

$$U_1^E(1) \subset \{e^x \mid x \in E\} \subset W .$$

Since W is obviously a subgroup of V, it is a clopen set of V. Hence

$$W = V .$$

Assume that E is commutative. By Proposition 2.2.3.7 a),

$$W = \{e^x \mid x \in E\}.$$

Let $\mathbb{K} = \mathbb{C}$ and take $x \in U$ with $x^n \in V$ for some $n \in \mathbb{N}$. By the above result, there is a $y \in E$ such that

$$x^n = e^y.$$

Given $\alpha \in \mathbb{C}$, put

$$z := xe^{-\frac{1}{n}y}, \quad z_\alpha := \alpha z + (1-\alpha)1.$$

Then

$$z^n = x^n e^{-y} = 1,$$

so that $\sigma(z)$ is finite (Theorem 2.1.3.4 a)) and $\{\alpha \in \mathbb{C} \mid z_\alpha \in U\}$ is connected. Since

$$\mathbb{C} \longrightarrow E, \quad \alpha \longmapsto z_\alpha$$

is continuous and

$$z_0 = 1, \quad z_1 = z,$$

it follows that $z \in V$ and

$$x = ze^{\frac{1}{n}y} \in V,$$

which proves the last assertion. ■

Proposition 2.2.4.19 (0) *Let E be a Banach algebra and $x' : E \to \mathbb{K}$ an algebra homomorphism. Then x' is continuous with norm at most 1. If E is quasiunital, then $x' = 0$ or $\|x'\| = 1$.*

First assume that E is a unital Banach algebra. Given $x \in E$,

$$x'(x) \in \{x'(x)\} = \sigma_{\mathbb{K}}(x'(x)) \subset \sigma_E(x)$$

(Proposition 2.1.3.2, Corollary 2.1.3.13), so that

$$|x'(x)| \leq r_E(x) \leq \|x\|$$

(Proposition 2.2.4.1). Hence x' is continuous and $\|x'\| \leq 1$.

Now suppose that E is not unital. Let $\mathbb{K} \times E$ be a unital Banach algebra associated to E (Definition 2.2.1.4). The map

$$y' : \mathbb{K} \times E \longrightarrow \mathbb{K}, \quad (\alpha, x) \longmapsto \alpha + x'(x)$$

is a unital algebra homomorphism (Proposition 2.1.1.7). By the above proof, y' is continuous with norm at most 1. Hence x' is continuous with norm at most one.

Now suppose that E is quasiunital and $x' \neq 0$. Take $x \in E$ with

$$x'(x) \neq 0$$

and let $\mathfrak{F}$ be an approximate unit of E. Since x' is continuous,

$$x'(x) = \lim_{y,\mathfrak{F}} x'(xy) = \lim_{y,\mathfrak{F}} x'(x)x'(y) = x'(x) \lim_{y,\mathfrak{F}} x'(y),$$

$$\lim_{y,\mathfrak{F}} x'(y) = 1.$$

$$\|x'\| \geq \lim_{y,\mathfrak{F}} |x'(y)| = 1, \quad \|x'\| = 1. \qquad \blacksquare$$

Definition 2.2.4.20 (0) *The element x of the normed algebra E is called* ***quasinilpotent if***

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = 0.$$

Every nilpotent element is quasinilpotent.

Proposition 2.2.4.21 (0) *Let E be a Banach algebra, x a quasinilpotent element of E and take $y \in \{x\}^c$.*

a) *xy is quasinilpotent.*

b) *If E is unital and y invertible then $x + y$ is invertible.*

c) $\sigma(x + y) = \sigma(y)$.

d) $\sigma(x) = \{0\}$.

a) For $n \in \mathbb{N}$,

$$\|(xy)^n\|^{\frac{1}{n}} = \|x^n y^n\|^{\frac{1}{n}} \leq \|x^n\|^{\frac{1}{n}} \|y^n\|^{\frac{1}{n}} \leq \|x^n\|^{\frac{1}{n}} \|y\|$$

(Proposition 2.2.3.1), so that

$$\lim_{n\to\infty} \|(xy)^n\|^{\frac{1}{n}} = 0\,.$$

b) Since $x \in \{y\}^c$, $x \in \{y^{-1}\}^c$ (Proposition 2.1.2.8) and $y^{-1} \in \{x\}^c$. By a), xy^{-1} is quasinilpotent, so that $1 + xy^{-1}$ is invertible (Proposition 2.2.4.1). Since

$$x + y = (1 + xy^{-1})y\,,$$

$x + y$ is invertible (Corollary 2.1.2.6).

c) We may assume E to be unital. Take $\alpha \in \mathbb{K}$. By b), $\alpha 1 - y$ is invertible iff $(\alpha 1 - y) - x$ is invertible. Thus

$$\sigma(x + y) = \sigma(y)\,.$$

d) By c),

$$\sigma(x) = \sigma(x + 0) = \sigma(0) = \{0\}$$

(Proposition 2.1.3.2). ■

Example 2.2.4.22 (Volterra, 1896) *Let*

$$T := \{(s,t) \in [0,1]^2 \mid t \leq s\}$$

and take $k \in \mathcal{C}(T)$. *Given* $x \in \mathcal{C}([0,1])$, *set*

$$kx : [0,1] \longrightarrow \mathbb{K}\,, \quad s \longmapsto \int_0^s k(s,t)x(t)dt\,.$$

Then

$$\widetilde{k} : \mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1])\,, \quad x \longmapsto kx$$

is a quasinilpotent element of $\mathcal{L}(\mathcal{C}([0,1]))$ *and given* $\alpha \in \mathbb{K}\backslash\{0\}$ *and* $y \in \mathcal{C}([0,1])$, *there is exactly one* $x \in \mathcal{C}([0,1])$, *such that*

$$\alpha x - kx = y\,.$$

This equation is called the ***Volterra Integral Equation.***

It is obvious that kx is continuous for every $x \in \mathcal{C}([0,1])$ and that $\widetilde{k}$ is linear. Set

$$\alpha := \|k\|_\infty\,.$$

Then

$$|kx(s)| \leq \int_0^s |k(s,t)|\,|x(t)|dt \leq \alpha\|x\|_\infty$$

for every $x \in C([0,1])$. Therefore, $\widetilde{k}$ is continuous.

We define the sequence $(k_n)_{n\in\mathbb{N}}$ in $C(T)$ inductively by putting $k_1 := k$ and

$$k_{n+1} : T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto \int_0^s k(s,r)k_n(r,t)dr$$

for every $n \in \mathbb{N}$. By complete induction,

$$\widetilde{k}^n x(s) = \int_0^s k_n(s,t)x(t)dt$$

for every $n \in \mathbb{N}$, $x \in C([0,1])$, and $s \in [0,1]$. Furthermore,

$$|k_n(s,t)| \leq \frac{\alpha^n s^{n-1}}{(n-1)!}$$

for every $n \in \mathbb{N}$ and $(s,t) \in T$. It follows that

$$\|\widetilde{k}^n\| \leq \frac{\alpha^n}{(n-1)!}$$

for every $n \in \mathbb{N}$ and that

$$\lim_{n\to\infty} \|\widetilde{k}^n\|^{\frac{1}{n}} = 0.$$

Hence $\widetilde{k}$ is quasinilpotent and $\sigma(\widetilde{k}) = \{0\}$ (Proposition 2.2.4.21 d)). The final assertion now follows. ■

Example 2.2.4.23 *Given $x, y \in C([0,1])$, set*

$$x * y : [0,1] \longrightarrow \mathbb{K}, \quad s \longmapsto \int_0^s x(s-t)y(t)dt.$$

The Banach space $C([0,1])$ endowed with the bilinear map

$$C([0,1]) \times C([0,1]) \longrightarrow C([0,1]), \quad (x,y) \longmapsto x * y$$

is a commutative Banach algebra all of whose elements are quasinilpotent.

Take $x, y, z \in C([0,1])$. Then

$$(x * (y * z))(s) = \int_0^s x(s-t)(y * z)(t)dt =$$

$$= \int_0^s x(s-t) \left(\int_0^t y(t-r)z(r)dr \right) dt =$$

$$= \int_0^s \left(\int_r^s x(s-t)y(t-r)dt \right) z(r)dr =$$

$$= \int_0^s \left(\int_0^{s-r} x(s-r-u)y(u)du \right) z(r)dr =$$

$$= \int_0^s (x * y)(s-r)z(r)dr = ((x * y) * z)(s)$$

for every $s \in [0,1]$, so

$$x * (y * z) = (x * y) * z .$$

Hence $C([0,1])$ is an algebra with respect to the above bilinear map. It is easy to see that it is a commutative Banach algebra. Now, by complete induction,

$$|x^n(s)| \leq \|x\|^n \frac{s^{n-1}}{(n-1)!}$$

for every $s \in [0,1]$ and $n \in \mathbb{N}$, so that

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \lim_{n\to\infty} \frac{\|x\|}{(n!)^{\frac{1}{n}}} = 0 .$$

Hence x is quasinilpotent. ■

Definition 2.2.4.24 (0) *Let E be a normed algebra. An element x of E is called a* ***topological zero-divisor*** *if there is a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that*

$$\limsup_{n\to\infty} \|x_n\| > 0$$

and either $(xx_n)_{n\in\mathbb{N}}$ or $(x_nx)_{n\in\mathbb{N}}$ converges to 0.

Every zero-divisor of 0 is a topological zero-divisor.

Proposition 2.2.4.25 (0) *Let x be a topological zero-divisor in the normed unital algebra E. Then x is not invertible.*

There is a sequence $(x_n)_{n\in\mathbb{N}}$ in E, with

$$\limsup_{n\to\infty} \|x_n\| > 0$$

and either $(xx_n)_{n\in\mathbb{N}}$ or $(x_nx)_{n\in\mathbb{N}}$ converges to 0. Assume x to be invertible. Since

$$x_n = (x_nx)x^{-1} = x^{-1}(xx_n)$$

for every $n \in \mathbb{N}$, $(x_n)_{n\in\mathbb{N}}$ must converge to 0 (Proposition 2.2.1.6), which is a contradiction. ■

Proposition 2.2.4.26 (0) *Let E be a unital Banach algebra and $(x_n)_{n\in\mathbb{N}}$ a convergent sequence of invertible elements of E. If $x := \lim_{n\to\infty} x_n$ is not invertible, then there is a sequence $(y_n)_{n\in\mathbb{N}}$ in E such that*

$$\lim_{n\to\infty} xy_n = \lim_{n\to\infty} y_nx = 0$$

and

$$\|y_n\| = 1$$

for every $n \in \mathbb{N}$. In particular, x it is a topological zero-divisor.

Now

$$\sup_{n\in\mathbb{N}} \|x_n^{-1}\| = \infty$$

(Corollary 2.2.4.10) and we may assume that

$$\lim_{n\to\infty} \|x_n^{-1}\| = \infty.$$

Given $n \in \mathbb{N}$, set

$$y_n := \frac{1}{\|x_n^{-1}\|} x_n^{-1}.$$

Then

$$\|y_n\| = 1$$

for every $n \in \mathbb{N}$,

$$\lim_{n\to\infty} y_nx_n = 0,$$

and

$$\limsup_{n\to\infty} \|y_n(x - x_n)\| \leq \lim_{n\to\infty} \|x - x_n\| = 0\,.$$

Thus

$$\lim_{n\to\infty} y_n(x - x_n) = 0\,,$$

$$\lim_{n\to\infty} y_n x = \lim_{n\to\infty} y_n(x - x_n) + \lim_{n\to\infty} y_n x_n = 0\,.$$

The relation

$$\lim x y_n = 0$$

is provided similarly. Hence, x is a topological zero–divisor. ■

Corollary 2.2.4.27 (0) (Šilov) *Let F be a closed subalgebra of the Banach algebra E. Take $x \in F$. Then the boundary of $\sigma_F(x)$ is contained in the boundary of $\sigma_E(x)$. In particular,*

$$r_E(x) = r_F(x)\,,$$

$$\sigma_E(x) \subset \mathbb{R}_+ \Longrightarrow \sigma_F(x) \subset \mathbb{R}_+\,.$$

If $\sigma_F(x)$ has not interior points or if $\mathbb{K} = \mathbb{C}$ and $\sigma_E(x) \subset \mathbb{R}$, then

$$\sigma_E(x) = \sigma_F(x)\,.$$

We may assume that E is unital (Definition 2.2.1.4) and that F is a unital subalgebra of E.

Let α be a boundary point of $\sigma_F(x)$. Let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{K}\backslash\sigma_F(x)$ converging to α. Then $(\alpha_n 1 - x)_{n\in\mathbb{N}}$ is a sequence of invertible elements of F converging to $\alpha 1 - x$. Since $\alpha 1 - x$ is not invertible in F (Corollary 2.2.4.5), it is a topological zero–divisor in F (Proposition 2.2.4.26). A fortiori, it is a topological zero–divisor in E. Hence $\alpha 1 - x$ is not invertible in E (Proposition 2.2.4.25). Thus α belongs to $\sigma_E(x)$. Since

$$\sigma_E(x) \subset \sigma_F(x)$$

(Corollary 2.1.3.14 a)), α is a boundary point of $\sigma_E(x)$.

Assume that $\sigma_F(x)$ has no interior points. Then

$$\sigma_E(x) \subset \sigma_F(x) = \text{boundary of} \sigma_F(x) \subset \text{ boundary of} \sigma_E(x) \subset \sigma_E(x)$$

and hence

$$\sigma_E(x) = \sigma_F(x) .$$

Now suppose that $\mathbb{K} = \mathbb{C}$ and $\sigma_E(x) \subset \mathbb{R}$. Since the boundary of $\sigma_F(x)$ is contained in the boundary of $\sigma_E(x)$, $\sigma_F(x) \subset \mathbb{R}$. By the above result,

$$\sigma_E(x) = \sigma_F(x) . \qquad \blacksquare$$

Proposition 2.2.4.28 (2) *Let (Ω, Λ) be a unital Banach category. Take $E, F \in \Omega$, and $x, y \in \Lambda(E, F)$. If x is invertible and*

$$\|x - y\| < \frac{1}{\|x^{-1}\|} ,$$

then y is invertible,

$$\|1_E - x^{-1}y\| < 1 ,$$

and

$$y^{-1} = \frac{1}{1_E - (1_E - x^{-1}y)} x^{-1} = \sum_{n=0}^{\infty} (1_E - x^{-1}y)^n x^{-1} .$$

Now

$$\|1_E - x^{-1}y\| = \|x^{-1}(x - y)\| \leq \|x^{-1}\| \, \|x - y\| \leq 1 .$$

Since

$$x^{-1}y = 1_E - (1_E - x^{-1}y) ,$$

it follows that

$$(x^{-1}y)^{-1} = \frac{1}{1_E - (1_E - x^{-1}y)}$$

(Proposition 2.2.4.1). From

$$y = x(x^{-1}y)$$

we deduce further that

$$y^{-1} = (x^{-1}y)^{-1}x^{-1} = \frac{1}{1_E - (1_E - x^{-1}y)} x^{-1} = \sum_{n=0}^{\infty} (1_E - x^{-1}y)^n x^{-1}$$

(Corollary 1.5.1.7, Proposition 1.5.1.3 b)). $\blacksquare$

Theorem 2.2.4.29 (2) *Let (Ω, Λ) be a unital Banach category. Take $E, F \in \Omega$. The set of invertible (left invertible, right invertible) morphisms of $\Lambda(E,F)$ is open. If U denotes the set of invertible elements of $\Lambda(E,F)$ then*

$$U \longrightarrow \Lambda(F,E), \quad x \longmapsto x^{-1}$$

is continuous.

Let x_0 be a left invertible morphism of $\Lambda(E,F)$. Take $y \in \Lambda(F,E)$ with

$$yx_0 = 1_E .$$

Take $x \in \Lambda(E,F)$ with

$$\|x - x_0\| < \frac{1}{\|y\|} .$$

Then

$$\|yx - 1_E\| = \|yx - yx_0\| = \|y(x - x_0)\| \leq \|y\| \, \|x - x_0\| < 1 .$$

By Proposition 2.2.4.1, yx is an invertible element of $\Lambda(E)$. Thus

$$(yx)^{-1}yx = 1_E ,$$

i.e. x is left invertible. Hence the set of left invertible morphisms of $\Lambda(E,F)$ is open. Similarly, the set of right invertible morphisms of $\Lambda(E,F)$ is also open. Hence the set of invertible morphisms of $\Lambda(E,F)$ is open.

Now take $x \in U$. From

$$\lim_{y \to x} (1 - x^{-1}y) = 1 - 1 = 0$$

(Proposition 1.5.1.3 a)), we deduce that

$$\lim_{y \to x} \frac{1}{1 - (1 - x^{-1}y)} = 1$$

(Proposition 2.2.3.2). Thus

$$\lim_{U \ni y \to x} y^{-1} = \lim_{U \ni y \to x} \frac{1}{1 - (1 - x^{-1}y)} x^{-1} = x^{-1}$$

(Proposition 1.5.1.3 a), Proposition 2.2.4.28). Hence the map

$$U \longrightarrow \Lambda(F,E), \quad x \longmapsto x^{-1}$$

is continuous. ■

Proposition 2.2.4.30 (0) *Let E be a unital Banach algebra. Take $x \in E$, and $\alpha \in \sigma(x)$. Then there is an $x' \in E'$ such that*

$$\|x'\| = x'(1) = 1\,, \quad x'(x) = \alpha\,.$$

Let F be the vector subspace of E generated by $\{x, 1\}$ and consider

$$y' : F \longrightarrow \mathbb{K}\,, \quad \beta x + \gamma 1 \longmapsto \alpha\beta + \gamma\,.$$

y' is well–defined even if F is one–dimensional. Indeed, in this case, there is a $\delta \in \mathbb{K}$ such that $x = \delta 1$. Since $\alpha \in \sigma(x)$, $\delta = \alpha$. Given $\beta, \gamma \in \mathbb{K}$,

$$\beta x + \gamma 1 = (\alpha\beta + \gamma)1\,,$$

so that the expression $\alpha\beta + \gamma$ is uniqely determined by $\beta x + \gamma 1$.

Take $\beta, \gamma \in \mathbb{K}$. Then

$$\alpha\beta + \gamma \in \beta\sigma(x) + \gamma = \sigma(\beta x + \gamma 1)\,,$$

$$|y'(\beta x + \gamma 1)| = |\alpha\beta + \gamma| \leq r(\beta x + \gamma 1) \leq \|\beta x + \gamma 1\|$$

(Proposition 2.2.4.1). Hence

$$\|y'\| \leq 1\,.$$

By the Hahn-Banach Theorem, there is a continuous linear extension x' of y' to E with

$$\|x'\| = \|y'\|\,.$$

We have

$$x'(x) = y'(x) = \alpha$$

and

$$\|x'\| \leq 1 = y'(1) = x'(1) \leq \|x'\|\,\|1\| = \|x'\|\,,$$

i.e.

$$\|x'\| = x'(1) = 1\,. \qquad \blacksquare$$

Corollary 2.2.4.31 (0) *Let E be a unital Banach algebra. Take $x \in E$, and put*

$$K := \{x'(x) \mid x' \in E',\ \|x'\| = x'(1) = 1\}.$$

If

$$\|x + \alpha 1\| = r(x + \alpha 1)$$

for every $\alpha \in \mathbb{K}$, then K is the convex hull of $\sigma(x)$.

By Proposition 2.2.4.30, $\sigma(x) \subset K$. Put

$$A := \{(\alpha, r) \in \mathbb{K} \times]0, \infty[\mid \sigma(x) \subset U_r^{\mathbb{K}}(\alpha)\}.$$

Take $(\alpha, r) \in A$ and $x' \in E'$ such that

$$\|x'\| = x'(1) = 1.$$

Then

$$\sigma(x - \alpha 1) = \sigma(x) - \alpha \subset U_r^{\mathbb{K}}(0),$$

so that

$$\|x - \alpha 1\| = r(x - \alpha 1) < r.$$

Hence

$$|x'(x) - \alpha| = |x'(x - \alpha 1)| \leq \|x'\|\,\|x - \alpha 1\| < r,$$

$$x'(x) \in U_r^{\mathbb{K}}(\alpha).$$

Since x' and (α, r) are arbitrary, it follows that

$$K \subset \bigcap_{(\alpha,r)\in A} U_r^{\mathbb{K}}(\alpha).$$

But $\bigcap_{(\alpha,r)\in A} U_r^{\mathbb{K}}(\alpha)$ is the convex hull of $\sigma(x)$. Since K is convex it must, in fact, be the convex hull of $\sigma(x)$. ■

Example 2.2.4.32 *Take $p \in \{0\} \cup [1, \infty]$ and let u_r, u_ℓ be the right and the left shift operators of ℓ^p, respectively. Then*

$$\sigma(u_r) = \sigma(u_\ell) = \mathbb{K}^{\#}.$$

This relation still holds for c in place of ℓ^p.

Take $\alpha \in \mathbb{K}$, with $|\alpha| < 1$ and put

$$y := (\alpha^n)_{n\in\mathbb{N}} \in \ell^p .$$

Then

$$(\alpha 1 - u_\ell)y = \alpha y - u_\ell y = 0 ,$$

so that

$$\alpha 1 - u_\ell$$

is not injective. Suppose that

$$y \in \mathrm{Im}(\alpha 1 - u_r) .$$

Then there is an $x \in \ell^p$ such that

$$(\alpha 1 - u_r)x = y .$$

By Example 1.2.2.9 d),

$$x_n = \frac{1}{\alpha^{n+1}} \sum_{k=1}^{n} \alpha^{2k}$$

for every $n \in \mathbb{N}$. Hence

$$\lim_{n\to\infty} |x_n| = \infty ,$$

which is a contradiction. Thus

$$y \notin \mathrm{Im}(\alpha 1 - u_r)$$

and $\alpha 1 - u_r$ is not surjectiv. It follows that

$$\{\alpha \in \mathbb{K} \mid |\alpha| < 1\} \subset \sigma(u_r) \cap \sigma(u_\ell) .$$

Since

$$\|u_r\| = \|u_\ell\| = 1 ,$$

it follows, by Proposition 2.2.4.1 and Corollary 2.2.4.5, that

$$\sigma(u_r) = \sigma(u_\ell) = \mathbb{K}^\# . \qquad \blacksquare$$

Definition 2.2.4.33 *Let E be a unital Banach algebra and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in E . We put*

$$\prod_{n\in\emptyset} x_n := 1 .$$

If $A \in \mathfrak{P}_f(\mathbb{N})$ then

$$\prod_{n\in A} x_n$$

is defined inductively using the order induced on A by the natural order of $\mathbb{N}$.

If the limit

$$\lim_{A,\mathfrak{F}_{\mathbb{N}}} \prod_{n\in A} x_n$$

exists, then the sequence $(x_n)_{n\in\mathbb{N}}$ is called ***multipliable*** *and the above limit is called* ***the product of the sequence*** *$(x_n)_{n\in\mathbb{N}}$ and it is denoted by*

$$\prod_{n\in\mathbb{N}} x_n \qquad (\text{in } E) .$$

Take $x \in E$. The relation

$$x = \prod_{n\in\mathbb{N}} x_n$$

is equivalent to the following: For every $\varepsilon > 0$ there is a finite subset A of $\mathbb{N}$ such that for every finite subset B of $\mathbb{N}$ containing A,

$$\| \prod_{x\in B} x_n - x \| < \varepsilon .$$

Proposition 2.2.4.34 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in the unital Banach algebra E. Let (C) denote the following statement: For every $\varepsilon > 0$, there is a finite subset A of $\mathbb{N}$ such that*

$$\| \prod_{n\in B} x_n - 1 \| < \varepsilon$$

whenever B is a finite subset of $\mathbb{N}\backslash A$.

a) *If $(x_n)_{n\in\mathbb{N}}$ is multipliable and $\prod_{n\in\mathbb{N}} x_n$ is invertible, then (C) is fulfilled.*

b) *If (C) is fulfilled, then $(x_n)_{n\in\mathbb{N}}$ is multipliable.*

c) *If* (C) *is fulfilled and each* x_n $(n \in \mathbb{N})$ *is invertible, then* $\prod_{n\in\mathbb{N}} x_n$ *is invertible.*

Given $A \in \mathfrak{P}_f(\mathbb{N})$, put

$$x_A := \prod_{n\in A} x_n .$$

a) Put

$$x := \prod_{n\in\mathbb{N}} x_n .$$

Take $\varepsilon > 0$. By the definition of the product and by Theorem 2.2.4.14, there is a $p \in \mathbb{N}$ such that for every $A \in \mathfrak{P}_f(\mathbb{N})$ with $\mathbb{N}_p \subset A$, x_A is invertible and

$$\|x_A^{-1} - x^{-1}\| < 1 ,$$

$$\|x_A - x\| < \frac{\varepsilon}{2(1 + \|x^{-1}\|)} .$$

Thus

$$\|x_A^{-1}\| \leq \|x_A^{-1} - x^{-1}\| + \|x^{-1}\| < 1 + \|x^{-1}\| ,$$

$$\|x_A - x_{\mathbb{N}_p}\| \leq \|x_A - x\| + \|x - x_{\mathbb{N}_p}\| < \frac{\varepsilon}{1 + \|x^{-1}\|}$$

for every $A \in \mathfrak{P}_f(\mathbb{N})$ with $\mathbb{N}_p \subset A$.

Take $B \in \mathfrak{P}_f(\mathbb{N}\backslash\mathbb{N}_p)$. Put

$$A := \mathbb{N}_p \cup B .$$

Then

$$\|x_B - 1\| = \|x_{\mathbb{N}_p}^{-1}(x_{\mathbb{N}_p} x_B - x_{\mathbb{N}_p})\| \leq$$

$$\leq \|x_{\mathbb{N}_p}^{-1}\| \, \|x_A - x_{\mathbb{N}_p}\| < (1 + \|x^{-1}\|)\frac{\varepsilon}{1 + \|x^{-1}\|} = \varepsilon .$$

Hence (C) is fulfilled.

b) By (C), there is a $p \in \mathbb{N}$ such that

$$\|x_A - 1\| < 1$$

for every $A \in \mathfrak{P}_f(\mathbb{N}\backslash\mathbb{N}_p)$. Hence, if $A \in \mathfrak{P}_f(\mathbb{N})$ and $\mathbb{N}_p \subset A$, then

$$\|x_A - x_{\mathbb{N}_p}\| = \|x_{\mathbb{N}_p}(x_{A\setminus\mathbb{N}_p} - 1)\| \leq \|x_{\mathbb{N}_p}\|\,\|x_{A\setminus\mathbb{N}_p} - 1\| \leq \|x_{\mathbb{N}_p}\|\,,$$

$$\|x_A\| \leq \|x_A - x_{\mathbb{N}_p}\| + \|x_{\mathbb{N}_p}\| \leq 2\|x_{\mathbb{N}_p}\|\,.$$

Take $\varepsilon > 0$. By (C), there is a $q \in \mathbb{N}$ such that

$$\|x_A - 1\| < \frac{\varepsilon}{2(1 + \|x_{\mathbb{N}_p}\|)}$$

for every $A \in \mathfrak{P}_f\,(\mathbb{N}\setminus\mathbb{N}_q)$. We may assume that $q \geq p$. Then

$$\|x_A - x_B\| \leq \|x_A - 1\| + \|1 - x_B\| < \frac{\varepsilon}{1 + \|x_{\mathbb{N}_p}\|}$$

for all $A, B \in \mathfrak{P}_f(\mathbb{N}\setminus\mathbb{N}_q)$ and

$$\|x_A - x_B\| = \|x_{\mathbb{N}_q}(x_{A\setminus\mathbb{N}_q} - x_{B\setminus\mathbb{N}_q})\| \leq \|x_{\mathbb{N}_q}\|\,\|x_{A\setminus\mathbb{N}_q} - x_{B\setminus\mathbb{N}_q}\| \leq$$

$$\leq \|x_{\mathbb{N}_p}\|\,\frac{\varepsilon}{1 + \|x_{\mathbb{N}_p}\|} < \varepsilon$$

for all $A, B \in \mathfrak{P}_f(\mathbb{N})$ with $\mathbb{N}_q \subset A \cap B$. Hence the image of $\mathfrak{F}_{\mathbb{N}}$ under the map

$$\mathfrak{P}_f(\mathbb{N}) \longrightarrow E\,, \quad A \longmapsto x_A$$

is a Cauchy filter. Thus the family $(x_n)_{n\in\mathbb{N}}$ is multipliable.

c) By (C), there is a $p \in \mathbb{N}$ such that

$$\|x_A - 1\| < \frac{1}{2}$$

for every $A \in \mathfrak{P}_f(\mathbb{N}\setminus\mathbb{N}_p)$. By b), the sequence $(x_{p+n})_{n\in\mathbb{N}}$ is multipliable and so, by the above inequality,

$$\|\prod_{n\in\mathbb{N}} x_{p+n} - 1\| \leq \frac{1}{2} < 1\,.$$

By Corollary 2.2.4.4, $\prod\limits_{n\in\mathbb{N}} x_{p+n}$ is invertible. From

$$\prod_{n\in\mathbb{N}} x_n = x_{\mathbb{N}_p}\Big(\prod_{n\in\mathbb{N}} x_{p+n}\Big)$$

we deduce that $\prod\limits_{n\in\mathbb{N}} x_n$ is also invertible. ■

Corollary 2.2.4.35 *Let $(x_n)_{n\in\mathbb{N}}$ be a multipliable sequence in a unital Banach algebra the product of which is invertible.*

a) $\lim_{n\to\infty} x_n = 1$.

b) *Every subsequence of* $(x_n)_{n\in\mathbb{N}}$ *is multipliable.*

a) follows from Proposition 2.2.4.34 a).
b) follows from Proposition 2.2.4.34 a), b). ■

Corollary 2.2.4.36 *Let* E *be a unital Banach algebra and* $(x_n)_{n\in\mathbb{N}}$ *an absolutely summable sequence in* E *. Then* $(1+x_n)_{n\in\mathbb{N}}$ *is a multipliable sequence in* E*. If in addition* $1+x_n$ *is invertible for every* $n\in\mathbb{N}$*, then* $\prod_{n\in\mathbb{N}}(1+x_n)$ *is also invertible.*

Given $A\in\mathfrak{P}_f(\mathbb{N})$, put

$$x_A := \prod_{n\in A} x_n .$$

Take $\varepsilon > 0$. Since $(\|x_n\|)_{n\in\mathbb{N}}$ is summable in $\mathbb{R}$, the sequence

$$(1+\|x_n\|)_{n\in\mathbb{N}}$$

is multipliable in $\mathbb{R}$. By Proposition 2.2.4.34 a), there is an $A\in\mathfrak{P}_f(\mathbb{N})$ such that

$$\prod_{n\in B}(1+\|x_n\|) - 1 < \varepsilon$$

for every $B\in\mathfrak{P}_f(\mathbb{N}\backslash A)$. If $B\in\mathfrak{P}_f(\mathbb{N}\backslash A)$, then

$$\|\prod_{n\in B}(1+x_n) - 1\| = \|\sum_{\substack{C\subset B\\ C\neq\emptyset}} x_C\| \leq$$

$$\leq \sum_{\substack{C\subset B\\ C\neq\emptyset}} \prod_{n\in C}\|x_n\| = \prod_{n\in B}(1+\|x_n\|) - 1 < \varepsilon$$

and the claim now follows from Proposition 2.2.4.34 b), c). ■

2.2.5 The Theorems of Riesz and Gelfand

Theorem 2.2.5.1 (0) (F. Riesz, 1913) *Let E be a unital Banach algebra and take $x \in E$. Then the resolvent of x*

$$\mathbb{K}\backslash\sigma(x) \longrightarrow E, \quad \alpha \longmapsto (\alpha 1 - x)^{-1}$$

is analytic (Corollary 2.2.4.5) and

$$\lim_{\alpha\to\infty} (\alpha 1 - x)^{-1} = 0 .$$

Take $\alpha_0 \in \mathbb{K}\backslash\sigma(x)$ and set

$$r := \frac{1}{\|(\alpha_0 1 - x)^{-1}\|} .$$

Take $\alpha \in U_r^{\mathbb{K}}(\alpha_0)$ and set

$$X := \alpha_0 1 - x, \quad Y := \alpha 1 - x .$$

Then

$$\|X - Y\| = |\alpha - \alpha_0| < \frac{1}{\|X^{-1}\|},$$

$$1 - X^{-1}Y = X^{-1}(X - Y) = (\alpha_0 - \alpha)X^{-1},$$

$$\|1 - X^{-1}Y\| < 1 .$$

By Corollary 2.2.4.4,

$$(\alpha 1 - x)^{-1} = Y^{-1} = \sum_{n=0}^{\infty} (1 - X^{-1}Y)^n X^{-1} = \sum_{n=0}^{\infty} (\alpha - \alpha_0)^n X^{-n-1},$$

where the radius of convergence of the power series

$$\sum_{n=0}^{\infty} t^n X^{-n-1}$$

is

$$\frac{1}{\limsup_{n\to\infty} \|X^{-n-1}\|^{\frac{1}{n}}} \geq \frac{1}{\|X^{-1}\|} = r$$

(Proposition 2.2.4.1). Hence the map

$$\mathbb{K}\backslash\sigma(x) \longrightarrow E, \quad \alpha \longmapsto (\alpha 1 - x)^{-1}$$

is analytic.

Given $\alpha \in \mathbb{K}\backslash\sigma(x)$, $\alpha \neq 0$,

$$(\alpha 1 - x)^{-1} = \frac{1}{\alpha}\left(1 - \frac{1}{\alpha}x\right)^{-1}.$$

Since

$$\lim_{\alpha\to\infty} \left\|\frac{1}{\alpha}x\right\| = \lim_{\alpha\to\infty} \frac{\|x\|}{|\alpha|} = 0,$$

we see that

$$\lim_{\alpha\to\infty} \left(1 - \frac{1}{\alpha}x\right)^{-1} = 1$$

(Theorem 2.2.4.14) and so

$$\lim_{\alpha\to\infty} (\alpha 1 - x)^{-1} = 0. \qquad \blacksquare$$

Corollary 2.2.5.2 (0) *Let E be a unital Banach algebra. Take $x \in E$ and let U be an open set of $\mathbb{K}$ containing $\sigma(x)$. Then there is an $\varepsilon > 0$ such that*

$$\sigma(y) \subset U$$

for every $y \in U_\varepsilon^E(x)$.

Since

$$\lim_{\alpha\to\infty} (\alpha 1 - x)^{-1} = 0$$

(Theorem 2.2.5.1),

$$\beta := \sup_{\alpha\in\mathbb{K}\backslash U} \|(\alpha 1 - x)^{-1}\| < \infty.$$

Set

$$\varepsilon := \frac{1}{\beta}.$$

Take $y \in U_\varepsilon(x)$ and $\alpha \in \mathbb{K}\backslash U$. Then

$$\|(\alpha 1 - x) - (\alpha 1 - y)\| = \|x - y\| < \varepsilon = \frac{1}{\beta} \leq \frac{1}{\|(\alpha 1 - x)^{-1}\|}.$$

By Corollary 2.2.4.4, $\alpha 1 - y$ is invertible, i.e. $\alpha \notin \sigma(y)$. Hence

$$\sigma(y) \subset U. \qquad \blacksquare$$

Corollary 2.2.5.3 (0) *Let E be a unital Banach algebra and $(x_n)_{n\in\mathbb{N}}$ a sequence in E converging to $x \in E$. Then*

$$\sigma(x) \cup \left(\bigcup_{n\in\mathbb{N}} \sigma(x_n) \right)$$

is compact.

Put

$$K := \sigma(x) \cup \left(\bigcup_{n\in\mathbb{N}} \sigma(x_n) \right) .$$

Let $\mathfrak{U}$ be an open covering of K. Then there is a finite subset $\mathfrak{V}$ of $\mathfrak{U}$ with

$$\sigma(x) \subset \bigcup_{V\in\mathfrak{V}} V .$$

(Corollary 2.2.4.5). By Corollary 2.2.5.2, there is an $m \in \mathbb{N}$, such that

$$\bigcup_{n=m}^{\infty} \sigma(x_n) \subset \bigcup_{V\in\mathfrak{V}} V .$$

Since $\bigcup_{m=1}^{m} \sigma(x_n)$ is compact (Corollary 2.2.4.5), there is a finite subset $\mathfrak{W}$ of $\mathfrak{U}$ with

$$\bigcup_{n=1}^{m} \sigma(x_n) \subset \bigcup_{W\in\mathfrak{W}} W .$$

Thus

$$K \subset \bigcup_{V\in\mathfrak{V}\cup\mathfrak{W}} V ,$$

and hence K is compact. ■

Theorem 2.2.5.4 (0) (Gelfand, 1941) *Let E be a Banach algebra and $x \in E$. For $\mathbb{K} = \mathbb{R}$, we assume in addition $\sigma_{\overset{\circ}{E}}((x,0)) \subset \mathbb{R}$. Then*

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\| ,$$

and

$$\sigma(x) \neq \emptyset .$$

We may assume that E is unital (Proposition 2.2.1.3) and $\mathbb{K} = \mathbb{C}$ (Proposition 2.2.1.19, Corollary 2.1.5.14). Put

$$U := \left\{\alpha \in \mathbb{C} \mid |\alpha| > \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}\right\},$$

$$V := \{\alpha \in \mathbb{C} \mid |\alpha| > r(x)\},$$

$$f : U \longrightarrow E, \quad \alpha \longmapsto \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} x^n,$$

$$g : V \longrightarrow E, \quad \alpha \longmapsto (\alpha 1 - x)^{-1}$$

(Proposition 2.2.3.1, Theorem 1.1.6.23). By Proposition 2.2.4.1, $U \subset V$ and $f = g|U$, and by Theorem 2.2.5.1, g is analytic. Take $x' \in E'$. Then $x' \circ g$ is analytic (Proposition 1.3.10.4) and

$$x' \circ g(\alpha) = x' \circ f(\alpha) = \sum_{n=0}^{\infty} \frac{x'(x^n)}{\alpha^{n+1}}$$

for every $\alpha \in U$ (Proposition 1.2.1.16). By the classical theory of analytic functions, it follows that $\left(\frac{x'(x^n)}{\alpha^{n+1}}\right)_{n\in\mathbb{N}}$ is summable for every $\alpha \in V$. Take $\alpha \in V$. Then

$$\lim_{n\to\infty} \frac{x'(x^n)}{\alpha^{n+1}} = 0,$$

so that

$$\sup_{n\in\mathbb{N}} \left| x' \left(\frac{1}{\alpha^{n+1}} x^n \right) \right| < \infty .$$

Since α is arbitrary,

$$\beta := \sup_{n\in\mathbb{N}} \left\| \frac{1}{\alpha^{n+1}} x^n \right\| < \infty$$

(Corollary 1.4.1.4). Given $n \in \mathbb{N}$,

$$\|x^n\|^{\frac{1}{n}} \leq (\beta|\alpha|)^{\frac{1}{n}} |\alpha|,$$

so that

$$\lim_{n\in\mathbb{N}} \|x^n\|^{\frac{1}{n}} \leq \alpha .$$

Since α is arbitrary, we deduce that

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq r(x)$$

and

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\|$$

(Proposition 2.2.4.1).

Assume that $\sigma(x)$ is empty. Set

$$f : \mathbb{C} \longrightarrow E, \quad \alpha \longmapsto (\alpha 1 - x)^{-1}.$$

Then f is analytic and

$$\lim_{\alpha\to\infty} f(\alpha) = 0$$

(Theorem 2.2.5.1). It follows that

$$x^{-1} = -f(0) = 0$$

(Corollary 1.3.10.7), which is a contradiction. ■

Remark. a) The above theorem does not hold in the real case without the supplementary condition $\sigma_{\overset{\circ}{E}}((x,0)) \subset \mathbb{R}$. Indeed, $\mathbb{C}$ is a real unital Banach algebra and we have that

$$i \in \mathbb{C}, \quad \sigma(i) = \emptyset, \quad r(i) = 0,$$

$$\lim_{n\to\infty} \|i^n\|^{\frac{1}{n}} = 1.$$

b) Example 2.1.4.4 shows that any nonempty compact set of $\mathbb{K}$ can be the spectrum of an element of a unital Banach algebra.

c) The above theorem was proved by Beurling in 1938, for $E := L^1(\mathbb{R})$.

d) $r(x)$ depends on the algebraic structure of E only. The above theorem shows therefore that the number

$$\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}}$$

does not depend on the particular norm of E , provided it defines a Banach algebra on E.

Corollary 2.2.5.5 (0) (Gelfand 1939, Mazur 1938) *Let E be a Banach algebra. For $\mathbb{K} = \mathbb{R}$, we assume in addition*

$$\sigma_{\overset{\circ}{E}}((x,0)) \subset \mathbb{R}$$

for every $x \in E$. If E is a division algebra, then

$$E = \mathbb{K}1.$$

Take $x \in E$ and choose $\alpha \in \sigma(x)$ (Theorem 2.2.5.4). Since $\alpha 1 - x$ is not invertible and E is a division algebra,

$$\alpha 1 - x = 0\,, \qquad x = \alpha 1\,.$$

Hence

$$E = \mathbb{K}1\,. \qquad \blacksquare$$

Remark. By the above corollary, $\mathbb{C}$ is the only complex Banach algebra which is a division algebra. In particular, there is no norm on the algebra of rational functions with complex coefficients making it a complex Banach algebra.

Corollary 2.2.5.6 (Le Page, Hirschfeld-Želazko) *Let E be a unital complex Banach algebra. If there is an $\alpha \in \mathbb{R}_+$ with*

$$\|x\|^2 \leq \alpha \|x^2\|$$

for all $x \in E$, then E is commutative.

Take $x \in E$. Then, by complete induction

$$\|x\|^{2^n} \leq \alpha^{2^n - 1} \|x^{2^n}\|\,,$$

$$\|x\| \leq \sqrt{\alpha} \|x^{2^n}\|^{2^{-n}}$$

for every $n \in \mathbb{N}$, so that

$$\|x\| \leq \lim_{n \to \infty} \sqrt{\alpha} \|x^{2^n}\|^{2^{-n}} = \sqrt{\alpha} r(x)$$

(Theorem 2.2.5.4). By Corollary 2.2.4.3, E is commutative. ■.

Corollary 2.2.5.7 *Let E be a finite-dimensional complex Banach algebra. If no element of $E \backslash \{0\}$ is nilpotent, then E is commutative.*

Let $\mathbb{C} \times E$ be a unital Banach algebra associated to E. Take $(\alpha, x) \in \mathbb{C} \times E$. Assume that (α, x) is nilpotent. Then there is an $n \in \mathbb{N}$ such that

$$(\alpha, x)^n = 0\,.$$

Then

$$\left(\alpha^n, \sum_{k=1}^{n} \binom{n}{k} \alpha^{n-k} x^k\right) = (\alpha, x)^n = 0\,,$$

so that, successively,

$$\alpha^n = 0\,, \quad \alpha = 0\,, \quad x^n = 0\,, \quad x = 0\,, \quad \text{and} \quad (\alpha, x) = 0\,.$$

Put

$$\beta := \inf\left\{\|x^2\| \mid x \in \mathbb{C} \times E\,,\ \|x\| = 1\right\}\,.$$

Since $\mathbb{C} \times E$ is finite-dimensional, $\{x \in \mathbb{C} \times E \mid \|x\| = 1\}$ is compact, so $\beta > 0$. Thus

$$\|x\|^2 \leq \frac{1}{\beta}\|x^2\|$$

for every $x \in E$. By Corollary 2.2.5.6, E is commutative. ■

Corollary 2.2.5.8 (Wielandt, 1949) *If E is a complex unital normed algebra, then*

$$xy - yx \neq 1$$

for every $x, y \in E$.

By Corollary 2.2.1.13, we may assume that E is complete. Assume that

$$xy - yx = 1\,.$$

Take $\alpha \in \sigma(yx)\backslash\mathbb{R}_+$. Then

$$\alpha \in \sigma(xy) = \sigma(yx + 1) = \sigma(yx) + 1$$

(Proposition 2.1.3.10 b)), so that

$$\alpha - 1 \in \sigma(yx)\backslash\mathbb{R}_+\,.$$

By complete induction,

$$\alpha - n \in \sigma(yx)\,,$$

and

$$|\alpha - n| \leq r(yx) \leq \|yx\|$$

for every $n \in \mathbb{N}$ (Proposition 2.2.4.1), which is a contradiction.

Take $\alpha \in \sigma(yx) \cap \mathbb{R}_+$. Then

$$\alpha \in \sigma(xy-1) = \sigma(xy) - 1\,,$$

$$\alpha + 1 \in \sigma(xy)\,,$$

$$\alpha + 1 \in \sigma(yx) \cap \mathbb{R}_+$$

(Proposition 2.1.3.10 b)). By complete induction,

$$\alpha + n \in \sigma(yx)\,,$$

and

$$\alpha + n \leq r(yx) \leq \|yx\|$$

for every $n \in \mathbb{N}$ (Proposition 2.2.4.1), which is a contradiction.

By the above proof, $\sigma(yx) = \emptyset$, and this contradicts Theorem 2.2.5.4. ■

Corollary 2.2.5.9 *Let E be a complex unital Banach algebra and take $x \in E$ such that*

$$\sigma(x) \subset \{\alpha \in \mathbb{C} \mid \operatorname{re}\alpha > 0\}\,.$$

Then there is an $\alpha > 0$ with

$$r(1-\alpha x) = \lim_{n\to\infty} \|(1-\alpha x)^n\|^{\frac{1}{n}} < 1$$

and for any such α and for any $n \in \mathbb{N}$,

$$\left(\alpha^{-\frac{1}{n}}\,[1-(1-\alpha x)]^{\frac{1}{n}}\right)^n = x\,.$$

Since $\sigma(x)$ is compact (Corollary 2.2.4.5) and

$$\sigma(x) \subset \bigcup_{\alpha>0} U_\alpha^{\mathbb{C}}(\alpha)\,,$$

there is an $\alpha > 0$ with

$$\sigma(x) \subset U_{\frac{1}{\alpha}}^{\mathbb{C}}(\frac{1}{\alpha})\,.$$

Thus

$$\sigma(\alpha x) \subset U_1^{\mathbb{C}}(1)\,,$$

$$\sigma(1-\alpha x) = 1 - \sigma(\alpha x) \subset 1 - U_1^{\mathbb{C}}(1) = U_1^{\mathbb{C}}(0)\,,$$

$$\lim_{n\to\infty} \|(1-\alpha x)^n\|^{\frac{1}{n}} = r(1-\alpha x) < 1$$

(Theorem 2.2.5.4). By Proposition 2.2.3.13 a),c),

$$\left(\alpha^{-\frac{1}{n}}\,[1-(1-\alpha x)]^{\frac{1}{n}}\right)^n = \frac{1}{\alpha}(1-(1-\alpha x)) = x\,.$$ ■

2.2.6 Poles of Resolvents

Proposition 2.2.6.1 (6) *Let E be a complex unital Banach algebra. Take $x \in E$, $\alpha_0 \in \mathbb{C}$ and $r_1, r_2 \in \mathbb{R}_+$, such that*

$$U := \left\{ \alpha \in \mathbb{C} \mid r_1 < |\alpha - \alpha_0| < r_2 \right\} \subset \mathbb{C} \backslash \sigma(x) .$$

Let

$$\sum_{n=-\infty}^{\infty} (t - \alpha_0)^n x_n$$

be the Laurent series of the resolvent of x in U (Theorems 2.2.5.1 and 1.3.10.8).

a) $\{x_n \mid n \in \mathbb{Z}\} \subset \{x_n \mid n \in \mathbb{Z}\}^c = \{x\}^c$.

b) *For every $n \in \mathbb{N}$,*

$$x_{-1} x_{n-1} = x_{n-1} x_{-1} = 0 ,$$

$$(x - \alpha_0 1)^n x_{n-1} = x_{-1} - 1 ,$$

$$x_{-n} = (x - \alpha_0 1)^{n-1} x_{-1} .$$

c) *x_{-1} is idempotent (i.e. $x_{-1}^2 = x_{-1}$).*

d) $\alpha \in U \Rightarrow x_{-1} = (\alpha 1 - x) \sum_{n=1}^{\infty} (\alpha - \alpha_0)^{-n} x_{-n} = 1 - (\alpha 1 - x) \sum_{n=0}^{\infty} (\alpha - \alpha_0)^n x_n$.

a) Take $y \in E$ and $\alpha \in U$. By Corollary 2.2.1.11,

$$\sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n y = \left(\sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n \right) y ,$$

$$\sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n y x_n = y \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n .$$

If $y \in \{x\}^c$, then

$$\left(\sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n \right) y = y \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n ,$$

by Proposition 2.1.3.9. Hence

$$\sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n y = \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n y x_n$$

and so

$$x_n y = y x_n ,$$

by Theorem 1.3.10.8. Thus

$$y \in \{x_n \mid n \in \mathbb{Z}\}^c , \quad \{x\}^c \subset \{x_n \mid n \in \mathbb{Z}\}^c ,$$

$$x \in \{x_n \mid n \in \mathbb{Z}\}^c , \quad \{x_n \mid n \in \mathbb{Z}\} \subset \{x\}^c .$$

If $y \in \{x_n \mid n \in \mathbb{Z}\}^c$, then

$$\sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n y = \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n y x_n$$

and so $y \in \{(\alpha 1 - x)^{-1}\}^c$ by the above relations. By Proposition 2.1.3.9,

$$y \in \{x\}^c , \quad \{x_n \mid n \in \mathbb{N}\}^c \subset \{x\}^c , \quad \{x_n \mid n \in \mathbb{Z}\}^c = \{x\}^c .$$

b), c), and d). Now

$$1 = (\alpha 1 - x) \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n =$$

$$= (\alpha - \alpha_0) \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n + (\alpha_0 1 - x) \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_n =$$

$$= \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n x_{n-1} + \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n (\alpha_0 1 - x) x_n =$$

$$= \sum_{n=-\infty}^{\infty} (\alpha - \alpha_0)^n (x_{n-1} + (\alpha_0 1 - x) x_n)$$

for every $\alpha \in U$ (Corollary 2.2.1.11, Proposition 1.1.6.11). Hence

$$x_{-1} + (\alpha_0 1 - x) x_0 = 1$$

and

$$x_{n-1} = (x - \alpha_0 1)x_n$$

for every $n \in \mathbb{Z}\backslash\{0\}$ (Theorem 1.3.10.8). It follows, by complete induction, that

$$x_{-n} = (x - \alpha_0 1)^{n-1} x_{-1},$$

$$x_{-1} - 1 = (x - \alpha_0 1)^n x_{n-1}$$

for every $n \in \mathbb{N}$. Then

$$1 - (\alpha 1 - x)\sum_{n=0}^{\infty}(\alpha - \alpha_0)^n x_n = (\alpha 1 - x)\sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n} x_{-n} =$$

$$= (\alpha - \alpha_0)\sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n} x_{-n} + (\alpha_0 1 - x)\sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n} x_{-n} =$$

$$= x_{-1} + \sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n} x_{-n-1} + \sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n}(\alpha_0 1 - x)x_{-n} =$$

$$= x_{-1} + \sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n}\left(x_{-n-1} + (\alpha_0 1 - x)x_{-n}\right) = x_{-1},$$

$$\sum_{n=-\infty}^{\infty}(\alpha - \alpha_0)^n x_n x_{-1} = (\alpha 1 - x)^{-1} x_{-1} =$$

$$= \sum_{n=-\infty}^{\infty}(\alpha - \alpha_0)^n x_n - \sum_{n=0}^{\infty}(\alpha - \alpha_0)^n x_n = \sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n} x_{-n},$$

$$\sum_{n=-\infty}^{\infty}(\alpha - \alpha_0)^n x_{-1} x_n = x_{-1}(\alpha 1 - x)^{-1} =$$

$$= \sum_{n=-\infty}^{\infty}(\alpha - \alpha_0)^n x_n - \sum_{n=0}^{\infty}(\alpha - \alpha_0)^n x_n = \sum_{n=1}^{\infty}(\alpha - \alpha_0)^{-n} x_{-n}$$

for every $\alpha \in U$ (Corollary 2.2.1.11, Proposition 1.1.6.11). Hence

$$x_{-1}^2 = x_{-1},$$

and

$$x_n x_{-1} = x_{-1} x_n = 0$$

for every $n \in \mathbb{N} \cup \{0\}$ (Theorem 1.3.10.8). ■

Corollary 2.2.6.2 (6) *Let E be a complex unital Banach algebra. Take $x \in E$ and let α_0 be a pole of order p of the resolvent of x. Let y be the residue of x in α_0.*

a) *The principal part of the resolvent of x in α_0 is*

$$\sum_{n=1}^{p} (t-\alpha_0)^{-n}(x-\alpha_0 1)^{n-1} y .$$

b) $(x-\alpha_0 1)^n y \neq 0$ *for all* $n \in \mathbb{N}_{p-1}$ *and*

$$(x-\alpha_0 1)^p y = 0 .$$

The assertion follows immediately from Proposition 2.2.6.1 b). ■

Corollary 2.2.6.3 (6) *Let E be a complex unital Banach algebra. Take $x, y \in E$, $\alpha, \beta \in \mathbb{C}$, r_1 and $r_2 \in \mathbb{R}_+$ such that*

$$U := \{\gamma \in \mathbb{C} \mid r_1 < |\gamma - \alpha| < r_2\} \subset \mathbb{C}\backslash\sigma(x) ,$$

$$V := \{\gamma \in \mathbb{C} \mid r_1 < |\gamma - \beta| < r_2\} \subset \mathbb{C}\backslash\sigma(y) .$$

Let

$$\sum_{n=-\infty}^{\infty} (t-\alpha)^n x_n , \quad \sum_{n=-\infty}^{\infty} (t-\beta)^n y_n$$

be the Laurent series of the resolvents of x and y in U and V, respectively. If

$$xy = yx ,$$

then

$$x_m y_n = y_n x_m$$

for every $m, n \in \mathbb{Z}$.

By applying Proposition 2.2.6.1 a) twice, we see that

$$y \in \{x\}^c = \{x_n \mid n \in \mathbb{Z}\}^c$$

and

$$\{x_n \mid n \in \mathbb{Z}\} \subset \{y\}^c = \{y_n \mid n \in \mathbb{Z}\}^c . \quad ■$$

Corollary 2.2.6.4 (6) *Let E be a complex unital Banach algebra. Take $x \in E$ such that $\mathbb{C}\backslash\sigma(x)$ is connected. Let α, β be two distinct isolated points of $\sigma(x)$ and y, z the residues of the resolvent of x in α and β, respectively. Then*

$$yz = 0 .$$

Let

$$\sum_{n=-\infty}^{\infty} (t-\alpha)^n x_n$$

be the Laurent series of x in α. By Proposition 2.2.6.1 d),

$$y = (\gamma 1 - x) \sum_{n=1}^{\infty} (\gamma - \alpha)^{-n} x_{-n}$$

for every $\gamma \neq \alpha$ in a neighbourhood of α. Since the radius or convergence of the power series

$$\sum_{n=1}^{\infty} t^n x_{-n}$$

is infinite (Theorem 1.3.10.8), the function

$$f : \mathbb{C}\backslash\sigma(x) \longrightarrow E , \quad \gamma \longmapsto (\gamma 1 - x)^{-1} y - \sum_{n=1}^{\infty} (\gamma - \alpha)^{-n} x_{-n}$$

is well-defined. Take $x' \in E'$. The function $x' \circ f$ is analytic (Theorem 2.2.5.1, Proposition 1.3.10.4) and vanishes on a nonempty open set. Hence it vanishes everywhere. It follows that

$$\langle (\gamma 1 - x)^{-1} y, x' \rangle = \langle \sum_{n=1}^{\infty} (\gamma - \alpha)^{-n} x_n, x' \rangle$$

for every $\gamma \in \mathbb{C}\backslash\sigma(x)$ and so

$$\lim_{\substack{\gamma \to \beta \\ \gamma \in \mathbb{C}\backslash\sigma(x)}} \langle (\gamma 1 - x)^{-1} y, x' \rangle = \langle \sum_{n=1}^{\infty} (\beta - \alpha)^{-n} x_n, x' \rangle$$

(Theorem 1.1.6.23). From Corollary 2.2.1.11, Proposition 1.2.1.16, and the classical theory of analytic functions, we see that

$$\langle zy, x' \rangle = 0 .$$

Since x' is arbitrary,

$$zy = 0$$

(Corollary 1.3.3.8). ■

Theorem 2.2.6.5 (6) *Let E be a complex unital Banach algebra. Take $x \in E$ and $\alpha \in \sigma(x)$. α is a pole of the resolvent of x iff there is an idempotent element y of E satisfying:*

1) $xy = yx$.

2) $(x - \alpha 1)y$ *is nilpotent.*

3) *There is a* $z \in E$ *with*

$$yz = zy = 0,$$

$$z(x - \alpha 1) = (x - \alpha 1)z = 1 - y.$$

If these conditions are fulfilled, then the order of the pole is the smallest number $p \in \mathbb{N}$, such that

$$((x - \alpha 1)y)^p = 0,$$

and

$$\sum_{n=-1}^{-p} (t - \alpha)^n (x - \alpha 1)^{-n-1} y - \sum_{n=0}^{\infty} (t - \alpha)^n z^{n+1}$$

is the Laurent series of the resolvent of x in α. It is defined in

$$\left\{ \beta \in \mathbb{C} \,\middle|\, 0 < |\beta - \alpha| < \frac{1}{\|z\|} \right\},$$

and y is its residue.

We first assume that α is a pole of order p of the resolvent of x. Let y be the corresponding residue. By Proposition 2.2.6.1 a), b), c), y is idempotent, $xy = yx$, and p is the smallest number $p \in \mathbb{N}$ such that

$$((x - \alpha 1)y)^p = 0.$$

In particular, $(x - \alpha 1)y$ is nilpotent. Let z be the residue of

$$\mathbb{C} \backslash \sigma(x) \longrightarrow E, \quad \beta \longmapsto -(\beta - \alpha)^{-1}(\beta 1 - x)^{-1}$$

at α. Then

$$yz = zy = 0,$$

$$z(x-\alpha 1)=(x-\alpha 1)z=1-y$$

(Proposition 2.2.6.1 a), b)).

Now assume that the above conditions 1), 2), 3) are fulfilled. Let p be the smallest natural number with

$$((x-\alpha 1)y)^p=0\,.$$

Then

$$((\beta-\alpha)1-(x-\alpha 1)y)^{-1}=\sum_{n=1}^{p}(\beta-\alpha)^{-n}((x-\alpha 1)y)^{n-1}$$

for every $\beta\in\mathbb{C}\backslash\{\alpha\}$ (Proposition 2.1.3.16 b)). Define

$$f:\mathbb{C}\backslash\{\alpha\}\longrightarrow,\quad \beta\longmapsto((\beta-\alpha)1-(x-\alpha 1)y)^{-1}y$$

$$g:U^{\mathbb{C}}_{\frac{1}{\|x\|}}(\alpha)\longrightarrow E\,,\quad \beta\longmapsto -z(1-(\beta-\alpha)z)^{-1}$$

(Proposition 2.2.4.1). Then

$$(1-y)f(\beta)=0\,,$$

$$((\beta-\alpha)1-(x-\alpha 1)y)f(\beta)=y$$

for every $\beta\in\mathbb{C}\backslash\{\alpha\}$, and

$$yg(\beta)=0\,,$$

$$g(\beta)((\beta-\alpha)(1-y)-(x-\alpha 1)(1-y))=$$

$$=-z(1-(\beta-\alpha)z)^{-1}(\beta-\alpha)(1-y)+z(1-(\beta-\alpha)z)^{-1}(x-\alpha 1)(1-y)=$$

$$=(1-y)(1-(\beta-\alpha)z)^{-1}(1-(\beta-\alpha)z)=1-y$$

for every $\beta\in U^{\mathbb{C}}_{\frac{1}{\|x\|}}(\alpha)$ (Proposition 2.1.2.8). It follows that

$$(f(\beta)+g(\beta))(\beta 1-x)=(f(\beta)+g(\beta))((\beta-\alpha)1-(x-\alpha 1))=$$

$$=(f(\beta)+g(\beta))(\beta-\alpha)y-(x-\alpha 1)y+(\beta-\alpha)(1-y)-(x-\alpha 1)(1-y))=$$

$$= f(\beta)((\beta-\alpha)y-(x-\alpha 1)y)+g(\beta)((\beta-\alpha)(1-y)-(x-\alpha 1)(1-y))=$$

$$= y+1-y=1$$

for every $\beta \in U^{\mathbb{C}}_{\frac{1}{\|x\|}}(\alpha)\backslash\{\alpha\}$. Therefore

$$\sum_{n=-1}^{-p}(t-\alpha)^n((x-\alpha 1)y)^{-n-1}-\sum_{n=0}^{\infty}(t-\alpha)^n z^{n+1}$$

is the Laurent series of the resolvent of x at α. All the other assertions now follow.

■

Theorem 2.2.6.6 (6) *Let E be a Banach space and take $u \in \mathcal{L}(E)$. Given $n \in \mathbb{N}\cup\{0\}$, put*

$$F_n := \operatorname{Ker} u^n, \quad G_n := \operatorname{Im} u^n.$$

a)

$$n \in \mathbb{N} \Longrightarrow \begin{cases} F_{n-1} \subset F_n, & u(F_n) \subset F_{n-1}, \\ G_n \subset G_{n-1}, & u(G_{n-1}) \subset G_n. \end{cases}$$

b) *If $F_p = F_{p+1}$ for a $p \in \mathbb{N}$, then $F_p = F_n$ for every $n \in \mathbb{N}$, $n > p$, and the map*

$$G_p \longrightarrow G_p, \quad x \longmapsto ux$$

is injective.

c) *If $G_p = G_{p+1}$ for a $p \in \mathbb{N}$, then $G_p = G_n$ for every $n \in \mathbb{N}$, $n > p$, and the map*

$$G_p \longrightarrow G_p, \quad x \longmapsto ux$$

is surjective.

d) *Given $p \in \mathbb{N}$, the following are equivalent:*

d$_1$) *G_p is closed and $F_p = F_{p+1}$, $G_p = G_{p+1}$.*

d$_2$) *There are $v, w \in \mathcal{L}(E)$ such that*

$$v^2 = v, \quad uv = vu, \quad (uv)^p = 0,$$

$$vw = wv = 0, \quad uw = wu = 1 - v.$$

If these conditions are fulfilled, then

$$E = F_p \oplus G_p ,$$

v is the projection of E onto F_p with $G_p = \operatorname{Ker} v$, the map

$$G_p \longrightarrow G_p , \quad x \longmapsto ux$$

is an isomorphism, and

$$G_p \longrightarrow G_p , \quad x \longmapsto wx$$

is its inverse. Defining

$$i : G_p \longrightarrow E , \quad x \longmapsto x ,$$

$$j : G_p \longrightarrow G_p , \quad x \longmapsto ux ,$$

and

$$(1 - v)_0 : E \longrightarrow G_p , \quad x \longmapsto (1 - v)x ,$$

it follows that

$$w = i \circ j^{-1} \circ (1 - v)_0 .$$

In particular, v and w in 2) are unique.

e) *If $\mathbb{K} = \mathbb{C}$ and $0 \in \sigma(u)$, then 0 is a pole of the resolvent of u iff there is a $p \in \mathbb{N}$ for which the above equivalent conditions d_1) and d_2) of d) are fulfilled. In this case, the order of the pole is the smallest $p \in \mathbb{N}$ for which these assertions hold and*

$$\sum_{n=1}^{p} t^{-n} u^{n-1} v - \sum_{n=0}^{\infty} t^n w^{n+1}$$

is the Laurent series of the resolvent of u in 0. It converges in

$$\left\{ \alpha \in \mathbb{C} \,\middle|\, 0 < |\alpha| < \frac{1}{\|w\|} \right\}$$

and v is its residue.

a) is obvious.

b) We first show, by complete induction, that $F_p = F_n$ for every $n \in \mathbb{N}$ with $n > p$. Assume that $F_{n-1} = F_p$. Then, for $x \in F_n$,

$$u^{n-1}ux = u^n x = 0, \qquad ux \in F_{n-1} = F_p, \qquad x \in F_{p+1} = F_p,$$

i.e. $F_n = F_p$.

Now take $x \in G_p$ with $ux = 0$. There is a $y \in E$ such that $u^p y = x$. Thus

$$u^{p+1}y = ux = 0,$$

$$y \in F_{p+1} = F_p,$$

$$x = u^p y = 0.$$

Hence the map

$$G_p \longrightarrow G_p, \quad x \longmapsto ux$$

is injective.

c) Take $x \in G_p$. There is a $y \in E$ such that $u^{p+1}y = x$. Then $u^p y \in G_p$ and

$$x = uu^p y \in u(G_p).$$

Hence the map

$$G_p \longrightarrow G_p, \quad u \longmapsto ux$$

is surjective. It follows that

$$G_n = u^{n-p}(G_p) = G_p$$

for every $n \in \mathbb{N}$ with $n > p$.

d) First suppose that d_1) is fulfilled. By b) and c), the map

$$G_p \longrightarrow G_p, \quad x \longmapsto ux$$

is bijective and therefore an isomorphism (Corollary 1.4.2.4).

Take $x \in F_p \cap G_p$. Then

$$u^p x = 0.$$

Thus $x = 0$. Hence

$$F_p \cap G_p = \{0\}.$$

Now take $x \in E$. There is a $y \in G_p$ with

$$u^p y = u^p x.$$

Since

$$u^p(x - y) = u^p x - u^p y = 0,$$

$x - y \in F_p$, so that the map

$$F_p \times G_p \longrightarrow E, \quad (x, y) \longmapsto x + y$$

is surjective. By Corollary 1.4.2.6,

$$E = F_p \oplus G_p.$$

Let v be the projection of E onto F_p with $\operatorname{Ker} v = G_p$ (Theorem 1.2.5.8 a $\Rightarrow$ b). Take $x \in E$. There is a $(y, z) \in F_p \times G_p$ such that

$$x = y + z.$$

Then $uy \in F_p$, $uz \in G_p$ by a), and so

$$uvy = uy = vuy,$$

$$uvz = 0 = vuz,$$

$$uvx = uvy + uvz = vuy + vuz = vux,$$

i.e.

$$uv = vu.$$

Thus

$$(uv)^p = v^p u^p = 0.$$

Define

$$w := i \circ j^{-1} \circ (1 - v)_0.$$

The map

$$G_p \longrightarrow G_p, \quad x \longmapsto wx$$

is obviously the inverse of the map

$$G_p \longrightarrow G_p, \quad x \longmapsto ux,$$

and

$$vw = wv = 0,$$

$$uw = u \circ \imath \circ j^{-1} \circ (1 - v)_0 = 1 - v,$$

$$wu = \imath \circ j^{-1} \circ (1 - v)_0 \circ u = 1 - v.$$

Now suppose that d_2) is fulfilled. Set

$$F := \operatorname{Im} v, \quad G := \operatorname{Ker} v.$$

Then

$$E = F \oplus G$$

(Theorem 1.2.5.8 b $\Rightarrow$ a) and

$$vu^p = u^p v = u^p v^p = (uv)^p = 0.$$

Thus $F \subset F_p$, $G_p \subset G$.

Now

$$wu^{p+1} = wuu^p = (1 - v)u^p = u^p - vu^p = u^p.$$

We deduce that

$$u^p x = wu^{p+1} x = 0,$$

so that $x \in F_p$ whenever $x \in F_{p+1}$. Hence $F_p = F_{p+1}$.

From

$$vw = 0,$$

and

$$uw = 1 - v\,,$$

it follows that

$$\operatorname{Im} w \subset G \subset u(G)\,,$$

$$G \subset u^{p+1}(G) \subset G_{p+1} \subset G_p \subset G\,.$$

Hence

$$G = G_p = G_{p+1}\,.$$

Thus, d_1) is proved. From the first part of the proof, we see that

$$E = F_p \oplus G_p\,,$$

so that

$$F = F_p\,.$$

From

$$uw = wu = 1 - v\,,$$

it follows that the map

$$G_p \longrightarrow G_p\,, \quad x \longmapsto wx$$

is the inverse of the map

$$G_p \longrightarrow G_p\,, \quad x \longmapsto ux\,.$$

From this and from

$$wv = 0\,,$$

we get that

$$w = \imath \circ j^{-1} \circ (1 - v)_0\,.$$

e) follows from Theorem 2.2.6.5. ■

2.2.7 Modules

The definitions and results of this subsection are special cases of the theory of Banach categories (Section 1.5).

Definition 2.2.7.1 (0) *Let E be a (unital) Banach algebra. A **(unital) E-module** is a Banach space F together with bilinear maps*

$$E \times F \longrightarrow F\,, \quad (x,a) \longmapsto xa\,,$$

$$F \times E \longrightarrow F\,, \quad (a,x) \longmapsto ax\,,$$

such that

$$(xy)a = x(ya)\,, \quad (xa)y = x(ay)\,, \quad a(xy) = (ax)y$$

$$\|xa\| \leq \|x\|\,\|a\|\,, \quad \|ax\| \leq \|a\|\,\|x\| \quad (1a = a1 = a)$$

*for every $x, y \in E$ and $a \in F$. An **E-submodule of F** is a Banach subspace G of F such that*

$$(x,a) \in E \times G \Longrightarrow xa, ax \in G\,.$$

*A **(unital) E-algebra** is a (unital) E-module F endowed with the structure of a (unital) Banach algebra such that*

$$x(ab) = (xa)b\,, \quad (ax)b = a(xb)\,, \quad (ab)x = a(bx)$$

for all $x \in E$ and $a, b \in F$.

*Let F, G be E-modules (E-algebras). $u \in \mathcal{L}(F,G)$ is called a **homomorphism of E-modules (of E-algebras)** if*

$$u(xa) = x(ua)\,, \quad u(ax) = (ua)x$$

for all $(x,a) \in E \times F$ (and u is an algebra homomorphism).

Every ideal of E is an E-algebra in a natural way.

Proposition 2.2.7.2 (0) *Let E be a (unital) Banach algebra and F a (unital) E-module. Given $(x,a') \in E \times F'$, define*

$$xa' : F \longrightarrow \mathbb{K}\,, \quad a \longmapsto \langle ax, a' \rangle\,,$$

$$a'x : F \longrightarrow \mathbb{K}\,, \quad a \longmapsto \langle xa, a' \rangle\,.$$

F' endowed with the bilinear maps

$$E \times F' \longrightarrow F'\,, \quad (x, a') \longmapsto xa'\,,$$

$$F' \times E \longrightarrow F'\,, \quad (a', x) \longmapsto a'x$$

is a (unital) E-module.

It is obvious that xa', $a'x$ belong to F', that

$$\|xa'\| \leq \|x\|\,\|a'\|\,, \quad \|a'x\| \leq \|a'\|\,\|x\|$$

for all $(x, a') \in E \times F'$, and that the corresponding maps are bilinear. Take $x, y \in E$ and $a' \in F'$. Then, given $a \in F$,

$$\langle a, (xy)a' \rangle = \langle a(xy), a' \rangle = \langle (ax)y, a' \rangle = \langle ax, ya' \rangle = \langle a, x(ya') \rangle\,,$$

$$\langle a, (xa')y \rangle = \langle ya, xa' \rangle = \langle (ya)x, a' \rangle = \langle y(ax), a' \rangle = \langle ax, a'y \rangle = \langle a, x(a'y) \rangle\,,$$

$$\langle a, a'(xy) \rangle = \langle (xy)a, a' \rangle = \langle x(ya), a' \rangle = \langle ya, a'x \rangle = \langle a, (a'x)y \rangle\,,$$

$$(\langle a, 1a' \rangle = \langle a1, a' \rangle = \langle a, a' \rangle, \quad \langle a, a'1 \rangle = \langle 1a, a' \rangle = \langle a, a' \rangle)\,,$$

so that

$$(xy)a' = x(ya')\,, \quad (xa')y = x(a'y)\,, \quad a'(xy) = (a'x)y \qquad (1a' = a'1 = a')\,. \quad \blacksquare$$

Example 2.2.7.3 *If T is locally compact, then, given $(x, x') \in C_0(T) \times C_0(T)'$, xx' is equal to $x \cdot x'$ in the sense of integration theory (Example 1.2.2.10).*

For $y \in C_0(T)$,

$$\langle y, xx' \rangle = \langle yx, x' \rangle = \int yx\,dx' = \int y\,d(x \cdot x') = \langle y, x \cdot x' \rangle$$

and so

$$xx' = x \cdot x'\,. \quad \blacksquare$$

Proposition 2.2.7.4 (0) *Let E be a Banach algebra and $x' \in E'$.*

a) $x \in E^c \Rightarrow xx' = x'x$.

b) *If* $\mathfrak{F}$ *is a approximate unit of* E *then*

$$x' = \lim_{x,\mathfrak{F}} xx' = \lim_{x \in \mathfrak{F}} x'x$$

in the topology of pointwise convergence.

a) For every $y \in E$,

$$\langle y, xx' \rangle = \langle yx, x' \rangle = \langle xy, x' \rangle = \langle y, x'x \rangle$$

so that

$$xx' = x'x .$$

b) For every $y \in E$,

$$\langle y, x' \rangle = \lim_{x,\mathfrak{F}} \langle yx, x' \rangle = \lim_{x,\mathfrak{F}} \langle y, xx' \rangle ,$$

$$\langle y, x' \rangle = \lim_{x,\mathfrak{F}} \langle xy, x' \rangle = \lim_{x,\mathfrak{F}} \langle y, x'x \rangle$$

so that

$$x' = \lim_{x,\mathfrak{F}} xx' = \lim_{x,\mathfrak{F}} x'x$$

in the topology of pointwise convergence. ■

Proposition 2.2.7.5 (0) *If* E *is a Banach algebra and* F *is an* E*-module, then the evaluation* $j_F : F \to F''$ *is a homomorphism of* E*-modules.*

By Proposition 2.2.7.2, F' and F'' are E-modules. Take $(x, a) \in E \times F$. Then, given $a' \in F'$,

$$\langle j_F(xa), a' \rangle = \langle xa, a' \rangle = \langle a, a'x \rangle = \langle j_F a, a'x \rangle = \langle x(j_F a), a' \rangle ,$$

$$\langle j_F(ax), a' \rangle = \langle ax, a' \rangle = \langle a, xa' \rangle = \langle j_F a, xa' \rangle = \langle (j_F a)x, a' \rangle$$

so that

$$j_F(xa) = x(j_F a) , \quad j_F(ax) = (j_F a)x . \qquad ■$$

Proposition 2.2.7.6 (0) *Let* F *and* G *be modules over the Banach algebra* E. *If* $u : F \to G$ *is a homomorphism of* E*-modules, then so is* u'.

Take $(x, b') \in E \times G'$. Given $a \in F$,

$$\langle a, u'(xb')\rangle = \langle ua, xb'\rangle = \langle (ua)x, b'\rangle =$$

$$= \langle u(ax), b'\rangle = \langle ax, u'b'\rangle = \langle a, x(u'b')\rangle ,$$

$$\langle a, u'(b'x)\rangle = \langle ua, b'x\rangle = \langle x(ua), b'\rangle =$$

$$= \langle u(xa), b'\rangle = \langle xa, u'b'\rangle = \langle a, (u'b')x\rangle .$$

Thus

$$u'(xb') = x(u'b') , \quad u'(b'x) = (u'b')x . \qquad \blacksquare$$

Proposition 2.2.7.7 (0) *Let E, F be Banach algebras and $u : E \to F$ a continuous algebra homomorphism. Then*

$$x(u'y') = u'((ux)y') , \quad (u'y')x = u'(y'(ux))$$

for every $(x, y') \in E \times F'$.

Take $y \in E$. Then

$$\langle y, x(u'y')\rangle = \langle yx, u'y'\rangle = \langle u(yx), y'\rangle =$$

$$= \langle (uy)(ux), y'\rangle = \langle uy, (ux)y'\rangle = \langle y, u'((ux)y')\rangle ,$$

$$\langle y, (u'y')x\rangle = \langle xy, u'y'\rangle = \langle u(xy), y'\rangle =$$

$$= \langle (ux)(uy), y'\rangle = \langle uy, y'(ux)\rangle = \langle y, u'(y'(ux))\rangle . \qquad \blacksquare$$

Definition 2.2.7.8 (0) *Let E be a Banach algebra and F an E-module. Given $(a, a') \in F \times F'$, define*

$$aa' : E \longrightarrow \mathbb{K} , \quad x \longmapsto \langle xa, a'\rangle ,$$

$$a'a : E \longrightarrow \mathbb{K} , \quad x \longmapsto \langle ax, a'\rangle .$$

If $E = F$ then the above definitions coincide with those introduced in Proposition 2.2.7.2.

Proposition 2.2.7.9 (0) *Let E be a Banach algebra and F an E-module. Then, given $(a, a') \in F \times F'$,*

$$aa', a'a \in E', \quad \|aa'\| \leq \|a\| \, \|a'\|, \quad \|a'a\| \leq \|a'\| \, \|a\|$$

and for $(x, a, a') \in E \times F \times F'$

$$(xa)a' = x(aa'), \quad (a'a)x = a'(ax),$$
$$(aa')x = a(a'x), \quad (xa')a = x(a'a),$$
$$(a'x)a = a'(xa), \quad (ax)a' = a(xa').$$

The first assertion is easy to verify. Take $y \in E$. Then

$$\langle y, (xa)a' \rangle = \langle y(xa), a' \rangle = \langle (yx)a, a' \rangle = \langle yx, aa' \rangle = \langle y, x(aa') \rangle,$$

$$\langle y, (a'a)x \rangle = \langle xy, a'a \rangle = \langle a(xy), a' \rangle = \langle (ax)y, a' \rangle = \langle y, a'(ax) \rangle,$$

$$\langle y, (aa')x \rangle = \langle xy, aa' \rangle = \langle (xy)a, a' \rangle = \langle x(ya), a' \rangle = \langle ya, a'x \rangle = \langle y, a(a'x) \rangle,$$

$$\langle y, (xa')a \rangle = \langle ay, xa' \rangle = \langle (ay)x, a' \rangle = \langle a(yx), a' \rangle = \langle yx, a'a \rangle = \langle y, x(a'a) \rangle,$$

$$\langle y, (a'x)a \rangle = \langle ay, a'x \rangle = \langle x(ay), a' \rangle = \langle (xa)y, a' \rangle = \langle y, a'(xa) \rangle,$$

$$\langle y, (ax)a' \rangle = \langle y(ax), a' \rangle = \langle (ya)x, a' \rangle = \langle ya, xa' \rangle = \langle y, a(xa') \rangle$$

(Proposition 2.2.7.2) and the last assertion now follows. ■

Proposition 2.2.7.10 (0) *If E is a Banach algebra and F an E-module then*

$$(j_F a)a' = aa', \quad a'(j_F a) = a'a$$

for any $(a, a') \in F \times F'$.

Take $x \in E$. Then

$$\langle x, (j_F a)a' \rangle = \langle j_F a, a'x \rangle = \langle a, a'x \rangle = \langle xa, a' \rangle = \langle x, aa' \rangle,$$

$$\langle x, a'(j_F a) \rangle = \langle j_F a, xa' \rangle = \langle a, xa' \rangle = \langle ax, a' \rangle = \langle x, a'a \rangle.$$

Hence

$$(j_F a)a' = aa' \quad a'(j_F a) = a'a. \qquad \blacksquare$$

Definition 2.2.7.11 (0) *Let E be a Banach algebra and F an E-module. Given $(x'', a') \in E'' \times F'$, define*

$$a'x'' : F \longrightarrow \mathbb{K}, \quad a \longmapsto \langle x'', aa' \rangle,$$

$$x''a' : F \longrightarrow \mathbb{K}, \quad a \longmapsto \langle x'', a'a \rangle$$

(Proposition 2.2.7.9).

It is easy to see that if $E = F$ then the above definition coincides with Definition 2.2.7.8.

Proposition 2.2.7.12 (0) *Let E be a Banach algebra and F an E-module. Then, given $x'' \in E''$, $a \in F$, and $a' \in F'$:*

a) $a'x'', x''a' \in F'$, $\|a'x''\| \leq \|a'\| \, \|x''\|$, $\|x''a'\| \leq \|x''\| \, \|a'\|$.

b) $a(a'x'') = (aa')x''$, $x''(a'a) = (x''a')a$.

c) *If u is defined by*

$$u : F \longrightarrow E', \quad a \longmapsto aa' \quad (resp.\ a'a),$$

then

$$u'x'' = a'x'' \quad (resp.\ x''a')$$

for every $x'' \in E''$.

a) is easy to see.

b) Given $x \in E$,

$$\langle x, a(a'x'') \rangle = \langle xa, a'x'' \rangle = \langle x'', (xa)a' \rangle = \langle x'', x(aa') \rangle = \langle x, (aa')x'' \rangle,$$

$$\langle x, x''(a'a) \rangle = \langle x'', (a'a)x \rangle = \langle x'', a'(ax) \rangle = \langle ax, x''a' \rangle = \langle x, (x''a')a \rangle$$

(Proposition 2.2.7.9). Thus

$$a(a'x'') = (aa')x'', \quad x''(a'a) = (x''a')a.$$

c) Given $a \in F$,

$$\langle a, u'x'' \rangle = \langle ua, x'' \rangle = \langle aa', x'' \rangle = \langle a, a'x'' \rangle$$

$$(\text{resp. } \langle a, u'x'' \rangle = \langle ua, x'' \rangle = \langle a'a, x'' \rangle = \langle a, x''a' \rangle).$$

Thus

$$u'x'' = a'x'' \quad (\text{resp. } x''a'). \qquad \blacksquare$$

Definition 2.2.7.13 (0) *Let E be a Banach algebra. For $x'', y'' \in E''$, define*

$$x'' \dashv y'' : E' \longrightarrow \mathbb{K}, \quad x' \longmapsto \langle y'', x'x'' \rangle,$$

$$x'' \vdash y'' : E' \longrightarrow \mathbb{K}, \quad x' \longmapsto \langle x'', y''x' \rangle$$

(Proposition 2.2.7.12). $\dashv$ and $\vdash$ are called ***the right and the left Arens multiplication****, respectively (Arens, 1951).*

Remark. $\dashv$ and $\vdash$ may be different (see Remark to Example 1.5.2.15) . It is easy to see that $x'' \dashv y''$, $x'' \vdash y'' \in E''$ and

$$\|x'' \dashv y''\| \leq \|x''\| \, \|y''\|, \quad \|x'' \vdash y''\| \leq \|x''\| \, \|y''\|$$

for every $x'', y'' \in E''$ and that the map

$$E'' \times E'' \longrightarrow E''$$

is bilinear.

Proposition 2.2.7.14 (0) *Let E be a Banach algebra and F an E-module. Then, given $x'', y'' \in E''$ and $a' \in F'$,*

$$a'(y'' \dashv x'') = (a'y'')x'', \quad (y'' \vdash x'')a' = y''(x''a').$$

By Proposition 2.2.7.12 b), given $a \in F$

$$\langle a, a'(y'' \dashv x'') \rangle = \langle aa', y'' \dashv x'' \rangle = \langle x'', (aa')y'' \rangle = \langle x'', a(a'y'') \rangle = \langle a, (a'y'')x'' \rangle$$

$$\langle a, (y'' \vdash x'')a' \rangle = \langle a'a, y'' \vdash x'' \rangle = \langle y'', x''(a'a) \rangle = \langle y'', (x''a')a \rangle = \langle a, y''(x''a') \rangle .$$

Thus

$$a'(y'' \dashv x'') = (a'y'')x'', \quad (y'' \vdash x'')a' = y''(x''a') . \qquad \blacksquare$$

Theorem 2.2.7.15 (0) *Let E be a (unital) Banach algebra.*

a) *E'' is a (unital) E-algebra with respect to $\dashv$ (with respect to $\vdash$) . We denote it by $E''_{\dashv}$ (by $E''_{\vdash}$).*

b) *The map*

$$E \longrightarrow E''_{\dashv} \ (\textit{resp. } E''_{\vdash}), \quad x \longmapsto j_E x$$

is a homomorphism of (unital) E-algebras.

c) $(x, x'') \in E \times E'' \Rightarrow \begin{cases} x''x = x'' \vdash (j_E x) = x'' \dashv (j_E x), \\ xx'' = (j_E x) \vdash x'' = (j_E x) \dashv x'' \end{cases}$.

d) *The map*

$$E''_{E'} \longrightarrow E''_{E'}, \quad x'' \longmapsto x'' \vdash y'' \quad (\textit{resp. } y'' \dashv x'')$$

is continuous for every $y'' \in E''$.

e) *If* $\vdash$ *and* $\dashv$ *coincide, then* F' *is an* $E''_{\vdash}$*-module whenever* F *is an* E*-module.*

f) *If* F *is an* E*-module and a predual of* E *such that*

$$\langle ax, y\rangle = \langle a, xy\rangle = \langle ya, x\rangle$$

for all $a \in F$, *and* $x, y \in E$, *then:*

f_1) $j_F(ax) = (j_F a)x$, $j_F(xa) = x(j_F a)$ *for every* $(a, x) \in F \times E''$.

f_2) $(j_F a)x'' = j_F(a(j'_F x''))$, $x''(j_F a) = j_F((j'_F x'')a)$ *for every* $(a, x'') \in F \times E''$.

f_3) $j'_F(x'' \vdash y'') = j'_F(x'' \dashv y'') = (j'_F x'')(j'_F y'')$ *for every* $x'', y'' \in E''$.

f_4) *If* $\vdash$ *and* $\dashv$ *coincide, then* j'_F *and the canonical projection of the tridual of* F *(Proposition 1.3.6.19 b)) are algebra homomorphisms.*

g) *If* E *is commutative, then*

g_1) $xx' = x'x$ *for every* $(x, x') \in E \times E'$.

g_2) $x'x'' = x''x'$ *for every* $(x', x'') \in E' \times E''$.

g_3) $x'' \vdash y'' = y'' \dashv x''$ *for every* $x'', y'' \in E''$.

In particular, if in addition $\vdash$ *and* $\dashv$ *coincide, then* E'' *is commutative.*

a) By Proposition 2.2.7.2, E'' is a (unital) E-module. We first show that $E''_{\dashv}$ and $E''_{\vdash}$ are (unital) Banach algebras.

Take $x'', y'', z'' \in E''$. By Proposition 2.2.7.14 (and Proposition 2.2.7.2),

$$\langle (x'' \dashv y'') \dashv z'', x'\rangle = \langle z'', x'(x'' \dashv y'')\rangle =$$

$$= \langle z'', (x'x'')y''\rangle = \langle y'' \dashv z'', x'x''\rangle = \langle x'' \dashv (y'' \dashv z''), x'\rangle,$$

$$\langle (x'' \vdash y'') \vdash z'', x' \rangle = \langle x'' \vdash y'', z''x' \rangle = \langle x'', y''(z''x') \rangle =$$

$$= \langle x'', (y'' \vdash z'')x' \rangle = \langle x'' \vdash (y'' \vdash z''), x' \rangle$$

for every $x' \in E'$, and so

$$(x'' \dashv y'') \dashv z'' = x'' \dashv (y'' \dashv z''),$$

$$(x'' \vdash y'') \vdash z'' = x'' \vdash (y'' \vdash z'').$$

Hence $\dashv$ and $\vdash$ are associative.

Now suppose that E is unital and take $x'' \in E''$. Then

$$\langle (j_E 1) \dashv x'', x' \rangle = \langle x'', x'(j_E 1) \rangle = \langle x'', x'1 \rangle = \langle x'', x' \rangle,$$

$$\langle x'' \dashv (j_E 1), x' \rangle = \langle j_E 1, x'x'' \rangle = \langle 1, x'x'' \rangle = \langle x'', 1x' \rangle = \langle x'', x' \rangle,$$

$$\langle (j_E 1) \vdash x'', x' \rangle = \langle j_E 1, x''x' \rangle = \langle 1, x''x' \rangle = \langle x'', x'1 \rangle = \langle x'', x' \rangle,$$

$$\langle x'' \vdash (j_E 1), x' \rangle = \langle x'', (j_E 1)x' \rangle = \langle x'', 1x' \rangle = \langle x'', x' \rangle$$

for every $x' \in E'$ (Proposition 2.2.7.10, Proposition 2.2.7.2), so

$$(j_E 1) \dashv x'' = x'' \dashv (j_E 1) = x'',$$

$$(j_E 1) \vdash x'' = x'' \vdash (j_E 1) = x''.$$

Hence $j_E 1$ is the unit of $E''_\dashv$ and $E''_\vdash$.

We now show that $E''_\dashv$ and $E''_\vdash$ are E-algebras. Let $x \in E$ and $x'', y'' \in E''$. By Proposition 2.2.7.9 (and Proposition 2.2.7.2)

$$\langle (xx') \dashv y'', x' \rangle = \langle y'', x'(xx'') \rangle = \langle y'', (x'x)x'' \rangle =$$

$$= \langle x'' \dashv y'', x'x \rangle = \langle x(x'' \dashv y''), x' \rangle,$$

$$\langle (xx'') \vdash y'', x' \rangle = \langle xx'', y''x' \rangle = \langle x'', (y''x')x \rangle =$$

$$= \langle x'', y''(x'x) \rangle = \langle x'' \vdash y'', x'x \rangle = \langle x(x'' \vdash y''), x' \rangle,$$

$$\langle (x'' \dashv y'')x, x' \rangle = \langle x'' \dashv y'', xx' \rangle = \langle y'', (xx')x'' \rangle =$$

$$= \langle y'', x(x'x'') \rangle = \langle y''x, x'x'' \rangle = \langle x'' \dashv (y''x), x' \rangle ,$$

$$\langle (x'' \vdash y'')x, x' \rangle = \langle x'' \vdash y'', xx' \rangle = \langle x'', y''(xx') \rangle =$$

$$= \langle x'', (y''x)x' \rangle = \langle x'' \vdash (y''x), x' \rangle ,$$

$$\langle (x''x) \dashv y'', x' \rangle = \langle y'', x'(x''x) \rangle = \langle y'', (x'x'')x \rangle =$$

$$= \langle xy'', x'x'' \rangle = \langle x'' \dashv (xy''), x' \rangle ,$$

$$\langle (x''x) \vdash y'', x' \rangle = \langle x''x, y''x' \rangle = \langle x'', x(y''x') \rangle =$$

$$= \langle x'', (xy'')x' \rangle = \langle x'' \vdash (xy''), x' \rangle$$

for every $x' \in E'$. Thus

$$(xx'') \dashv y'' = x(x'' \dashv y'') , \quad (xx'') \vdash y'' = x(x'' \vdash y'') ,$$

$$(x'' \dashv y'')x = x'' \dashv (y''x) , \quad (x'' \vdash y'')x = x'' \vdash (y''x) ,$$

$$(x''x) \dashv y'' = x'' \dashv (xy'') , \quad (x''x) \vdash y'' = x'' \vdash (xy'')$$

and $E''_{\dashv}, E''_{\vdash}$ are therefore E–algebras.

b) By Proposition 2.2.7.5, j_E is a homomorphism of E– modules. Take $x, y \in E$. By Proposition 2.2.7.10,

$$\langle j_E(xy), x' \rangle = \langle xy, x' \rangle = \langle x, yx' \rangle = \langle x, (j_E x)x' \rangle =$$

$$= \langle j_E x, (j_E y)x' \rangle = \langle (j_E x) \vdash (j_E y), x' \rangle ,$$

$$\langle j_E(xy), x' \rangle = \langle xy, x' \rangle = \langle y, x'x \rangle = \langle y, x'(j_E x) \rangle =$$

$$= \langle j_E y, x'(j_E x) \rangle = \langle (j_E x) \dashv (j_E y), x' \rangle$$

for every $x' \in E'$, so that

$$j_E(xy) = (j_E x) \vdash (j_E y) = (j_E x) \dashv (j_E y)\,.$$

Hence j_E is an algebra homomorphism. From the proof of a), we see that j_E is unital whenever E is unital.

c) By Proposition 2.2.7.10,

$$\langle x'' \vdash (j_E x), x' \rangle = \langle x'', (j_E x)x' \rangle = \langle x'', xx' \rangle = \langle x''x, x' \rangle\,,$$

$$\langle x'' \dashv (j_E x), x' \rangle = \langle j_E x, x'x'' \rangle = \langle x, x'x'' \rangle = \langle x''x, x' \rangle\,,$$

$$\langle (j_E x) \dashv x'', x' \rangle = \langle x'', x'(j_E x) \rangle = \langle x'', x'x \rangle = \langle xx'', x' \rangle\,,$$

$$\langle (j_E x) \vdash x'', x' \rangle = \langle j_E x, x''x' \rangle = \langle x, x''x' \rangle = \langle xx'', x' \rangle$$

for every $x' \in E'$. Hence

$$x'' \vdash (j_E x) = x''x = x'' \dashv (j_E x)\,,$$

$$(j_E x) \dashv x'' = xx'' = (j_E x) \vdash x''\,.$$

d) Given $x' \in E'$,

$$\lim_{x'' \to 0} \langle x'' \vdash y'', x' \rangle = \lim_{x'' \to 0} \langle x'', y''x' \rangle = 0\,,$$

$$\lim_{x'' \to 0} \langle y'' \dashv x'', x' \rangle = \lim_{y'' \to 0} \langle x'', x'y'' \rangle = 0\,,$$

where the limits are taken in $E''_{E'}$.

e) follows from Proposition 2.2.7.14.

f_1) Given $y \in E$,

$$\langle j_F(ax), y \rangle = \langle ax, y \rangle = \langle a, xy \rangle = \langle j_F a, xy \rangle = \langle (j_F a)x, y \rangle\,,$$

$$\langle j_F(xa), y \rangle = \langle xa, y \rangle = \langle a, yx \rangle = \langle j_F a, yx \rangle = \langle x(j_F a), y \rangle\,.$$

Thus

$$j_F(ax) = (j_F a)x\,, \quad j_F(xa) = x(j_F a)\,.$$

f_2) By f_1), if $x \in E$ then

$$\langle x, (j_F a)x'' \rangle = \langle x(j_F a), x'' \rangle = \langle j_F(xa), x'' \rangle = \langle xa, j'_F x'' \rangle =$$

$$= \langle x, a(j'_F x'') \rangle = \langle x, j_F(a(j'_F x'')) \rangle ,$$

$$\langle x, x''(j_F a) \rangle = \langle (j_F a)x, x'' \rangle = \langle j_F(ax), x'' \rangle = \langle ax, j'_F x'' \rangle =$$

$$= \langle x, (j'_F x'')a \rangle = \langle x, j_F((j'_F x'')a) \rangle .$$

Hence

$$(j_F a)x'' = j_F(a(j'_F x'')) , \quad x''(j_F a) = j_F((j'_F x'')a) .$$

f_3) By f_2), if $a \in F$, then

$$\langle a, j'_F(x'' \vdash y'') \rangle = \langle j_F a, x'' \vdash y'' \rangle = \langle y''(j_F a), x'' \rangle = \langle j_F((j'_F y'')a), x'' \rangle =$$

$$= \langle (j'_F y'')a, j'_F x'' \rangle = \langle a, (j'_F x'')(j'_F y'') \rangle = \langle a(j'_F x''), j'_F y'' \rangle =$$

$$= \langle j_F(a(j'_F x'')), y'' \rangle = \langle (j_F a)x'', y'' \rangle = \langle j_F a, x'' \dashv y'' \rangle = \langle a, j'_F(x'' \dashv y'') \rangle .$$

Hence

$$j'_F(x'' \vdash y'') = j'_F(x'' \dashv y'') = (j'_F x'')(j'_F y'') .$$

f_4) follows from b), f_3), and Proposition 1.3.6.19 c).

g_1) Since

$$\langle y, xx' \rangle = \langle yx, x' \rangle = \langle xy, x' \rangle = \langle y, x'x \rangle$$

for every $y \in E$, it follows that

$$xx' = x'x .$$

g_2) By g_1), if $x \in E$, then

$$\langle x, x'x'' \rangle = \langle xx', x'' \rangle = \langle x'x, x'' \rangle = \langle x, x''x' \rangle .$$

Hence

$$x'x'' = x''x' .$$

g3) By g2),

$$\langle x', x'' \vdash y'' \rangle = \langle y''x', x'' \rangle = \langle x'y'', x'' \rangle = \langle x', y'' \dashv x'' \rangle$$

for every $x' \in E'$. Hence

$$x'' \vdash y'' = y'' \dashv x'' . \qquad \blacksquare$$

Proposition 2.2.7.16 *Let E, F be Banach algebras, $u : E \to F$ a continuous algebra homomorphism, and $x \in E$, $x'' \in E''$, $y' \in F'$, $y'' \in E''$.*

a) $u''(xx'') = (ux)(u''x'')$, $\quad u''(x''x) = (u''x'')(ux)$.

b) $x''(u'y') = u'((u''x'')y')$, $\quad (u'y')x'' = u'(y'(u''x''))$.

c) $u''(x'' \dashv y'') = (u''x'') \dashv (u''y'')$, $\quad u''(x'' \vdash y'') = (u''x'') \vdash u''y''$.

a) By Proposition 2.2.7.7,

$$\langle u''(xx''), y' \rangle = \langle xx'', u'y' \rangle = \langle x'', (u'y')x \rangle = \langle x'', u'(y'(ux)) \rangle =$$

$$= \langle u''x'', y'(ux) \rangle = \langle (ux)(ux''), y' \rangle ,$$

$$\langle u''(x''x), y' \rangle = \langle x''x, u'y' \rangle = \langle x'', x(u'y') \rangle = \langle x'', u'(ux)y' \rangle =$$

$$= \langle u''x'', (ux)y' \rangle = \langle (u''x'')(ux), y' \rangle$$

for every $y' \in F'$.

b) By a),

$$\langle x, x''(u'y') \rangle = \langle xx'', u'y' \rangle = \langle u''(xx''), y' \rangle = \langle (ux)(u''x''), y' \rangle =$$

$$= \langle ux, (u''x'')y' \rangle = \langle x, u'((u''x'')y') \rangle ,$$

$$\langle x, (u'y')x'' \rangle = \langle x''x, u'y' \rangle = \langle u''(x''x), y' \rangle = \langle (u''x'')(ux), y' \rangle =$$

$$= \langle ux, y'(u''x'') \rangle = \langle x, u'(y'(u''x'')) \rangle$$

for every $x \in E$.

c) By b),

$$\langle u''(x'' \dashv y''), y' \rangle = \langle x'' \dashv y'', u'y' \rangle = \langle y'', (u'y')x'' \rangle =$$

$$= \langle y'', u'(y'(u''x'')) \rangle = \langle u''y'', y'(u''x'') \rangle = \langle (u''x'') \dashv (u''y''), y' \rangle ,$$

$$\langle u''(x'' \vdash y''), y' \rangle = \langle x'' \vdash y'', u'y' \rangle = \langle x'', y''(u'y') \rangle =$$

$$= \langle x'', u'((u''y'')y') \rangle = \langle u''x'', (u''y'')y' \rangle = \langle (u''x'') \vdash (u''y''), y' \rangle$$

for every $y' \in E'$. ■

Proposition 2.2.7.17 *Let E be a Banach algebra, $\widetilde{E}$ an associated unital Banach algebra, $\mathfrak{F}$ an approximate unit of E, and F an E-module such that*

$$\lim_{x,\mathfrak{F}} \|xa\| = \lim_{x,\mathfrak{F}} \|ax\| = \|a\|$$

for every $a \in F$. Then F admits a unique $\widetilde{E}$-module structure extending its structure as E-module.

We may assume that the unital algebra $\mathbb{K} \times E$ defined in Proposition 2.1.1.7 is the underlying unital algebra of $\widetilde{E}$. The uniqueness being trivial, we prove the existence. We endow F with the bilinear maps

$$\widetilde{E} \times F \longrightarrow F, \quad ((\alpha, x), a) \longmapsto \alpha a + xa ,$$

$$F \times \widetilde{E} \longrightarrow F, \quad (a, (\alpha, x)) \longmapsto \alpha a + ax ,$$

Take $(\alpha, x), (\beta, y) \in \widetilde{E}$ and $a \in F$. Then

$$((\alpha, x)(\beta, y))a = (\alpha\beta, \alpha y + \beta x + xy)a = \alpha\beta a + \alpha ya + \beta xa + xya =$$

$$= \alpha(\beta a + ya) + x(\beta a + ya) = (\alpha, x)((\beta, y)a) ,$$

$$((\alpha, x)a)(\beta, y) = (\alpha a + xa)(\beta, y) = \alpha\beta a + \beta xa + \alpha ay + xay =$$

$$= \alpha(\beta a + ay) + x(\beta a + ay) = (\alpha, x)(a(\beta, y)) ,$$

$$a((\alpha,x)(\beta,y)) = a(\alpha\beta, \alpha y + \beta x + xy) = \alpha\beta a + \alpha a y + \beta a x + a x y =$$

$$= \beta(\alpha a + a x) + (\alpha a + a x)y = (a(\alpha,x))(\beta,y)\,,$$

$$(1,0)a = a = a(1,0)\,.$$

Thus the algebraic axioms are fulfilled. Now

$$\|(a(\alpha,x))z\| = \|(\alpha a + a x)z\| = \|\alpha a z + a x z\| = \|a(\alpha z + x z)\| \leq$$

$$\leq \|a\|\,\|\alpha z + x z\| = \|a\|\,\|(\alpha,x)(0,z)\| \leq \|a\|\,\|(\alpha,x)\|\,\|(0,z)\| \leq \|x\|(\alpha,a)\|\,,$$

$$\|z((\alpha,x)a)\| = \|z(\alpha a + x a)\| = \|\alpha z a + z x a\| = \|(\alpha z + z x)a\| \leq$$

$$\leq \|\alpha z + z x\|\,\|a\| = \|(0,z)(\alpha,x)\|\,\|a\| \leq \|(0,z)\|\,\|(\alpha,x)\|\,\|a\| \leq \|(\alpha,x)\|\,\|a\|$$

for every $z \in E^{\#}$. Hence

$$\|a(\alpha,x)\| = \lim_{z,\mathfrak{F}} \|a(\alpha,x)z)\| \leq \|a\|\,\|(\alpha,x)\|\,,$$

$$\|(\alpha,x)a\| = \lim_{z,\mathfrak{F}} \|z((\alpha,x)a)\| \leq \|(\alpha,x)\|\,\|a\|$$

and the norm axioms are also fulfilled. ■

Proposition 2.2.7.18 *Let E be a Banach algebra, $\widetilde{E}$ an associated unital Banach algebra, $\mathfrak{F}$ an approximate unit of E, and F an E-module such that for every $a \in F$,*

$$\lim_{x,\mathfrak{F}} xa = \lim_{x,\mathfrak{F}} ax = a\,.$$

a) $\lim_{x,\mathfrak{F}} \|xa'\| = \lim_{x,\mathfrak{F}} \|a'x\| = \|a'\|$ *for every* $a' \in F'$.

b) F' *admits a unique* $\widetilde{E}$*-module structure extending its canonical* E*-module structure (Proposition 2.2.7.2).*

a) Take $a \in F^{\#}$. Then, given $x \in E$,

$$|\langle a, a'\rangle| \leq |\langle a - ax, a'\rangle| + |\langle a, xa'\rangle| \leq \|a - ax\|\,\|a'\| + \|xa'\|\,,$$

$$|\langle a, a' \rangle| \leq |\langle a - xa, a' \rangle| + |\langle a, a'x \rangle| \leq \|a - xa\| \, \|a'\| + \|a'x\| \,,$$

so that

$$|\langle a, a' \rangle| \leq \lim_{x,\mathfrak{F}} \inf \|xa'\| \leq \lim_{x,\mathfrak{F}} \sup \|xa'\| \leq \|a'\| \,,$$

$$|\langle a, a' \rangle| \leq \lim_{x,\mathfrak{F}} \inf \|a'x\| \leq \lim_{x,\mathfrak{F}} \sup \|a'x\| \leq \|a'\| \,.$$

Since a is arbitrary, it follows that

$$\lim_{x,\mathfrak{F}} \|xa'\| = \lim_{x,\mathfrak{F}} \|a'x\| = \|a'\| \,.$$

b) follows from a) and Proposition 2.2.7.17. ■

Remark. Assertion a) cannot be strengthened to $\lim_{x,\mathfrak{F}} xa' = \lim_{x,\mathfrak{F}} a'x = a'$ for every $a' \in F'$ as shown by the counterexample

$$E := c_0 \,, \qquad F := \ell^1 \,.$$

Corollary 2.2.7.19 *Let E be a quasiunital Banach algebra and $\widetilde{E}$ an associated unital Banach algebra. Then E' admits a unique unital $\widetilde{E}$-module structure extending its canonical E-module structure.* ■

Proposition 2.2.7.20 (0) *If E is a quasiunital Banach algebra, then*

$$|x'(x)| \leq \|xx'\| \,, \qquad \textit{and} \quad |x'(x)| \leq \|x'x\|$$

for every $(x, x') \in E \times E'$.

Let $\mathfrak{F}$ be an approximate unit of E. Then

$$|x'(yx)| = |\langle y, xx' \rangle| \leq \|xx'\| \,,$$

$$|x'(xy)| = |\langle y, x'x \rangle| \leq \|x'x\| \,,$$

for every $y \in E^{\#}$. Thus

$$|x'(x)| = \lim_{y,\mathfrak{F}} |x'(yx)| \leq \|xx'\| \,,$$

$$|x'(x)| = \lim_{y,\mathfrak{F}} |x'(xy)| \leq \|x'x\| \,.$$

■

Example 2.2.7.21 (0) *Let E be a (unital) Banach algebra, $(E_t)_{t\in T}$ a family of E-modules, $p \in [1,\infty] \cup \{0\}$,*

$$F := \left\{ \xi \in \prod_{t\in T} E_t \;\middle|\; (\|\xi(t)\|)_{t\in T} \in \ell^p(T) \right\},$$

and

$$q : F \longrightarrow \mathbb{R}_+, \quad \xi \longmapsto \|(\|\xi(t)\|)_{t\in T}\|_p .$$

a) *F is a vector subspace of $\prod_{t\in T} E_t$ and q is a complete norm.*

b) *F endowed with the maps*

$$E \times F \longrightarrow F, \quad (x,\xi) \longmapsto (x\xi(t))_{t\in T},$$

$$F \times E \longrightarrow F, \quad (\xi,x) \longmapsto (\xi(t)x)_{t\in T}$$

is a (unital) E-module.

a) was proved in Proposition 1.1.2.7.
b) is easy to verify. ■

Example 2.2.7.22 (0) *Let E be a (unital) Banach algebra, F a Banach space, and G a (unital) E- module. For all $x \in E$ and $u \in \mathcal{L}(F,G)$, put*

$$xu : F \longrightarrow G, \quad y \longmapsto x(uy),$$

$$ux : F \longrightarrow G, \quad y \longmapsto (uy)x .$$

Then $xu, ux \in \mathcal{L}(F,G)$ for all $x \in E$ and $u \in \mathcal{L}(F,G)$ and $\mathcal{L}(F,G)$ endowed with the maps

$$E \times \mathcal{L}(F,G) \longrightarrow \mathcal{L}(F,G), \quad (x,u) \longmapsto xu,$$

$$\mathcal{L}(F,G) \times E \longrightarrow \mathcal{L}(F,G), \quad (u,x) \longmapsto ux$$

is a (unital) E-module. ■

Definition 2.2.7.23 (0) *If E is a Banach algebra, F an E-module, and $m,n \in \mathbb{N}$, then we denote for every $x \in E$ and $a \in F_{m,n}$ by xa and ax the elements of $F_{m,n}$ defined by*

$$(xa)_{ij} := xa_{ij}, \quad (ax)_{ij} := a_{ij}x$$

for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$, respectively.

Proposition 2.2.7.24 *Let E be a quasiunital Banach algebra. For every $n \in \mathbb{N}$, we denote by E^n the corresponding E-module endowed with the euclidean norm (Example 2.2.7.21). Furthermore, we denote for every $m, n \in \mathbb{N}$ by $\mathcal{F}(E^n, E^m)$ the set of $u \in \mathcal{L}(E^n, E^m)$ such that*

$$u(\xi x) = (u\xi)x$$

for all $(\xi, x) \in E^n \times E$ and set

$$\widetilde{a} : E^n \longrightarrow E^m, \quad \xi \longmapsto \left(\sum_{j=1}^{n} a_{ij}\xi_j \right)_{i \in \mathbb{N}_m}$$

for every $a \in E_{m,n}$. For all $m, n, p \in \mathbb{N}$, we have:

a) *$\mathcal{F}(E^n, E^m)$ is a closed vector subspace of $\mathcal{L}(E^n, E^m)$ such that*

$$x \in E, u \in \mathcal{F}(E^n, E^m) \Longrightarrow xu \in \mathcal{F}(E^n, E^m).$$

b) *For all $a \in E_{m,n}$, the map $\widetilde{a}$ belongs to $\mathcal{F}(E^n, E^m)$ and*

$$\sup_{j \in \mathbb{N}_n} \sum_{i=1}^{m} \|a_{ij}\|^2 \leq \|\widetilde{a}\|^2 \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{ij}\|^2.$$

c) *The map*

$$E_{m,n} \longrightarrow \mathcal{F}(E^n, E^m), \quad a \longmapsto \widetilde{a}$$

is linear and injective and its image is a closed vector subspace of $\mathcal{F}(E^n, E^m)$. If E is unital the this map is bijective. We endow $E_{m,n}$ with the norm induced by $\mathcal{L}(E^n, E^m)$.

d) *$x \in E, a \in E_{m,n} \Rightarrow \widetilde{xa} = x\widetilde{a}$.*

e) *$E_{m,n}$ endowed with the maps*

$$E \times E_{m,n} \longrightarrow E_{m,n}, \quad (x, a) \longmapsto xa,$$

$$E_{m,n} \times E \longrightarrow E_{m,n}, \quad (a, x) \longmapsto ax$$

is an E-module. It is unital if E is unital.

f) *$a \in E_{m,n}, b \in E_{n,p} \Rightarrow \widetilde{ab} = \widetilde{a}\,\widetilde{b}$.*

g) *We set*

$$\widetilde{a'} : E_{m,n} \longrightarrow \mathbb{K}, \quad a \longmapsto \sum_{i=1}^{m}\sum_{j=1}^{n} a'_{ji}(a_{ij})$$

for all $a' \in E'_{n,m}$. *Then* $\widetilde{a'} \in (E_{m,n})'$ *and*

$$\left(\sum_{i=1}^{m}\sum_{j=1}^{n}\|a'_{ji}\|^2\right)^{\frac{1}{2}} \leq \|\widetilde{a'}\| \leq \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\|a'_{ji}\|^2\right)^{\frac{1}{2}}$$

for every $a' \in E'_{m,n}$ *and the map*

$$E'_{n,m} \longrightarrow (E_{m,n})', \quad a' \longmapsto \widetilde{a'}$$

is linear and bijective.

h) $E_{n,n}$ *is an* E*-algebra and for all* $(a, a') \in E_{nn} \times E'_{n,n}$ *and* $i, j \in \mathbb{N}_n$

$$(aa')_{ij} = \sum_{k=1}^{n} a_{ik}a'_{kj}, \quad (a'a)_{ij} = \sum_{k=1}^{n} a'_{ik}a_{kj}.$$

If E *is unital then the matrix* $[\delta_{ij}1]_{i,j\in\mathbb{N}_n}$ *is the unit of* $E_{n,n}$.

a) is obvious.

b) $\widetilde{a}$ is obviously linear. For all $\xi \in E^n$,

$$\|\widetilde{a}\xi\|^2 = \sum_{i=1}^{m}\left\|\sum_{j=1}^{n} a_{ij}\xi_j\right\|^2 \leq$$

$$\leq \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\|a_{ij}\|\,\|\xi_j\|\right)^2 \leq \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\|a_{ij}\|^2\right)\left(\sum_{j=1}^{n}\|\xi_j\|^2\right) =$$

$$= \left(\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\|a_{ij}\|^2\right)\right)\|\xi\|^2,$$

so $\widetilde{a}$ is continuous and

$$\|\widetilde{a}\|^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\|a_{ij}\|^2.$$

Let $j \in \mathbb{N}_n$. For every $x \in E$, denote by $\varphi_j x$ the vector

$$(0, \ldots, 0, x, 0, \ldots, 0)$$

of E^n, where x is on the j^{th}– place. Then for every $x \in E^{\#}$ and $j \in \mathbb{N}_m$,

$$\|\widetilde{a}\|^2 \geq \|\widetilde{a}(\varphi_j x)\|^2 = \|(a_{ij}x)_{i \in \mathbb{N}_m}\|^2 = \sum_{i=1}^{m} \|a_{ij}x\|^2 .$$

Since E is quasiunital, we get

$$\|\widetilde{a}\|^2 \geq \sum_{i=1}^{m} \|a_{ij}\|^2 ,$$

so

$$\|\widetilde{a}\|^2 \geq \sup_{j \in \mathbb{N}_n} \sum_{i=1}^{m} \|a_{ij}\|^2 .$$

For all $(\xi, x) \in E^n \times E$,

$$(\widetilde{a}(\xi x))_i = \sum_{j=1}^{n} a_{ij}(\xi_j x) = \left(\sum_{j=1}^{n} a_{ij}\xi_j \right) x = (\widetilde{a}\xi)_i x$$

for all $i \in \mathbb{N}_m$, so

$$\widetilde{a}(\xi x) = (\widetilde{a}\xi)x$$

and $\widetilde{a} \in \mathcal{F}(E^n, E^m)$.

c) The first assertion follows from b). Assume now that E is unital and let $u \in \mathcal{F}(E^n, E^m)$. For every $i \in \mathbb{N}_m$ and $j \in \mathbb{N}_n$, put

$$a_{ij} := (u((\delta_{jk})_{k \in \mathbb{N}_n}))_i .$$

Then for every $\xi \in E^n$ and $i \in \mathbb{N}_m$,

$$(\widetilde{a}\xi)_i = \left(\sum_{j=1}^{n} a_{ij}\xi_j \right)_i = \left(\sum_{j=1}^{n} u((\delta_{jk})_{k \in \mathbb{N}_n})\xi_j \right)_i =$$

$$= \left(\sum_{j=1}^{n} u((\delta_{jk}\xi_j)_{k \in \mathbb{N}_n}) \right)_i = (u(\xi))_i ,$$

so

$$\widetilde{a} = u$$

and the map

$$E_{m,n} \longrightarrow \mathcal{F}(E^n, E^m), \quad a \longmapsto \tilde{a}$$

is surjective.

d) For every $\xi \in E^n$,

$$\widetilde{xa}\xi = \left(\sum_{j=1}^{n} x a_{ij}\xi_j\right)_{i\in\mathbb{N}_m} = x\left(\sum_{j=1}^{n} a_{ij}\xi_j\right)_{i\in\mathbb{N}_m} = x(\tilde{a}\xi) = (x\tilde{a})\xi\,.$$

e) follows from Example 2.2.7.22.

f) is easy to see.

g) By b),

$$|\tilde{a}'(a)| \leq \sum_{i=1}^{m}\sum_{j=1}^{n} |a'_{ji}(a_{ij})| \leq \sum_{i=1}^{m}\sum_{j=1}^{n} \|a'_{ji}\|\,\|a_{ij}\| \leq$$

$$\leq \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\|a'_{ji}\|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^{n}\|a_{ij}\|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \|\tilde{a}\|\sum_{i=1}^{m}\left(\sum_{j=1}^{n}\|a'_{ji}\|^2\right)^{\frac{1}{2}},$$

so $\tilde{a}' \in (E_{mn})'$ and

$$\|\tilde{a}'\| \leq \sum_{i=1}^{m}\left(\sum_{j=1}^{n}\|a'_{ji}\|^2\right)^{\frac{1}{2}}.$$

Let $\theta \in]0,1[$. For every $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$, there is a $b_{ij} \in E^{\#}$ such that

$$a'_{ji}(b_{ij}) \geq \theta\|a'_{ji}\|\,.$$

We set

$$a_{ij} = \|a'_{ji}\|b_{ij}$$

for all $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$. By b),

$$\|\tilde{a}\|^2 \leq \sum_{i=1}^{m}\sum_{j=1}^{n}\|a'_{ji}\|^2\,.$$

We get

$$\|\widetilde{a'}\| \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \|a'_{ji}\|^2 \right)^{\frac{1}{2}} \geq \|\widetilde{a'}\| \, \|\widetilde{a}\| \geq \widetilde{a'}(\widetilde{a}) =$$

$$= \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a'_{ji}(a_{ij}) \right) = \sum_{i=1}^{m} \sum_{j=1}^{n} \|a'_{ji}\| a'_{ji}(b_{ij}) \geq \theta \sum_{i=1}^{m} \sum_{j=1}^{n} \|a'_{ji}\|^2 ,$$

$$\|\widetilde{a'}\| \geq \theta \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \|a'_{ji}\|^2 \right)^{\frac{1}{2}} .$$

Since θ is arbitrary, it follows that

$$\|\widetilde{a'}\| \geq \left(\sum_{i=1}^{m} \sum_{j=1}^{n} \|a'_{ji}\|^2 \right)^{\frac{1}{2}} .$$

By Proposition 2.1.4.25, the map

$$E'_{n,m} \longrightarrow (E_{m,n})' , \quad a' \longmapsto \widetilde{a'}$$

is linear and bijective.

h) By c),f), and Example 2.2.7.22, $E_{n,n}$ is an E–algebra.

Take $b \in E_{n,n}$. Then

$$\langle b, aa' \rangle = \langle ba, a' \rangle = \sum_{i,k=1}^{n} \langle (ba)_{ik}, a'_{ki} \rangle =$$

$$= \sum_{i,k=1}^{n} \langle \sum_{j=1}^{n} b_{ij} a_{jk} \, , \, a'_{ki} \rangle =$$

$$= \sum_{i,j=1}^{n} \sum_{k=1}^{n} \langle b_{ij}, a_{jk} a'_{ki} \rangle =$$

$$= \sum_{i,j=1}^{n} \langle b_{ij}, \sum_{k=1}^{n} a_{jk} a'_{ki} \rangle ,$$

so

$$(aa')_{ij} = \sum_{k=1}^{n} a_{ik} a'_{kj}$$

for all $i, j \in \mathbb{N}_n$. Similary

$$(a'a)_{ij} = \sum_{k=1}^{n} a'_{ik} a_{ki}$$

for all $i, k \in \mathbb{N}_n$. ■

Proposition 2.2.7.25 (0) *Let E be a Banach algebra, $\mathfrak{F}$ an approximate unit of E, and F an E-submodule of E' (Proposition 2.2.7.2) such that E_F is Haussdorff.*

a) *If $\mathfrak{F}$ converges in E_F, then its limit is a unit of E and $\mathfrak{F}$ converges to this unit in norm.*

b) *If $E_F^{\#}$ is compact, then E is unital.*

a) Let x be the limit of $\mathfrak{F}$ in E_F and let $y \in E$. Then for every $x' \in F$,

$$\langle xy, x' \rangle = \langle x, yx' \rangle = \lim_{z,\mathfrak{F}} \langle z, yx' \rangle =$$

$$= \lim_{z,\mathfrak{F}} \langle zy, x' \rangle = \langle y, x' \rangle .$$

Since E_F is Hausdorff, it follows

$$xy = y .$$

Similarly,

$$yx = y .$$

Hence x is a unit of E. By Proposition 2.2.1.16, $\mathfrak{F}$ converges to x in norm.

b) Let $\mathfrak{G}$ be an ultrafilter on E finer than $\mathfrak{F}$. Since $E_F^{\#}$ is compact and $E^{\#} \in \mathfrak{G}$, $\mathfrak{G}$ converges in E_F. By a), E is unital.

Exercises

E 2.2.1 Let p be a norm on $\mathbb{K}$. Show that $\mathbb{K}$ is a normed algebra with respect to p iff there is an $\alpha \in [1, \infty[$ such that $p = \alpha|\cdot|$. It is a unital normed algebra if $\alpha = 1$.

E 2.2.2 Let E be a Banach algebra and take $x \in E$. Show that the following are equivalent:

a) $(x^n)_{n\in\mathbb{N}}$ is summable.

b) $(x^n)_{n\in\mathbb{N}}$ is absolutely summable.

c) $\|x^n\| < 1$ for some $n \in \mathbb{N}$.

E 2.2.3 Let E be a Banach algebra and a unital algebra such that for each $x \in E$ the maps

$$E \longrightarrow E, \quad y \longmapsto xy,$$

$$E \longrightarrow E, \quad y \longmapsto yx$$

are continuous. Show that there is a norm on E , which is equivalent to the initial norm, such that E is a unital Banach algebra with respect to this norm. Deduce that every finite-dimensional unital algebra F admits a norm, which renders F a unital Banach algebra.

E 2.2.4 Let E be a unital Banach algebra, U a subset of $\mathbb{K}$, and α a non-isolated point of U. Let $f, g : U \to E$ be maps which are differentiable at α. Show that fg is differentiable at α and that

$$(fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha).$$

Moreover, if

$$f(\alpha)f'(\alpha) = f'(\alpha)f(\alpha),$$

then

$$(f^n)'(\alpha) = nf(\alpha)^{n-1}f'(\alpha),$$

$$(e^f)'(\alpha) = e^{f(\alpha)}f'(\alpha)$$

for every $n \in \mathbb{N}$.

E 2.2.5 Let E_α be the unital algebra defined in E 2.1.3. Show:

a) E_α is a unital Banach algebra with respect to the norm

$$E_\alpha \longrightarrow \mathbb{R}_+, \quad (\xi, \eta) \longmapsto |\xi| + |\eta| .$$

b) Define

$$E_1 \longrightarrow \mathbb{R}_+, \quad (\xi, \eta) \longmapsto \sup\{|\xi - \eta|, |\xi + \eta|\} .$$

E_1 is a unital Banach algebra with respect to this norm and E_1, as such, is isometric to $C(\{0,1\})$.

E 2.2.6 Take $a, b \in \mathbb{R}$ with $a < b$, $n \in \mathbb{N} \cup \{0\}$, and

$$E := \{x \in C([a,b]) \mid x \text{ is } n\text{-times continuously differentiable}\}$$

Show:

a) E is a unital subalgebra of $C([a,b])$.

b) The map

$$E \longrightarrow \mathbb{R}_+, \quad x \longmapsto \sup_{t \in [a,b]} \sum_{k=0}^{n} \frac{1}{k!} |x^{(k)}(t)|$$

is a norm.

c) E is a unital Banach algebra with respect to the norm defined in b).

E 2.2.7 Let E be a normed space and a unital algebra. Suppose that the map

$$E \times E \longrightarrow E, \quad (x, y) \longmapsto xy$$

is continuous. For each $x \in E$, define

$$u_x : E \longrightarrow E, \quad y \longmapsto xy .$$

Show:

a) $u_x \in \mathcal{L}(E)$ for every $x \in E$.

b) The map

$$E \longrightarrow \mathcal{L}(E), \quad x \longmapsto u_x$$

is an injective, continuous unital algebra homomorphism.

c) The map

$$p : E \longrightarrow \mathbb{R}_+ , \quad x \longmapsto \|u_x\|$$

is a norm equivalent to the initial norm and E is a unital normed algebra with respect to p.

E 2.2.8 Let T be a group. Take $p \in [1, \infty] \cup \{0\}$, $x \in \ell^p(T)$, and define

$$\widetilde{x} : T \times T \longrightarrow \mathbb{K} , \quad (s,t) \longmapsto x(ts^{-1}) .$$

Show that $\widetilde{x} \in \ell^{\infty,p}(T,T)$ and

$$\|\widetilde{x}\| = \|x\|_p .$$

E 2.2.9 Generalize Theorem 2.2.2.10 to Hausdorff topological groups, without assuming local compactness.

E 2.2.10 Let E be a unital Banach algebra. Take $x \in E$ with $\|x\| < 1$. Prove that

$$\frac{1}{1+\|x\|} \leq \|(1-x)^{-1}\| \leq \frac{1}{1-\|x\|} .$$

E 2.2.11 Let E be a unital Banach algebra. Take $x \in E$ and

$$f : \mathbb{K} \longrightarrow E , \quad \alpha \longmapsto e^{\alpha x} .$$

Prove that f is differentiable and that

$$f' = xf = fx .$$

E 2.2.12 Let E be a Banach space. Take $u \in \mathcal{L}(E)$ and $x \in E$. Show that there is a unique differentiable map $f : \mathbb{K} \to E$ such that

$$f(0) = x , \quad f' = u \circ f .$$

E 2.2.13 Let E be a unital Banach algebra. Take $x \in E$, and $\alpha \in \mathbb{K} \backslash (\sigma(x) \cup \{0\})$. Prove: if x is invertible, then

$$\left(\frac{1}{\alpha} 1 - x^{-1}\right)^{-1} = -\alpha x(\alpha 1 - x)^{-1} .$$

E 2.2.14 Let U be the set of invertible elements of the unital Banach algebra E. Take $x \in U$ and $V := U \backslash \{x\}$. Prove that

$$\lim_{V \ni y \to x} \frac{\|y^{-1} - x^{-1} + x^{-1}(y-x)x^{-1} + x^{-1}(y-x)x^{-1}(y-x)x^{-1}\|}{\|y-x\|^2} = 0 .$$

E 2.2.15 Let E be a complex unital Banach algebra. Suppose that for some $\alpha \in \mathbb{R}_+$

$$\|x\| \, \|y\| \leq \alpha \|xy\| \, ,$$

for every $x, y \in E$. Show that E is isometric to $\mathbb{C}$.

E 2.2.16 Let E be a unital Banach algebra, F a closed unital subalgebra of E and U (resp. V) the set of invertible elements of E (resp. F). Show that the boundary of V in F is contained in the boundary of U in E.

E 2.2.17 Let F be a closed unital subalgebra of the unital Banach algebra E. Take $x \in F$. Show that if $\mathbb{K} \backslash \sigma_E(x)$ is connected, then $\sigma_E(x) = \sigma_F(x)$.

E 2.2.18 Let E be a unital Banach algebra. Show that

$$\{x' \in E \mid \|x'\| = x'(1)\}$$

is a sharp convex cone of E'.

E 2.2.19 Let E, F be Banach algebras and $u : E \to F$ an algebra homomorphism. We assume that F is unital and finite–dimensional, that

$$\|x^2\| = \|x\|^2$$

for every $x \in F$, and that the ideals of the subalgebras of F are trivial. Show that u is continuous with norm at most 1. If E is quasiunital, then $u = 0$ or $\|u\| = 1$ (generalization of Proposition 2.2.4.19).

2.3 Involutive Banach Algebras

All of Chapter IV is devoted to C^*-algebras, which are Banach algebras endowed with an involution, satisfying a very strong condition. The theory of involutive Banach algebras without this condition is of mathematical interest in its own right with applications in harmonic analysis for example.

2.3.1 Involutive Algebras

Definition 2.3.1.1 (0) *Let E be a set. An **involution on** E is a map*

$$E \longrightarrow E\,, \quad x \longmapsto x^*$$

such that

$$x^{**} = x$$

*for every $x \in E$. An **involutive space** is a set endowed with an involution.*

*Let E be an involutive space. For $x \in E$, we call x^* **the adjoint of** x. The elements of the set*

$$\mathrm{Re}\, E := \{x \in E \mid x = x^*\}$$

*are called **selfadjoint**. For $A \subset E$, define*

$$A^* := \{x^* \mid x \in A\}\,.$$

*A is called an **involutive set of** E if $A^* \subset A$ (in which case $A^* = A$).*

Let E, F be involutive spaces. Given $u \in F^E$, define

$$u^* : E \longrightarrow F\,, \quad x \longmapsto (ux^*)^*\,.$$

The map

$$F^E \longrightarrow F^E\,, \quad u \longmapsto u^*$$

*is an involution on F^E; it is called **the (canonical) involution of** F^E. The selfadjoint elements on F^E are called **involutive maps**. $u \in F^E$ is an involutive map iff*

$$ux^* = (ux)^*$$

for all $x \in E$.

Let E be an involutive space. Then $\operatorname{Re} E$ is an involutive set of E and every involutive set of E is an involutive space with respect to the induced structure.

Proposition 2.3.1.2 (0) *Let E, F be involutive spaces, A (resp. B) an involutive set of E (resp. F) and $u : E \to F$ an involutive map. Then $u(A)$ (resp. $\overset{-1}{u}(B)$) is an involutive set of F (resp. E) and $u(\operatorname{Re} E) \subset \operatorname{Re} F$.*

If $x \in A$, then

$$(ux)^* = ux^* \in u(A).$$

If $x \in \overset{-1}{u}(B)$, then

$$ux^* = (ux)^* \in B.$$

Finally, if $x \in \operatorname{Re} E$, then

$$(ux)^* = ux^* = ux. \qquad \blacksquare$$

Definition 2.3.1.3 (0) *An **involutive vector space** (resp. an **involutive algebra**) is a vector space (resp. an algebra) endowed with a conjugate linear involution, which, in the case of an algebra, satisfies*

$$(xy)^* = y^* x^*$$

*for every $x, y \in E$. An **isomorphism of involutive vector spaces (of involutive algebras)** is an isomorphism of vector spaces (of algebras), which is involutive.*

Let E be an involutive vector space. For $x \in E$, define

$$\operatorname{re} x := \frac{1}{2}(x + x^*).$$

*It is called **the real part of** x and is selfadjoint.*

Let E be an involutive algebra. The elements of the set

$$\operatorname{No} E := \{x \in E \mid x^* x = x x^*\}, \quad \operatorname{Sn} E := \left\{ x \in E \;\middle|\; \begin{array}{l} x \in \operatorname{No} E \ \text{ if } \mathbb{K} = \mathbb{C} \\ x \in \operatorname{Re} E \ \text{ if } \mathbb{K} = \mathbb{R} \end{array} \right\}$$

*are called **normal** (O. Toeplitz, 1918) and **self-normal**, respectively. If E is unital, then the elements of the set*

$$\operatorname{Un} E := \{x \in E \mid x^* x = x x^* = 1\}$$

*are called **unitary** (L. Autonne, 1902). A **conjugate involution on** E is an isomorphism of involutive real algebras $u : E \to E$ which is conjugate linear and for which u^2 is the identity map.*

If E is an involutive vector space, then from

$$0^* = 0 + 0^*$$

it follows that 0 is selfadjoint.

$\mathbb{K}$ endwoed with the involution

$$\mathbb{K} \longrightarrow \mathbb{K}, \quad x \longmapsto \overline{x}$$

is an involutive algebra.

Example 2.3.1.4 (0) *Let $(E_t)_{t\in T}$ be a family of involutive (unital) algebras. The algebra $\prod_{t\in T} E_t$ (Example 2.1.4.1) endowed with pointwise involution is an involutive (unital) algebra. If $E_t = \mathbb{K}$ for every $t \in T$ then the algebra $\mathbb{K}^T$ endowed with the involution*

$$\mathbb{K}^T \longrightarrow \mathbb{K}^T, \quad x \longmapsto \overline{x}$$

is an involutive commutative unital algebra. An element of $\mathbb{K}^T$ is selfadjoint iff it is real. The element x of $\mathbb{K}^T$ is unitary iff $|x| = 1$. $\ell^\infty(T)$ is an involutive unital subalgebra of $\mathbb{K}^T$. If T is a topological space, then $C(T)$ is an involutive unital subalgebra of $\ell^\infty(T)$. If T is a locally compact space, then $C_0(T)$ is an involutive ideal of $C(T)$. ■

Example 2.3.1.5 *The algebra $\mathbb{K}[s,t]$ (of polynomials in the variables s and t with coefficients in $\mathbb{K}$) endowed with the involution*

$$\mathbb{K}[s,t] \longrightarrow \mathbb{K}[s,t], \quad \sum_{i,j} \alpha_{ij} s^i t^j \longmapsto \sum_{i,j} \overline{\alpha_{ij}} s^j t^i$$

is an involutive commutative unital algebra. ■

Example 2.3.1.6 (0) *Let E, F be involutive vector spaces and $L(E,F)$ the vector space of linear maps of E into F. $L(E,F)$ is an involutive set of F^E. $L(E,F)$ is an involutive vector space with respect to the canonical involution.*

Take $u \in L(E,F)$, $x, y \in E$, and $\alpha, \beta \in \mathbb{K}$. Then

$$u^*(\alpha x + \beta y) = (u(\overline{\alpha} x^* + \overline{\beta} y^*))^* = (\overline{\alpha} u x^* + \overline{\beta} u y^*)^* =$$

$$= \alpha (u x^*)^* + \beta (u y^*)^* = \alpha u^* x + \beta u^* y$$

and so $u^* \in L(E,F)$.

Take $u,v \in L(E,F)$ and $\alpha,\beta \in \mathbb{K}$. Then

$$(\alpha u + \beta v)^* x = ((\alpha u + \beta v)x^*)^* = (\alpha u x^* + \beta v x^*)^* =$$

$$= \overline{\alpha}(ux^*)^* + \overline{\beta}(vx^*)^* = \overline{\alpha}u^*x + \overline{\beta}v^*x = (\overline{\alpha}u^* + \overline{\beta}v^*)x$$

for every $x \in E$, so that

$$(\alpha u + \beta v)^* = \overline{\alpha}u^* + \overline{\beta}v^*. \qquad \blacksquare$$

Proposition 2.3.1.7 (0) *If E,F are involutive vector spaces and $u : E \to F$ an involutive linear map, then*

$$u(\mathrm{re}\, x) = \mathrm{re}(ux)$$

for every $x \in E$. If, in addition, u is injective, then

$$\overset{-1}{u}(\mathrm{Re}\, F) \subset \mathrm{Re}\, E.$$

Now

$$u(\mathrm{re}\, x) = u(\frac{1}{2}(x + x^*)) = \frac{1}{2}(ux + ux^*) = \frac{1}{2}(ux + (ux)^*) = \mathrm{re}(ux).$$

Now suppose that u is injective and $x \in \overset{-1}{u}(\mathrm{Re}\, F)$. Then

$$u(\mathrm{re}\, x) = \mathrm{re}(ux) = ux,$$

so that

$$x = \mathrm{re}\, x \in \mathrm{Re}\, E. \qquad \blacksquare$$

Proposition 2.3.1.8 (0) *Let E be an involutive unital algebra. Then*

$$x^{-1} = x^*$$

for every unitary element x of E and the set of unitary elements of E is a subgroup of the multiplicative group of invertible elements of E.

If x,y are unitary elements of E, then

$$(xy)^* xy = y^* x^* xy = y^* y = 1,$$

$$(xy)(xy)^* = xyy^* x^* = xx^* = 1.$$

Hence xy is also unitary. $\blacksquare$

Proposition 2.3.1.9 (0) *Let E be a non-unital involutive algebra and $\mathbb{K} \times E$ the unital algebra associated to E. Then $\mathbb{K} \times E$ endowed with the involution*

$$\mathbb{K} \times E \longrightarrow \mathbb{K} \times E, \quad (\alpha, x) \longmapsto (\overline{\alpha}, x^*).$$

is an involutive unital algebra. It is called ***the involutive unital algebra associated to*** *E. If E is an involutive unital algebra, then* ***the involutive unital algebra associated to*** *E is E.*

It is obvious that the above involution is conjugate linear. Given (α, x), $(\beta, y) \in \mathbb{K} \times E$,

$$((\alpha.x)(\beta, y))^* = (\alpha\beta, \alpha y + \beta x + xy)^* = (\overline{\alpha}\overline{\beta}, \overline{\alpha}y^* + \overline{\beta}x^* + y^*x^*) =$$

$$= (\overline{\beta}, y^*)(\overline{\alpha}, x^*) = (\beta, y)^*(\alpha, x)^*. \qquad \blacksquare$$

Proposition 2.3.1.10 (0) *If E is an involutive vector space, then* $\operatorname{Re} E$ *is a real vector subspace of E.*

Take $x, y \in \operatorname{Re} E$ and $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha x + \beta y)^* = \alpha x + \beta y.$$

Hence $\alpha x + \beta y \in \operatorname{Re} E$. $\blacksquare$

Proposition 2.3.1.11 *If E is an involutive vector space and F a vector subspace of E, then F^* is also a vector subspace of E.*

Take $x, y \in F^*$ and $\alpha, \beta \in \mathbb{K}$. Then $x^*, y^* \in F$ and

$$\alpha x + \beta y = (\overline{\alpha}x^* + \overline{\beta}y^*)^* \in F^*. \qquad \blacksquare$$

Proposition 2.3.1.12 (0) *If E is an involutive algebra, then*

$$(x^n)^* = (x^*)^n$$

for every $x \in E$ and $n \in \mathbb{N}$.

Use proof by complete induction. $\blacksquare$

Proposition 2.3.1.13 (0) *Let E be an involutive algebra.*

a) *x^*x is selfadjoint for any $x \in E$.*

b) *If E is unital, then 1 is selfadjoint.*

c) *If $x \in E$ and*

$$xy = y$$

for every $y \in E$, then x is a unit of E.

a) We have that

$$(x^*x)^* = x^*x^{**} = x^*x\,.$$

b) $1^* = 1^*1$, so by a), $1 = 1^*$.

c) We have that

$$xx^* = x^*$$

and so

$$x^* = x$$

by a). It follows that for $y \in E$,

$$yx = (x^*y^*)^* = (xy^*)^* = y^{**} = y\,.$$

Thus x is a unit of E. ■

Proposition 2.3.1.14 (0) *Let E be an involutive unital algebra and take $x \in E$. If x is invertible, then*

$$(x^{-1})^* = (x^*)^{-1}\,.$$

In particular,

$$\sigma(x^*) = \{\overline{\alpha} \mid \alpha \in \sigma(x)\}$$

for every $x \in E$. If x is normal (selfadjoint) and invertible, then x^{-1} is also normal (selfadjoint).

It follows from

$$xx^{-1} = x^{-1}x = 1$$

that

$$(x^{-1})^*x^* = x^*(x^{-1})^* = 1^* = 1$$

(Proposition 2.3.1.13 b)), so

$$(x^{-1})^* = (x^*)^{-1}.$$

Hence, given $x \in E$,

$$\alpha \in \sigma(x^*) \Longleftrightarrow (\alpha 1 - x^* \text{ not invertible}) \Longleftrightarrow$$

$$\Longleftrightarrow (\overline{\alpha} 1 - x \text{ not invertible}) \Longleftrightarrow \overline{\alpha} \in \sigma(x).$$

If x is normal and invertible, then

$$(x^{-1})^* x^{-1} = (x^*)^{-1} x^{-1} = (xx^*)^{-1} = (x^* x)^{-1} = x^{-1}(x^*)^{-1} = x^{-1}(x^{-1})^*,$$

so that x^{-1} is normal. The corresponding assertion for selfadjoint is obvious. ■

Corollary 2.3.1.15 *The resolvent of a normal element of an involutive unital algebra is normal.*

Let x be a normal element of an involutive unital algebra and take $\alpha \in \mathbb{K} \backslash \sigma(x)$. Then $\alpha 1 - x$ is normal and by Proposition 2.3.1.14, $(\alpha 1 - x)^{-1}$ is also normal. ■

Proposition 2.3.1.16 (0) *Let E, F be involutive algebras and $u : E \to F$ an involutive algebra homomorphism. If G is an involutive subalgebra of E, then $u(G)$ is an involutive subalgebra of F. If x is a normal element of E, then ux is a normal element of F. If E and F are unital and u is an involutive unital algebra homomorphism, then ux is unitary for every unitary element x of E.* ■

Proposition 2.3.1.17 (0) *If E is an involutive algebra (resp. involutive unital algebra), F a subalgebra (resp. a unital subalgebra) of E, and G an ideal (resp. a proper ideal) of E, then the same is true for F^* and G^*.* ■

Proposition 2.3.1.18 (0) *Let E be an involutive (unital) algebra. If $(F_\iota)_{\iota \in I}$ is a nonempty family of involutive (unital) subalgebras of E, then $\bigcap_{\iota \in I} F_\iota$ is an involutive (unital) subalgebra of E. If A (resp. x) is a subset (resp. an element of E), then there is a smallest (unital) subalgebra of E containing A (resp. x), called* ***the involutive (unital) subalgebra of E generated by A*** *(resp. x). The same holds – mutatis mutandis – if we replace "(unital) algebra" and "(unital) subalgebra" by "vector space" and "vector subspace", respectively. The result is called* ***the involutive vector subspace of E generated by A*** *(resp. x).* ■

Proposition 2.3.1.19 (0) *Let E be an involutive (unital) algebra, A a subset of E, and F the involutive (unital) subalgebra generated by A. Then F is the (unital) subalgebra of E generated by $A \cup A^*$. If $A \cup A^*$ is commutative, then so is F.*

Let G be the (unital) subalgebra of E generated by $A \cup A^*$. Since

$$A^* \subset F^* \subset F,$$

it follows that $G \subset F$. G^* is a (unital) subalgebra of E (Proposition 2.3.1.17) containing $A \cup A^*$. Hence $G \subset G^*$, i.e. G is an involutive (unital) subalgebra of E. Thus $F \subset G$ and so $F = G$.

The final assertion follows from Proposition 2.1.1.17 f). ■

Proposition 2.3.1.20 (0) *Let E be an involutive (unital) algebra. Take $x \in E$ and let F be the involutive (unital) subalgebra of E generated by x. F is commutative iff x is normal, in which case the map*

$$u : G \longrightarrow E, \quad P \longmapsto P(x, x^*)$$

is an involutive (unital) algebra homomorphism and

$$F = u(G),$$

where

$$G := \{P \in \mathbb{K}[s,t] \mid P(0,0) = 0\},$$

$$(G := \mathbb{K}[s,t]).$$

By Proposition 2.3.1.19, if x is normal, then F is commutative. The converse is trivial.

Assume x to be normal. Then

$$\left(u\left(\sum_{i,j} \alpha_{ij} s^i t^j\right)\right)^* = \left(\sum_{i,j} \alpha_{ij} x^i x^{*j}\right)^* = \left(\sum_{i,j} \alpha_{ij} x^i x^{j*}\right)^* =$$

$$= \sum_{i,j} \overline{\alpha_{ij}} x^{i*} x^j = u\left(\sum_{i,j} \overline{\alpha_{ij}} s^j t^i\right) = u\left(\left(\sum_{i,j} \alpha_{ij} s^i t^j\right)^*\right)$$

for every $\sum_{i,j} \alpha_{ij} s^i t^j \in \mathbb{K}[s,t]$ (Proposition 2.3.1.12 and 2.3.1.13 b)). Hence u is an involutive (unital) algebra homomorphism (Proposition 2.1.1.19). By Proposition 2.3.1.16, $u(G)$ is an involutive (unital) subalgebra of E and so

$$F \subset u(G).$$

The reverse inclusion is trivial. ■

Proposition 2.3.1.21 (0) *Let E be an involutive vector space, F an involutive vector subspace of E, and $q : E \to E/F$ the quotient map.*

a) $X \in E/F \Rightarrow X^* \in E/F$.

b) *E/F endowed with the involution*

$$E/F \longrightarrow E/F, \quad X \longmapsto X^*$$

is an involutive vector space and q is involutive.

c) *If E is an involutive (unital) algebra and F is an involutive (proper) ideal of E, then E/F is an involutive (unital) algebra (Proposition 2.1.1.13).*

a) Take $x, y \in X^*$. Then $x^*, y^* \in X$, so that $x^* - y^* \in F$. We get

$$x - y = (x - y)^{**} = (x^* - y^*)^* \in F^* = F .$$

Now take $z \in E$ with $x - z \in F$. Then

$$x^* - z^* \in F^* = F ,$$

$$z^* \in X ,$$

$$z \in X^* .$$

Hence $X^* \in E/F$.

b) Choose $X, Y \in E/F$ and $\alpha, \beta \in \mathbb{K}$. Further, take $x \in X$, $y \in Y$. Then $\alpha x + \beta y \in \alpha X + \beta Y$, $x^* \in X^*$, $y^* \in Y^*$, so

$$\overline{\alpha} X^* + \overline{\beta} Y^* \ni \overline{\alpha} x^* + \overline{\beta} y^* = (\alpha x + \beta y)^* \in (\alpha X + \beta Y)^* .$$

Hence

$$(\alpha X + \beta Y)^* = \overline{\alpha} X^* + \beta Y^* .$$

c) Let $X, Y \in E/F$ and $x \in X$, $y \in Y$. Then

$$xy \in XY , \; x^* \in X^* , \; y^* \in Y^* ,$$

so that

$$Y^* X^* \ni y^* x^* = (xy)^* \in (XY)^*$$

$$(XY)^* = Y^* X^* . \qquad \blacksquare$$

Proposition 2.3.1.22 (0) *Let E be an involutive complex vector space and take $x \in E$. Then there are $y, z \in \operatorname{Re} E$ uniquely such that*

$$x := y + iz .$$

Define

$$\operatorname{im} x := z$$

and call it ***the imaginary part of*** *x. Then*

$$y = \operatorname{re} x = \frac{1}{2}(x + x^*), \quad \operatorname{im} x = \frac{1}{2i}(x - x^*),$$

$$\operatorname{re} x = \operatorname{im}(ix), \quad \operatorname{im} x = -\operatorname{re}(ix),$$

and the map

$$E \longrightarrow \operatorname{Re} E, \quad x \longmapsto \operatorname{re} x \ \textit{(resp.}\ \operatorname{im} x)$$

is $\mathbb{R}$-linear.

Put

$$y := \frac{1}{2}(x + x^*), \quad z := \frac{1}{2i}(x - x^*) .$$

Then

$$y^* = \frac{1}{2}(x^* + x) = \frac{1}{2}(x + x^*) = y,$$

$$z^* = -\frac{1}{2i}(x^* - x) = \frac{1}{2i}(x - x^*) = z,$$

$$y + iz = \frac{1}{2}(x + x^*) + \frac{1}{2}(x - x^*) = x$$

which proves the existence.

Now take $y, z \in \operatorname{Re} E$ with

$$x = y + iz .$$

Then

$$x^* = y - iz,$$

$$\frac{1}{2}(x^* + x) = y$$

and

$$\frac{1}{2i}(x^* - x) = z,$$

proving the uniqueness.

The other assertions are trivial. ■

Corollary 2.3.1.23 (0) *Let $u : E \to F$ be a linear map between the involutive complex vector spaces E and F. Then the following are equivalent:*

a) *u is involutive.*

b) $u(\operatorname{Re} E) \subset \operatorname{Re} F$.

c) $u(\operatorname{re} x) = \operatorname{re}(ux)$, $u(\operatorname{im} x) = \operatorname{im}(ux)$ *for every* $x \in E$.

d) $u(\operatorname{re} x) = \operatorname{re}(ux)$ *for every* $x \in E$.

a $\Rightarrow$ b follows from Proposition 2.3.1.2.
b $\Rightarrow$ c. Now

$$ux = u(\operatorname{re} x + i \operatorname{im} x) = u(\operatorname{re} x) + iu(\operatorname{im} x).$$

By b), $u(\operatorname{re} x)$, $u(\operatorname{im} x) \in \operatorname{Re} F$ so that

$$\operatorname{re}(ux) = u(\operatorname{re} x), \quad \operatorname{im}(ux) = u(\operatorname{im} x).$$

c $\Rightarrow$ d is trivial.
d $\Rightarrow$ c. By Proposition 2.3.1.22,

$$u(\operatorname{im} x) = u(-\operatorname{re}(ix)) = -\operatorname{re}(u(ix)) = -\operatorname{re}(iux) = \operatorname{im}(ux).$$

c $\Rightarrow$ a. Given $x \in E$,

$$ux^* = u(\operatorname{re} x - i \operatorname{im} x) = u(\operatorname{re} x) - iu(\operatorname{im} x) =$$

$$= \operatorname{re}(ux) - i \operatorname{im}(ux) = (ux)^*. \qquad \blacksquare$$

Corollary 2.3.1.24 (0) *Let E be an involutive complex algebra. Take $x \in E$ and put*

$$y := \operatorname{re} x, \quad z := \operatorname{im} x.$$

a) $x \in \operatorname{Re} E \Leftrightarrow z = 0$.

b) $x^*x = y^2 + z^2 + i(yz - zy)$, $xx^* = y^2 + z^2 - i(yz - zy)$.

c) $x \in \operatorname{No} E \Leftrightarrow yz = zy \Leftrightarrow x^*x = y^2 + z^2$.

d) *If E is unital, then*

$$x \text{ unitary} \Leftrightarrow (yz = zy \text{ and } y^2 + z^2 = 1).$$

a) The assertion follows from

$$z = \frac{1}{2i}(x - x^*) .$$

b)

$$x^*x = (y - iz)(y + iz) = y^2 + z^2 + i(yz - zy) ,$$

$$xx^* = (y + iz)(y - iz) = y^2 + z^2 - i(yz - zy) .$$

c) and d) follow from b). ■

Remark. The following calculation would be false:

$$x^*x = (x^*x)^* = y^2 + z^2 - i(yz - zy) = xx^* ,$$

i.e. all elements are normal! The error arises from the fact that $yz - zy$ is not selfadjoint.

Corollary 2.3.1.25 *If all the elements of the involutive complex algebra E are normal, then E is commutative.*

By Corollary 2.3.1.24 c), $\operatorname{Re} E$ is commutative. Hence $\dot{E}$ is commutative (Proposition 2.3.1.22). ■

Definition 2.3.1.26 (0) *An involutive algebra E is called* ***symmetric (strongly symmetric)*** *if*

$$\sigma(x^*x) \subset \mathbb{R}_+$$

for every $x \in \operatorname{No} E$ ($x \in E$).

If E is commutative, then the notions "symmetric" and "strongly symmetric" coincide.

Example 2.3.1.27 *With respect to the involution*

$$\mathbb{K}^2 \longrightarrow \mathbb{K}^2 , \quad (\alpha, \beta) \longmapsto (\overline{\beta}, \overline{\alpha}) ,$$

the commutative unital algebra $\mathbb{K}^2$ (pointwise multiplication) is an involutive algebra, which is not symmetric.

Given $(\alpha, \beta) \in \mathbb{K}^2$,

$$\sigma((\alpha,\beta)^*(\alpha,\beta)) = \sigma((\alpha\overline{\beta}, \overline{\alpha}\beta)) = \{\alpha\overline{\beta}, \overline{\alpha}\beta\}$$ ■

Remark. Another example of a non–symmetric involutive algebra is provided in Example 2.4.3.7 e).

Proposition 2.3.1.28 (0) *Let E be a (strongly) symmetric involutive algebra. Then every involutive ideal of E is (strongly) symmetric.*

The assertion follows immediately from Corollary 2.1.3.15. ■

Proposition 2.3.1.29 (0) *If E is a symmetric involutive algebra then*

$$\sigma(x) \subset \mathbb{R},$$

for every $x \in \operatorname{Re} E$.

Take $\alpha \in \sigma(x)$ and let F be the involutive unital algebra associated to E. Then $\alpha 1 - x$ is not invertible in F. Since

$$\alpha^2 1 - x^2 = (\alpha 1 - x)(\alpha 1 + x),$$

it follows that $\alpha^2 1 - x^2$ is not invertible in F (Corollary 2.1.2.6). Thus

$$\alpha^2 \in \sigma_F(x^2) = \sigma_E(x^*x) \subset \mathbb{R}_+.$$

Hence $\alpha \in \mathbb{R}$ and

$$\sigma(x) \subset \mathbb{R}.$$ ■

Definition 2.3.1.30 (0) *Let E be an involutive vector space and take $m, n \in \mathbb{N}$. We denote for every $m \times n$–matrix a with entries from E (i.e. for every $a \in E_{m,n}$) by a^* the $n \times m$–matrix with entries from E (i.e. $a^* \in E_{n,m}$) defined by*

$$(a^*)_{ij} = (a_{ji})^*$$

for all $i \in \mathbb{N}_n$ and $j \in \mathbb{N}_m$.

Proposition 2.3.1.31 (0) *Let E be an involutive (unital) algebra and take $m, n, p \in \mathbb{N}$.*

a) *The map*

$$E_{m,n} \longrightarrow E_{n,m}\,, \quad a \longmapsto a^*$$

is conjugate linear.

b) $a \in E_{m,n} \Rightarrow a^{**} = a$.

c) $a \in E_{m,n}\,,\ b \in E_{n,p} \Rightarrow (ab)^* = b^*a^*$.

d) *The algebra* $E_{n,n}$ *(Proposition 2.1.4.24 b)) endowed with the involution*

$$E_{n,n} \longrightarrow E_{n,n}\,, \quad a \longmapsto a^*$$

is an involutive algebra.

e) *If* $(E_t)_{t\in T}$ *is a family of involutive (unital) algebras then for every* $n \in \mathbb{N}$ *the map*

$$\prod_{t\in T}(E_t)_{n,n} \longrightarrow \left(\prod_{t\in T} E_t\right)_{n,n}\,,$$

$$([x_{t,j,k}]_{j,k\in\mathbb{N}_n})_{t\in T} \longmapsto [(x_{t,j,k})_{t\in T}]_{j,k\in\mathbb{N}_n}$$

is an isomorphism of involutive (unital) algebras. ■

Example 2.3.1.32 (0) *Let* $n \in \mathbb{N}$. $\mathbb{K}_{n,n}$ *is a strongly symmetric involutive unital algebra (Proposition 2.1.4.24 b),c) and 2.3.1.31 d)). In particular, for every* $a \in \mathbb{K}_{n,n}$ *with*

$$a_{ij} = \overline{a}_{ji}$$

for all $i, j \in \mathbb{N}_n$ *the zeros of the polynomial*

$$\mathrm{Det}[a_{ij} - \delta_{ij}t]$$

are real.

By Example 1.2.2.7 a) (and Example 2.1.4.10 a)), $\mathbb{K}_{n,n}$ is strongly symmetric. The last assertion follows from Proposition 2.3.1.29. ■

Example 2.3.1.33 *Take $n \in \mathbb{N}$ and define*

$$\widetilde{a} : \mathbb{K}_{n,n} \longrightarrow \mathbb{K}, \quad [\alpha_{ij}] \longmapsto \sum_{i,j=1}^{n} \alpha_{ij} a_{ji}$$

for $a := [a_{ij}] \in \mathbb{K}_{n,n}$ (Example 2.3.1.32). Then $\widetilde{a} \in (\mathbb{K}_{n,n})'$, whenever $a \in \mathbb{K}_{n,n}$ and the map

$$\mathbb{K}_{n,n} \longrightarrow (\mathbb{K}_{n,n})', \quad a \longmapsto \widetilde{a}$$

is an isomorphism of involutive vector spaces (Example 2.3.1.6), where $(\mathbb{K}_{n,n})'$ denotes the algebraic dual of $\mathbb{K}_{n,n}$.

Let $\alpha := [\alpha_{ij}] \in \mathbb{K}_{n,n}$. Then

$$\langle [\alpha_{ij}], \widetilde{a}^* \rangle = \langle \overline{[\alpha_{ij}]^*}, \widetilde{a} \rangle = \sum_{i,j=1}^{n} \alpha_{ji} \overline{a_{ji}} = \langle [\alpha_{ij}], \widetilde{a^*} \rangle$$

for every $[\alpha_{ij}] \in \mathbb{K}_{n,n}$. Hence

$$\widetilde{a}^* = \widetilde{a^*}$$

and the map

$$\mathbb{K}_{n,n} \longrightarrow (\mathbb{K}_{n,n})', \quad a \longmapsto \widetilde{a}$$

is an isomorphism of involutive vector spaces. ■

Example 2.3.1.34 *The matrix*

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

is a normal element of the involutive algebra $\mathbb{K}_{2,2}$ (Example 2.3.1.32) iff

$$|\beta| = |\gamma|, \quad \overline{\gamma}(\alpha - \delta) = \beta(\overline{\alpha} - \overline{\delta}).$$

x is an idempotent selfadjoint element of $\mathbb{K}_{2,2}$ (Example 2.3.1.30) iff either

$$\beta = \gamma = 0, \quad \alpha, \delta \in \{0, 1\}$$

or

$$\alpha, \delta \in \mathbb{R}_+, \quad \alpha + \delta = 1, \quad \beta = \overline{\gamma}, \quad |\beta|^2 = \alpha\delta.$$

Note that

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{bmatrix} = \begin{bmatrix} |\alpha|^2 + |\beta|^2 & \alpha\overline{\gamma} + \beta\overline{\delta} \\ \overline{\alpha}\gamma + \overline{\beta}\delta & |\gamma|^2 + |\delta|^2 \end{bmatrix},$$

$$\begin{bmatrix} \overline{\alpha} & \overline{\gamma} \\ \overline{\beta} & \overline{\delta} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} |\alpha|^2 + |\gamma|^2 & \beta\overline{\alpha} + \delta\overline{\gamma} \\ \alpha\overline{\beta} + \gamma\overline{\delta} & |\beta|^2 + |\delta|^2 \end{bmatrix}.$$

The first assertion now follows.

For the second assertion assume first that the matrix is idempotent and selfadjoint. Then

$$\alpha, \delta \in \mathbb{R}, \quad \beta = \overline{\gamma}.$$

Moreover,

$$\begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \overline{\beta} & \delta \end{bmatrix}^2 = \begin{bmatrix} \alpha^2 + |\beta|^2 & \beta(\alpha + \delta) \\ \overline{\beta}(\alpha + \delta) & |\beta|^2 + \delta^2 \end{bmatrix}.$$

Thus

$$\begin{cases} \alpha = \alpha^2 + |\beta|^2 \\ \beta = \beta(\alpha + \delta) \\ \delta = \delta^2 + |\beta|^2. \end{cases}$$

If $\beta = 0$, then $\alpha, \delta \in \{0, 1\}$. If $\beta \neq 0$, then

$$\alpha, \delta \in \mathbb{R}_+, \quad \alpha + \delta = 1,$$

$$|\beta|^2 = \alpha - \alpha^2 = \alpha(1 - \alpha) = \alpha\delta.$$

The converse is easy to see. ■

Lemma 2.3.1.35 *Take $n \in \mathbb{N}$ and define*

$$\langle x|y\rangle := \sum_{i=1}^{n} x_i \overline{y}_i$$

for $x, y \in \mathbb{K}^n$. If E is a vector subspace of $\mathbb{K}^n$, then there is a finite subset A of E such that

$$\langle x|y\rangle = \delta_{x,y}$$

for every $x, y \in A$ *and*

$$x = \sum_{y\in A}\langle x|y\rangle y$$

for every $x \in E$. A *is an algebraic basis of* E.

Let $\mathfrak{A}$ be the set of subsets A of E such that

$$\langle x|y\rangle = \delta_{x,y}$$

whenever $x, y \in A$ and let A be a maximal element of $\mathfrak{A}$ with respect to inclusion. Since A is linearly independent, it is finite. Take $x \in E$ and let

$$z := x - \sum_{y\in A}\langle x|y\rangle y\,.$$

If $z \neq 0$, then

$$\langle z|z\rangle > 0$$

and

$$A \cup \left\{\frac{1}{\langle z|z\rangle^{\frac{1}{2}}} z\right\} \in \mathfrak{A}$$

which contradicts the maximality of A. Hence $z = 0$,

$$x = \sum_{y\in A}\langle x|y\rangle y$$

and A is an algebraic basis of E. ■

Example 2.3.1.36 *Take* $n \in \mathbb{N}$ *and define*

$$\langle x|y\rangle := \sum_{i=1}^{n} x_i\overline{y}_i\,,$$

$$[x|y] := \left[x_i\overline{y}_j\right]_{i,j\in\mathbb{N}_n} \in \mathbb{K}_{n,n}$$

for $x, y \in \mathbb{K}^n$. *A matrix* u *is an idempotent and selfadjoint element of* $\mathbb{K}_{n,n}$ *(Example 2.3.1.32) iff there is a finite subset* A *of* $\mathbb{K}^n$ *such that*

$$\langle x|y\rangle = \delta_{x,y}$$

whenever $x, y \in A$, *and*

$$u = \sum_{x\in A}[x|x]\,.$$

Assume u is idempontent and selfadjoint. Put $E := \operatorname{Im} u$. By Lemma 2.3.1.35, there is a finite subset A of E such that

$$\langle x|y\rangle = \delta_{x,y}$$

for every $x, y \in A$ and

$$y = \sum_{x\in A}\langle y|x\rangle x$$

for every $y \in E$. Hence

$$uy = y = \sum_{x\in A}\langle y|x\rangle x = \sum_{x\in A}[x|x]y$$

for every $y \in E$.

Take $y \in \mathbb{K}^n$. Then $uy \in E$ and

$$\langle y|x\rangle = \langle y|ux\rangle = \sum_{i=1}^{n} y_i\overline{(ux)_i} = \sum_{i=1}^{n} y_i \sum_{j=1}^{n} \overline{u_{i,j}}\overline{x}_j =$$

$$= \sum_{i=1}^{n} y_i \sum_{j=1}^{n} u_{j,i}\overline{x}_j = \sum_{j=1}^{n}\left(\sum_{i=1}^{n} u_{j,i}y_i\right)\overline{x}_j = \sum_{j=1}^{n}(uy)_j\overline{x}_j = \langle uy|x\rangle$$

for every $x \in E$, so that

$$\sum_{x\in A}[x|x]y = \sum_{x\in A}\langle y|x\rangle x = \sum_{x\in A}\langle uy|x\rangle x = \sum_{x\in A}[x|x]uy = uuy = uy.$$

Hence

$$u = \sum_{x\in A}[x|x].$$

Now assume that there is a finite subset A of $\mathbb{K}^n$ such that

$$\langle x|y\rangle = \delta_{x,y}$$

for every $x, y \in A$ and

$$u = \sum_{x\in A}[x|x].$$

Since $[x|x]$ is selfadjoint for every $x \in \mathbb{K}^n$, u is also selfadjoint. Since

$$[x|x]\,[y \mid y] = \delta_{x,y}[x \mid x]$$

for every $x, y \in A$, it follows

$$u^2 = u. \qquad \blacksquare$$

Proposition 2.3.1.37 *Let E be a strongly symmetric involutive algebra and u a surjective involutive algebra homomorphism of E into an involutive algebra F. Then F is also strongly symmetric.*

Take $y \in F$ and $x \in u^{-1}(y)$. Then

$$u(x^*x) = (ux^*)(ux) = y^*y\,,$$

so that

$$\sigma_F(y^*y) \subset \sigma_E(x^*x) \cup \{0\} \subset \mathbb{R}_+$$

(Corollary 2.1.3.12). Hence F is strongly symmetric. ■

Proposition 2.3.1.38 (0) *Let E be an involutive vector space and $\overset{\circ}{E}$ the complexification of the underlying vector space of E (Lemma 2.1.5.12)). Then $\overset{\circ}{E}$ endowed with the involution*

$$\overset{\circ}{E} \longrightarrow \overset{\circ}{E}\,, \quad (x,y) \longmapsto (x^*, -y^*)$$

is an involutive complex vector space called ***the complexification of*** *E.*

Let $(a,b),(c,d) \in \overset{\circ}{E}$ and $\alpha,\beta \in \mathbb{R}$. We have

$$((a,b)+(c,d))^* = (a+c, b+d)^* = (a^*+c^*, -b^*-d^*) =$$

$$= (a^*,-b^*) + (c^*,-d^*) = (a,b)^* + (c,d)^*\,,$$

$$((\alpha+i\beta)(a,b))^* = (\alpha a - \beta b, \beta a + \alpha b)^* = (\alpha a^* - \beta b^*, -\beta a^* - \alpha b^*) =$$

$$= (\alpha - i\beta)(a^*,-b^*) = \overline{\alpha+i\beta}(a,b)^*\,.$$

■

Proposition 2.3.1.39 (0) *Let E,F be involutive real vector spaces, $L(E,F)$ the involutive real space of the linear maps of E into F (Example 2.3.1.6), $\overset{\circ}{E}$, $\overset{\circ}{F}$, and $\overset{\circ}{L}(E,F)$ the complexifications of E,F, and $L(E,F)$ respectively, and $L(\overset{\circ}{E},\overset{\circ}{F})$ the involutive complex vector space of the linear maps of $\overset{\circ}{E}$ into $\overset{\circ}{F}$ (Example 2.3.1.6). For $(u,v) \in \overset{\circ}{L}(E,F)$ define*

$$\widetilde{(u,v)} : \overset{\circ}{E} \longrightarrow \overset{\circ}{F}\,, \quad (x,y) \longmapsto (ux - vy, uy + vx)\,.$$

Then the linear bijective map

$$\overset{\circ}{L}(E,F) \longrightarrow L(\overset{\circ}{E},\overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

(Lemma 2.1.5.3) is involutive. (We may thus identify $\overset{\circ}{L}(E,F)$ *and* $L(\overset{\circ}{E},\overset{\circ}{E})$*). If* $(u,v) \in \overset{\circ}{L}(E,F)$*, then* $\widetilde{(u,v)}$ *is selfadjoint iff* $u = u^*, v = -v^*$.

If $\overset{\circ}{E}$ *and* $\overset{\circ}{F}$ *are endowed with norms such that the canonical bijections*

$$\overset{\circ}{E} \longrightarrow E \times E, \quad \overset{\circ}{F} \longrightarrow F \times F$$

are isomorphisms of real normed spaces, then we may replace L *by* $\mathcal{L}$ *in the above enunciation.*

Choose $(u,v) \in \overset{\circ}{L}(E,F)$. Then, for every $(x,y) \in \overset{\circ}{E}$,

$$\left(\widetilde{(u,v)}\right)^*((x,y)) = \left(\widetilde{(u,v)}((x,y)^*)\right)^* = \left(\widetilde{(u,v)}((x^*,-y^*))\right)^* =$$

$$= (ux^* + vy^*, -uy^* + vx^*)^* = ((ux^*)^* + (vy^*)^*, (uy^*)^* - (vx^*)^*) =$$

$$= (u^*x + v^*y, u^*y - v^*x) = \widetilde{(u^*,-v^*)}((x,y)) = \widetilde{(u,v)}^*((x,y)).$$

Thus

$$\left(\widetilde{(u,v)}\right)^* = \widetilde{(u,v)}^*.$$

Hence the map

$$\overset{\circ}{L}(E,F) \longrightarrow L(\overset{\circ}{E},\overset{\circ}{F}), \quad (u,v) \longmapsto \widetilde{(u,v)}$$

is involutive. In particular, $\widetilde{(u,v)}$ is selfadjoint iff

$$(u^*,-v^*) = (u,v)$$

i.e. iff $u = u^*$, $v = -v^*$.

The last assertion follows from Proposition 2.1.5.6. ■

Proposition 2.3.1.40 (0) *If* E *is an involutive real algebra, then the complex algebra* $\overset{\circ}{E}$ *(Proposition 2.1.5.7) endowed with the involution*

$$\overset{\circ}{E} \longrightarrow \overset{\circ}{E}, \quad (x,y) \longmapsto (x^*,-y^*)$$

is an involutive complex algebra called the **complexification of** E. *If* $\overset{\circ}{E}$ *is (strongly) symmetric, then* E *is also (strongly) symmetric.*

If F *is an involutive subalgebra of* E, *then* $F \times F$ *is an involutive subalgebra of* $\overset{\circ}{E}$ *and is therefore a complexification of* F.

For every $n \in \mathbb{N}$ *the map*

$$\widehat{\overset{\circ}{E}_{n,n}} \longrightarrow \left(\overset{\circ}{E}\right)_{n,n}, \quad (x,y) \longmapsto x + iy$$

is an isomorphism of involutive complex algebras.

If $(a,b),(c,d) \in \overset{\circ}{E}$, then

$$((a,b)(c,d))^* = (ac - bd, ad + bc)^* = (c^*a^* - d^*b^*, -d^*a^* - c^*b^*) =$$

$$= (c^*, -d^*)(a^*, -b^*) = (c,d)^*(a,b)^*.$$

This proves the first assertion (Proposition 2.3.1.38). By Corollary 2.1.5.14, if $\overset{\circ}{E}$ is (strongly) symmetric, then E is also (strongly) symmetric. The last assertions follow easily from Propositions 2.1.5.10 and 2.3.1.31 d). ■

Proposition 2.3.1.41 (0) *Let* E, F *be involutive real algebras,* $\overset{\circ}{E}, \overset{\circ}{F}$ *their complexifications, and* $u : E \to F$ *an involutive algebra homomorphism. Then*

$$\overset{\circ}{u} : \overset{\circ}{E} \longrightarrow \overset{\circ}{F}, \quad (x,y) \longmapsto (ux, uy)$$

is an involutive algebra homomorphism, surjective iff u *is surjective.*

We have

$$\overset{\circ}{u}((x,y)^*) = \overset{\circ}{u}((x^*, -y^*)) = (ux^*, -uy^*) =$$

$$= ((ux)^*, -(uy)^*) = (ux, uy)^* = (\overset{\circ}{u}((x,y)))^*$$

for every $(x,y) \in \overset{\circ}{E}$. Thus $\overset{\circ}{u}$ is involutive. The other assertions follow from Proposition 2.1.5.11. ■

Proposition 2.3.1.42 (0) *Let* E *be an involutive algebra and* F *an involutive ideal of* E. *Then the map*

$$\widehat{\overset{\circ}{E/F}} \longrightarrow \overset{\circ}{E}/\overset{\circ}{F}$$

defined in Proposition 2.1.5.10 c) is an isomorphism of involutive algebras. In particular, $\widehat{\overset{\circ}{E/F}}$ *and* $\overset{\circ}{E}/\overset{\circ}{F}$ *may be identified.*

The proof is by direct verification. ■

Proposition 2.3.1.43 (0) *Let E be an involutive complex algebra and $\mathcal{U}$ the set of conjugate involutions on E. Given $u \in \mathcal{U}$, define*

$$F_u := \{z \in E | uz = z\}.$$

Further, let $\mathcal{F}$ be the set of involutive real subalgebras F of E for which there is an isomorphism of involutive complex algebras $v : \overset{\circ}{F} \longrightarrow E$ such that

$$v((x,0)) = x$$

for every $x \in F$.

a) *If $u \in \mathcal{U}$, then $F_u \in \mathcal{F}$ and the map*

$$\overset{\circ}{F_u} \longrightarrow E, \quad (x,y) \longmapsto x + iy$$

is the isomorphism appearing in the definition of $\mathcal{F}$.

b) *The map*

$$\mathcal{U} \longrightarrow \mathcal{F}, \quad u \longmapsto F_u$$

is bijective.

c) *If G is an involutive real algebra such that $\overset{\circ}{G}$ is isomorph to E, then G is isomorph to an $F \in \mathcal{F}$.*

d) *For every $u \in \mathcal{U}$, the structure of F_u may be extended to that of an involutive complex algebra iff there is a linear map $v : E \to E$, such that*

$$v^2 = -1, \quad uv = vu$$

and

$$v(xy) = (vx)y = x(vy), \quad vx^* = -(vx)^*$$

for all $x, y \in E$; in this case the complex structure on F_u is defined by the map

$$\mathbb{C} \times F_u \longrightarrow F_u, \quad (\alpha + i\beta, x) \longmapsto \alpha x + \beta vx,.$$

e) *For all* $u, v \in \mathcal{U}$, *the involutive real algebras* F_u *and* F_v *are isomorphic iff there is an isomorphism of involutive complex algebras* $w : E \to E$ *such that*

$$w \circ u = v \circ w .$$

a) It is easy to see that F_u is an involutive real subalgebra of E. Define

$$v : \overset{\circ}{F}_u \longrightarrow E , \quad (x, y) \longmapsto x + iy ,$$

and take $(x, y) \in \overset{\circ}{F}_u$. Then

$$v(i(x, y)) = v((-y, x)) = -y + ix = i(x + iy) = iv((x, y)) ,$$

$$v((x, y)^*) = v((x^*, -y^*)) = x^* - iy^* = (x + iy)^* = v((x, y))^* .$$

Hence v is linear and involutive. If

$$v((x, y)) = 0 ,$$

then

$$x + iy = 0$$

and

$$x - iy = ux - iuy = u(x + iy) = 0 ,$$

so that $(x, y) = 0$, whence v is injective.

If (x_1, y_1), $(x_2, y_2) \in \overset{\circ}{F}_u$, then

$$v((x_1, y_1)(x_2, y_2)) = v((x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)) =$$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) = (x_1 + iy_1)(x_2 + iy_2) = v((x_1, y_1))v((x_2, y_2)) .$$

Hence v is an injective homomorphism of involutive complex algebras.

Finally, we show that v is surjective. Let $z \in E$. Put

$$\begin{cases} x := \frac{1}{2}(z + uz) \\ y := \frac{1}{2i}(z - uz) . \end{cases}$$

Then $x, y \in F_u$ and

$$v((x,y)) = x + iy = z\,.$$

b) Take $u, v \in \mathcal{U}$ such that $F_u = F_v$. Take $z \in E$ and put

$$\begin{cases} x := \frac{1}{2}(z + uz) \\ y := \frac{1}{2i}(z - uz)\,. \end{cases}$$

Then $x, y \in F_u$ and $z = x + iy$. We have

$$uz = ux - iuy = x - iy = vx - ivy = vz\,.$$

Hence $u = v$ and the map

$$\mathcal{U} \longrightarrow \mathcal{F}, \quad u \longmapsto F_u$$

is injective.

Take $F \in \mathcal{F}$. We identify $\overset{\circ}{F}$ and E using the isomorphism $\overset{\circ}{F} \to E$ described in the definition of $\mathcal{F}$. Define

$$u : E \longrightarrow E, \quad (x,y) \longmapsto (x,-y)\,.$$

Then for any $(x,y) \in E$

$$u(i(x,y)) = u((-y,x)) = (-y,-x) = -i(x,-y) = -iu((x,y))\,,$$

$$u((x,y)^*) = u((x^*,-y^*)) = (x^*,y^*) = (x,-y)^* = u((x,y))^*\,,$$

so that u is involutive and conjugate linear. If $(x_1,y_1)(x_2,y_2) \in E$, then

$$u((x_1,y_1)(x_2,y_2)) = u((x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)) =$$

$$= (x_1x_2 - y_1y_2, -x_1y_2 - y_1x_2) = (x_1,-y_1)(x_2,-y_2) = u((x_1,y_1))u((x_2,y_2))\,.$$

Since u is bijective it is an isomorphism of real algebras. Moreover, $u^2 = 1$. Hence $u \in \mathcal{U}$. Since $F = F_u$ the map

$$\mathcal{U} \longrightarrow \mathcal{F}, \quad u \longmapsto F_u$$

is surjective.

c) Let $v : \overset{\circ}{G} \to E$ be an isomorphism of involutive algebras and put

$$F := v(G \times \{0\})\,.$$

Then $F \in \mathcal{F}$ and the map

$$G \longrightarrow F, \quad x \longmapsto v((x,0))$$

is an isomorphism of involutive algebras.

d) We identify $\overset{\circ}{F}_u$ with E via the isomorphism

$$\overset{\circ}{F}_u \longrightarrow E, \quad (x,y) \longmapsto x + iy.$$

First assume that the structure of F_u may be extended to that of an involutive complex algebra. Consider

$$v : \overset{\circ}{F}_u \longrightarrow \overset{\circ}{F}_u, \quad (x,y) \longmapsto (ix, iy).$$

Take $(x,y) \in \overset{\circ}{F}_u$. Then

$$v(i(x,y)) = v((-y,x)) = (-iy, ix) = i(ix, iy) = iv((x,y)),$$

$$v((x,y)^*) = v((x^*, -y^*)) = (ix^*, -iy^*) = -((ix)^*, -(iy)^*) =$$

$$= -(ix, iy)^* = -v((x,y))^*,$$

$$v^2((x,y)) = v((ix, iy)) = (i^2x, i^2y) = -(x,y),$$

$$vu((x,y)) = v((x,-y)) = (ix, -iy) = u((ix, iy)) = uv(x,y)).$$

Take $(x_1,y_1), (x_2,y_2) \in \overset{\circ}{F}_u$. Then

$$(v((x_1,y_1))(x_2,y_2) = (ix_1, iy_1)(x_2,y_2) = (ix_1x_2 - iy_1y_2, ix_1y_2 + iy_1x_2) =$$

$$= v((x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)) = v((x_1,y_1)(x_2,y_2)),$$

$$(x_1,y_1)v((x_1,y_2)) = (x_1,y_1)(ix_2, iy_2) = (ix_1x_2 - iy_1y_2, ix_1y_2 + iy_1x_2) =$$

$$= v((x_1,y_1)(x_2,y_2)).$$

Assume next that there is a v with the above properties. Take $x \in F_u$. Then

$$uvx = vux = vx,$$

so that $vx \in F_u$. We endow F_u with the map

$$\mathbb{C} \times F_u \longrightarrow F_u, \quad (\alpha + \imath\beta, x) \longmapsto \alpha x + \beta v x.$$

Take $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $x \in F_u$. Then

$$((\alpha + i\beta)(\gamma + i\delta))x = ((\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma))x =$$

$$= (\alpha\gamma - \beta\delta)x + (\alpha\delta + \beta\gamma)vx = \alpha\gamma x + \beta\delta v^2 x + \alpha\delta vx + \beta\gamma vx =$$

$$= \alpha(\gamma x + \delta vx) + \beta v(\gamma x + \delta vx) = (\alpha + \imath\beta)((\gamma + i\delta)x),$$

$$(ix)^* = (vx)^* = -vx^* = -\imath x^*.$$

Thus F_u is an involutive complex vector space. If $x, y \in F_u$ then

$$(ix)y = (vx)y = v(xy) = i(xy),$$

$$x(iy) = xvy = v(xy) = i(xy).$$

hence F_u is an involutive complex algebra.

e) Let $w_0 : F_u \to F_v$ be an isomorphism of involutive real algebras. Then

$$w : \overset{\circ}{F}_u \longrightarrow \overset{\circ}{F}_v, \quad (x, y) \longmapsto (w_0 x, w_0 y)$$

is an isomorphism of involutive complex algebras (Proposition 2.3.1.41). We canonically identify E with $\overset{\circ}{F}_u$ and $\overset{\circ}{F}_v$. Then for all $x, y \in \overset{\circ}{F}_u$,

$$w \circ u(x + iy) = w(ux - iuy) = w(x - iy) = w_0 x - iw_0 y =$$

$$= vw_0 x - ivw_0 y = v(w_0 x + iw_0 y) = v \circ w(x + \imath y)$$

so that $w \circ u = v \circ w$.

Conversely, let $w : E \to E$ be an isomorphism of involutive algebras such that $w \circ u = v \circ w$. Then for each $x \in F_u$,

$$wx = wux = vwx,$$

so that $wx \in F_v$. Hence the map

$$F_u \longrightarrow F_v, \quad x \longmapsto wx$$

is an isomorphism of involutive real algebras. ■

Remark. Let E be the involutive complex algebra $\mathbb{C}_{n,n}$ (Example 2.3.1.31 d)) and u the map

$$\mathbb{C}_{n,n} \longrightarrow \mathbb{C}_{n,n}, \quad [\alpha_{ij}] \longmapsto [\overline{\alpha}_{ij}].$$

Then u is an isomorphism of involutive real algebras, which is conjugate linear and satisfies $u^2 = 1$. We have

$$\mathbb{R}_{n,n} = \{x \in \mathbb{C}_{n,n} | ux = x\}$$

and $\mathbb{C}_{n,n}$ is the complexification of $\mathbb{R}_{n,n}$ (Example 2.1.5.9 and Example 2.3.1.31 d)).

Example 2.3.1.44 *We take* $\mathbb{K} = \mathbb{C}$. *Let* T *be a set,* Φ *the set of involutions of* T, *and* $\mathcal{F}$ *the set of involutive real subalgebras* F *of* $\ell^\infty(T)$ *such that the map*

$$\overset{\circ}{F} \longrightarrow \ell^\infty(T), \quad (x, y) \longmapsto x + iy$$

is an isomorphism of involutive complex algebras. For each $\varphi \in \Phi$, *put*

$$F_\varphi := \{x \in \ell^\infty(T) | x = \overline{x} \circ \varphi\}.$$

a) $F_\varphi \in \mathcal{F}$ *for every* $\varphi \in \Phi$.

b) *The map*

$$\Phi \longrightarrow \mathcal{F}, \quad \varphi \longmapsto F_\varphi$$

is bijective.

c) *Take* $\varphi \in \Phi$, *put*

$$S := \{t \in T | \varphi(t) = t\},$$

and let R *be a subset of* $T \backslash S$ *such that*

$$\varphi(R) = T \backslash (R \cup S).$$

Then F_φ *is isomorphic to* $\ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S)$, *where* $\ell^\infty_{\mathbb{C}}(R)$ *denotes the underlying involutive real algebra of* $\ell^\infty(R)$ *and* $\ell^\infty_{\mathbb{R}}(S)$ *the corresponding involutive real algebra* $\ell^\infty(S)$ *constructed over* $\mathbb{R}$.

d) *For each $\varphi \in \Phi$ the real involutive algebra structure of F_φ may be extended to that of a complex involutive algebra iff φ has no fixed-point. (This implies that* $\operatorname{Card} T$ *is not an odd natural number).*

e) *Given $\varphi, \psi \in \Phi$, the following are equivalent:*

e$_1$) *F_φ and F_ψ are isomorphic.*

e$_2$) *There is a bijective map $f : T \to T$ such that*

$$\varphi \circ f = f \circ \psi .$$

e$_3$) $\operatorname{Card}\{t \in T | \varphi(t) = t\} = \operatorname{Card}\{t \in T | \psi(t) = t\}$.

f) *Let $\mathcal{E}$ be the class of involutive real algebras the complexifications of which are isomorphic to $\ell^\infty(T)$. Given $F, G \in \mathcal{E}$, define*

$$F \sim G :\Longleftrightarrow F \text{ and } G \text{ are isomorphic.}$$

Then

$$\operatorname{Card} \mathcal{E}/\sim \; = \begin{cases} 1 + \frac{\operatorname{Card} T - |\sin(\frac{\pi}{2} \operatorname{Card} T)|}{2} & \text{if } T \text{ is finite} \\ \sup\{\aleph_0, \operatorname{Card} \xi\} & \text{if } \operatorname{Card} T = \aleph_\xi \end{cases}$$

(where ξ is an ordinal number). We may replace ℓ^∞ by c_0 or c in the above.

Let $\mathcal{U}$ be the set of conjugate involutions on $\ell^\infty(T)$. Given $u \in \mathcal{U}$, put

$$E_u := \{x \in \ell^\infty(T) | ux = x\} .$$

Then by Proposition 2.3.1.43 a),b), $E_u \in \mathcal{F}$ and the map

$$\mathcal{U} \longrightarrow \mathcal{F}, \quad u \longmapsto E_u$$

is bijective.

For $\varphi \in \Phi$, define

$$u_\varphi : \ell^\infty(T) \longrightarrow \ell^\infty(T), \quad x \longmapsto \bar{x} \circ \varphi .$$

It is easy to see that $u_\varphi \in \mathcal{U}$ and $F_\varphi = E_{u_\varphi}$.

a) follows from the above considerations.

b) It is easy to see that the map

$$\Phi \longrightarrow \mathcal{F}, \quad \varphi \longmapsto F_\varphi$$

is injective. Take $F \in \mathcal{F}$. By the above considerations, there is a $u \in \mathcal{U}$ such that $F = F_u$. Define

$$v : \ell^\infty(T) \longrightarrow \ell^\infty(T), \quad x \longmapsto u\overline{x}.$$

Then v is an isomorphism of complex algebras. By Proposition 2.1.4.22, there is a bijective map $\varphi : T \to T$ such that

$$vx = x \circ \varphi$$

for every $x \in \ell^\infty(T)$. Thus

$$ux = \overline{x} \circ \varphi$$

for every $x \in \ell^\infty(T)$. Take $t \in T$. Then

$$ue_t = \overline{e}_t \circ \varphi = e_{\varphi^{-1}(t)},$$

$$e_t = u^2 e_t = e_{\varphi^{-1}(\varphi^{-1}(t))},$$

so that

$$\varphi^{-1}(\varphi^{-1}(t)) = t.$$

Hence φ is an involution, i.e. $\varphi \in \Phi$. We get $u = u_\varphi$, $E_u = F_\varphi$, whence the map

$$\Phi \longrightarrow \mathcal{F}, \quad \varphi \longmapsto F_\varphi$$

is surjective.

c) For $(x, y) \in \ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S)$, define

$$(x,y)' : T \longrightarrow \mathbb{C}, \quad t \longmapsto \begin{cases} x(t) & \text{if } t \in R \\ \overline{x(\varphi(t))} & \text{if } t \in T \backslash (R \cup S) \\ y(t) & \text{if } t \in S \end{cases}$$

and consider

$$u : \ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S) \longrightarrow \ell^\infty(T), \quad (x,y) \longmapsto (x,y)'.$$

Then u is an injective homomorphism of involutive real algebras.

Take $(x, y) \in \ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S)$. Then

$$\overline{(x,y)'} \circ \varphi(t) = x(\varphi^2(t)) = x(t) = (x,y)'(t)$$

for $t \in R$,

$$\overline{(x,y)'} \circ \varphi(t) = \overline{x}(\varphi(t)) = (x,y)'(t)$$

for $t \in T\backslash(R \cup S)$, and

$$\overline{(x,y)'} \circ \varphi(t) = \overline{(x,y)'}(t) = \overline{y}(t) = y(t) = (x,y)'(t)$$

for $t \in S$. Hence

$$\overline{(x,y)'} \circ \varphi = (x,y)',$$

i.e. $(x,y)' \in F_\varphi$. It follows that

$$u(\ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S)) \subset F_\varphi.$$

Take $z \in F_\varphi$. Then $\overline{z} \circ \varphi = z$. Define

$$x : R \longrightarrow \mathbb{C}, \quad t \longmapsto z(t),$$

$$y : S \longrightarrow \mathbb{R}, \quad t \longmapsto z(t).$$

Then

$$(x,y)'(t) = \overline{x(\varphi(t))} = \overline{z(\varphi(t))} = z(t)$$

for all $t \in T\backslash(R \cup S)$. Hence

$$z = (x,y)' = u((x,y)) \in u(\ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S)),$$

$$u(\ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S)) = F_\varphi.$$

Hence F_φ is isomorphic to $\ell^\infty_{\mathbb{C}}(R) \times \ell^\infty_{\mathbb{R}}(S)$.

d) follows from c).

$e_1 \Rightarrow e_2$. By Proposition 2.3.1.43 e), there is an isomorphism of involutive complex algebras $v : \ell^\infty(T) \to \ell^\infty(T)$ such that

$$v \circ u_\varphi = u_\psi \circ v.$$

By Proposition 2.1.4.22, there is a bijective map $f : T \to T$ such that

$$vx = x \circ f$$

for every $x \in \ell^\infty(T)$. Hence

$$x \circ \varphi \circ f = x \circ f \circ \psi$$

for every $x \in \ell^\infty(T)$. It follows that

$$\varphi \circ f = f \circ \psi .$$

$e_2 \Rightarrow e_3$. Take $t \in T$ such that $\psi(t) = t$. Then

$$\varphi(f(t)) = f(\psi(t)) = f(t),$$

so that

$$f(\{t \in T | \psi(t) = t\}) \subset \{t \in T | \varphi(t) = t\}.$$

The reverse inclusion follows by symmetry, and so

$$\text{Card}\,\{t \in T | \varphi(t) = t\} = \text{Card}\,\{t \in T \mid \psi(t) = t\}.$$

$e_3 \Rightarrow e_1$ follows from c).

f) follows from b) and $e_1 \Rightarrow e_3$ (and Proposition 2.3.1.43 c)). ■

Remark. a) and b) will be generalized in Proposition 4.3.5.6.

Proposition 2.3.1.45 *Let E be an involutive complex algebra E. Then the following are equivalent:*

a) *E is the complexification of an involutive complex algebra (i.e. there is an involutive complex algebra F such that if $\overset{\circ}{F}$ denotes the complexification of its underlying involutive real algebra, then E and $\overset{\circ}{F}$ are isomorphic).*

b) *There is a homomorphism of involutive real algebras $p : E \to E$ such that*

$$p^2 = p, \quad p(E) \cap ip(E) = \{0\}, \quad p(E) + ip(E) = E$$

and

$$p(ip(ix)) = -px$$

for all $x \in E$.

a $\Rightarrow$ b. We identify E and $\overset{\circ}{F}$ and consider

$$p : \overset{\circ}{F} \longrightarrow \overset{\circ}{F}, \quad (x, y) \longmapsto (x + iy, 0).$$

Then

$$p^2 = p, \quad p(\overset{\circ}{F}) \cap ip(\overset{\circ}{F}) = \{(0,0)\}$$

are obvious.

Take $(x, y) \in \overset{\circ}{F}$. Then

$$p((x,y)^*) = p((x^*, -y^*)) = (x^* - iy^*, 0) =$$

$$= ((x + iy)^*, 0) = (x + iy, 0)^* = p((x,y))^*,$$

$$(x,y) = (x,0) + i(-y,0) = p((x,0)) + ip((0,iy)) \in p(\overset{\circ}{F}) + ip(\overset{\circ}{F}),$$

$$p(ip(i(x,y))) = p(ip((-y,x))) = p(i(-y + ix, 0)) =$$

$$= p((0, -y + ix)) = (-iy - x, 0) = -p((x,y)).$$

Hence p is involutive,

$$p(\overset{\circ}{F}) + ip(\overset{\circ}{F}) = \overset{\circ}{F},$$

and the last condition is fulfilled.

Take $(x_1, y_1), (x_2, y_2) \in \overset{\circ}{F}$. Then

$$p((x_1,y_1)(x_2,y_2)) = p((x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)) =$$

$$= ((x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2), 0) = ((x_1 + iy_1)(x_2 + iy_2), 0) =$$

$$= (x_1 + iy_1, 0)(x_2 + iy_2, 0) = p((x_1,y_1))p((x_2,y_2)).$$

Thus p is a homomorphism of involutive real algebras.

b $\Rightarrow$ a. We put $F := p(E)$. F is an involutive real subalgebra of E. We extend the scalars of F to $\mathbb{C}$ by defining

$$\alpha \circ x = p(\alpha x)$$

for all $(\alpha, x) \in \mathbb{C} \times F$.

Take $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ and $x \in F$. Then

$$(\alpha + i\beta) \circ ((\gamma + i\delta) \circ x) = (\alpha + i\beta) \circ (\gamma x + \delta p(ix)) =$$

$$= \alpha\gamma x + \alpha\delta p(ix) + \beta\gamma p(ix) + \beta\delta p(\imath p(ix)) =$$

$$= (\alpha\gamma - \beta\delta)x + (\alpha\delta + \beta\gamma)p(ix) =$$

$$= ((\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma)) \circ x = ((\alpha + i\beta)(\gamma + i\delta)) \circ x\,,$$

$$((\alpha + i\beta) \circ x)^* = (\alpha x + \beta p(ix))^* =$$

$$= \alpha x^* + \beta p(-ix^*) = \alpha x^* - \beta p(ix^*)) = (\alpha - i\beta) \circ x^*\,.$$

Take $\alpha \in \mathbb{C}$ and $x, y \in F$. Then

$$\alpha \circ (xy) = p(\alpha(xy)) = p((\alpha x)y) = p(\alpha x)py = (\alpha \circ x)y\,,$$

$$\alpha \circ (xy) = p(\alpha(xy)) = p(x(\alpha y)) = (px)p(\alpha y) = x(\alpha \circ y)\,.$$

Hence F endowed with the map

$$\mathbb{C} \times F \longrightarrow F\,, \quad (\alpha, x) \longmapsto \alpha \circ x$$

is an involutive complex algebra.

Define

$$u : \overset{\circ}{F} \longrightarrow E\,, \quad (x, y) \longmapsto x + iy\,.$$

By the assumption on p, u is bijective. For every $(x, y) \in \overset{\circ}{F}$,

$$u(i(x, y)) = u((-y, x)) = -y + ix = \imath(x + iy) = iu((x, y))\,,$$

$$u((x, y)^*) = u((x^*, -y^*)) = x^* - iy^* = (x + iy)^* = u((x, y))^*\,.$$

Hence u is linear and involutive.

Take $(x_1, y_1), (x_2, y_2) \in \overset{\circ}{F}$. Then

$$u((x_1, y_1)(x_2, y_2)) = u((x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)) =$$

$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2) = (x_1 + iy_1)(x_2 + iy_2) = u((x_1, y_1))u((x_2, y_2))\,.$$

Thus u is an isomorphism of involutive complex algebras. ∎

Example 2.3.1.46 (0) *We identify the algebra* $\mathbb{H}$ *of quaternions with the unital real subalgebra of* $\mathbb{C}_{2,2}$ *generated by*

$$I := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad J := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad K := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

(Proposition 2.1.4.16, Corollary 2.1.4.17). $\mathbb{H}$ *is an involutive real subalgebra of* $\mathbb{C}_{2,2}$ *(Proposition 2.3.1.31 d)) such that*

$$(\alpha 1 + \beta I + \gamma J + \delta K)^*(\alpha 1 + \beta I + \gamma J + \delta K) =$$

$$= (\alpha 1 + \beta I + \gamma J + \delta K)(\alpha 1 + \beta I + \gamma J + \delta K)^* = (\alpha^2 + \beta^2 + \gamma^2 + \delta^2)1$$

for all $\alpha 1 + \beta I + \gamma J + \delta K \in \mathbb{H}$ *and such that the map*

$$\overset{\circ}{\mathbb{H}} \longrightarrow \mathbb{C}_{2,2}, \quad (x, y) \longmapsto x + iy$$

is an isomorphism of involutive complex algebras. In particular, for every $n \in \mathbb{N}$ *the map*

$$\overbrace{\overset{\circ}{\mathbb{H}}_{n,n}} \longrightarrow \mathbb{C}_{2n,2n}, \quad (x, y) \longmapsto x + iy$$

is an isomorphism of involutive complex algebras.

The last assertion follows from the preceding one and from the final assertion of Proposition 2.3.1.40. ■

Remark. If E is a unital involutive algebra and a an element of E such that

$$a^2 = -1, \quad a^* = -a$$

then there is a unique unital algebra homomorphism $u : \mathbb{H} \to E_{2,2}$ such that

$$uI = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}, \quad uJ = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad uJ = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}.$$

u is injective and involutive (Proposition 2.3.1.31 d)).

Proposition 2.3.1.47 (0) *Let* E *be an involutive real algebra such that* $\operatorname{Re} E$ *is one-dimensional and such that*

$$x \in E\backslash\{0\} \Longrightarrow x^* x \neq 0.$$

Further let $\mathfrak{A}$ *be the set of finite subsets* A *of* $E\backslash\{0\}$ *such that*

$$x \in A \Longrightarrow x + x^* = 0,$$

$$x, y \in A, x \neq y \Longrightarrow xy + yx = 0.$$

a) *There is an* $a \in \operatorname{Re} E \backslash \{0\}$ *with*

$$a^2 = a .$$

b) *For every* $x \in E \backslash \{0\}$, *there is an* $\alpha_x \in \mathbb{R} \backslash \{0\}$ *such that*

$$x^* x = \alpha_x a .$$

c) *Take* $A \in \mathfrak{A}$ *and let* F *be the subalgebra of* E *generated by* $A \cup \{a\}$. *If* $E \neq F$, *then there is a* $z \in E \backslash F$ *such that* $A \cup \{z\} \in \mathfrak{A}$.

d) *If* E *is unital, then for every* $A \in \mathfrak{A}$ *and for distinct* $x, y \in A$, A *and* $\{x, y\}$ *generate the same subalgebra of* E.

e) *If* E *is unital and if for every* $x \in E$, *there is an* $\alpha \in \mathbb{R}_+$ *with* $x^* x = \alpha 1$, *then* E *is isomorphic to* $\mathbb{R}, \mathbb{C}$, *or* $\mathbb{H}$.

a) Take $b \in \operatorname{Re} E \backslash \{0\}$. Then $b^2 = b^* b \in \operatorname{Re} E$, so that there is an $\alpha \in \mathbb{R} \backslash \{0\}$ with

$$b^2 = \alpha b .$$

Put

$$a := \frac{1}{\alpha} b \in \operatorname{Re} E \backslash \{0\} .$$

Then

$$a^2 = \frac{1}{\alpha^2} b^2 = \frac{1}{\alpha} b = a .$$

b follows from $x^* x \in \operatorname{Re} E$.

c) Take $y \in E \backslash F$. Since

$$y + y^* \in \operatorname{Re} E ,$$

there is an $\alpha \in \mathbb{R}$ such that

$$y + y^* = \alpha a .$$

For $x \in A$,

$$(xy + yx - \frac{\alpha}{2}(ax + xa))^* = y^* x^* + x^* y^* - \frac{\alpha}{2}(x^* a + a x^*) =$$

$$= -(\alpha a - y)x - x(\alpha a - y) + \frac{\alpha}{2}(ax + xa) =$$

$$= -\alpha a x + yx - \alpha x a + xy + \frac{\alpha}{2}(ax + xa) =$$

$$= xy + yx - \frac{\alpha}{2}(ax + xa),$$

so that there is a $\beta_x \in \mathbb{R}$ with

$$xy + yx - \frac{\alpha}{2}(ax + xa) = \beta_x a.$$

Put

$$z := y - \frac{\alpha}{2}a + \frac{1}{2}\sum_{x \in A}\frac{\beta_x}{\alpha_x}x \in E \backslash F.$$

Then

$$z + z^* = y + y^* - \alpha a + \frac{1}{2}\sum_{x \in A}\frac{\beta_x}{\alpha_x}(x + x^*) = 0,$$

and for every $x \in A$,

$$xz + zx = xy + yx - \frac{\alpha}{2}(xa + ax) + \frac{\beta_x}{\alpha_x}x^2 =$$

$$= \beta_x a - \frac{\beta_x}{\alpha_x}x^* x = \beta_x a - \beta_x a = 0.$$

Hence $A \cup \{z\} \in \mathfrak{A}$.

d) Let $z \in A \backslash \{x, y\}$. Then

$$(xyz)^* = z^* y^* x^* = -zyx = xyz,$$

so that there is an $\alpha \in \mathbb{R}$ with

$$xyz = \alpha 1.$$

It follows that

$$\alpha xy = xyxyz = -x^2 y^2 z = -(x^* x)(y^* y)z = -\alpha_x \alpha_y z,$$

$$z = -\frac{\alpha}{\alpha_x \alpha_y}xy.$$

Hence A and $\{x, y\}$ generate the same subalgebra of E.

e) If $\emptyset$ is a maximal element of $\mathfrak{A}$, then by c), E is generated by a, so that it is isomorphic to $\mathbb{R}$. Otherwise, there is an $x \in E$ such that $\{x\} \in \mathfrak{A}$. By the supplementary hypothesis of e), we may assume that

$$x^2 = -x^*x = -1\,.$$

If $\{x\}$ is a maximal element of $\mathfrak{A}$, then by c), E is generated by x, so that it is isomorphic to $\mathbb{C}$. Otherwise, there is a $y \in E\backslash\{x\}$ such that $\{x,y\} \in \mathfrak{A}$ and we may assume

$$y^2 = -1\,.$$

Let F be the subalgebra of E generated by $\{x,y\}$. Suppose $E \neq F$. Then by c), there is a $z \in E\backslash F$ such that $\{x,y,z\} \in \mathfrak{A}$. This contradicts d). Hence $E = F$ and E is isomorphic to $\mathbb{H}$. ■

Proposition 2.3.1.48 (7) *Let E be an involutive complex algebra, $\overset{\circ}{E}$ the complexification of the underlying involutive real algebra of E, $E \times E$ the involutive complex algebra defined in Example 2.3.1.4, and $u : E \to E$ a homomorphism of involutive real algebras.*

a) *The map*

$$v : \overset{\circ}{E} \longrightarrow E\,, \quad (x,y) \longmapsto ux + iuy$$

is a homomorphism of involutive complex algebras.

b) *If u is a conjugate involution, then the map*

$$w : \overset{\circ}{E} \longrightarrow E \times E\,, \quad (x,y) \longmapsto (x + iy, ux + iuy)$$

is an isomorphism of involutive complex algebras.

c) *If E is the complexification of an involutive real algebra, then $\overset{\circ}{E}$ and $E \times E$ are isomorphic.*

a) For $(x,y)\,,\ (x_1,y_1)\,,\ (x_2,y_2) \in \overset{\circ}{E}$,

$$v(i(x,y)) = v((-y,x)) = -uy + iux = i(ux + iuy) = iv((x,y))\,,$$

$$v((x,y)^*) = v((x^*,-y^*)) = ux^* - iuy^* = (ux + iuy)^* = v((x,y))^*\,,$$

$$v((x_1,y_1)(x_2,y_2)) = v((x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)) =$$

$$= u(x_1x_2 - y_1y_2) + iu(x_1y_2 + y_1x_2) =$$

$$= (ux_1)(ux_2) - (uy_1)(uy_2) + i((ux_1)(uy_2) + (uy_1)(ux_2)) =$$

$$= (ux_1 + iuy_1)(ux_2 + iuy_2) = v((x_1, y_1))v((x_2, y_2)).$$

b) By a), w is a homomorphism of involutive complex algebras. For $(x, y) \in E \times E$,

$$w((\frac{1}{2}(x + uy), \frac{1}{2i}(x - uy)) =$$

$$= (\frac{1}{2}(x + uy) + \frac{i}{2i}(x - uy), u(\frac{1}{2}(x + uy)) + iu(\frac{1}{2i}(x - uy))) =$$

$$= (x, \frac{1}{2}ux + \frac{1}{2}y - \frac{1}{2}ux + \frac{1}{2}y) = (x, y),$$

so that w is surjective. From $(x, y) \in \overset{\circ}{E}$ and

$$w(x, y) = 0$$

it follows successively that

$$0 = ux + iuy = u(x - iy),$$

$$x - iy = 0,$$

$$(x + iy, x - iy) = 0,$$

$$(x, y) = 0,$$

i.e. w is injective. Hence w is an isomorphism of involutive complex algebras.

c) By Proposition 2.3.1.43 b), there is a conjugate involutive on E. By b), $\overset{\circ}{E}$ and $E \times E$ are isomorphic. ■

Proposition 2.3.1.49 (0) *Let E be an involutive algebra such that the real vector space* $\mathrm{Re}\, E$ *is finite-dimensional. Then the involutive subalgebra of E generated by any finite subset of E is finite-dimensional.*

By Proposition 2.3.1.22, we may assume $\mathbb{K} = \mathbb{R}$. Let A be a finite subset of E. We show that the involutive subalgebra of E generated by $A \cup \operatorname{Re} E$ is finite–dimensional. Since

$$y + y^* \in \operatorname{Re} E$$

for every $y \in A$, we may assume

$$y^* = -y$$

for each $y \in A$.

Let $(x_i)_{i \in \mathbb{N}_m}$ be an algebraic basis of the real vector space $\operatorname{Re} E$ and let $(y_k)_{k \in \mathbb{N}_n}$ be a family with

$$A = \{y_k | k \in \mathbb{N}_n\}.$$

For $i, j \in \mathbb{N}_m$ and $k, \ell \in \mathbb{N}_n$,

$$x_i x_j + x_j x_i\,,\ x_i y_k - y_k x_i\,,\ y_k y_\ell + y_\ell y_k \in \operatorname{Re} E\,.$$

It follows that the vector subspace of E generated by the elements of the form

$$x_{i_1} x_{i_2} \dots x_{i_a} y_{k_1} y_{k_2} \dots y_{k_b}\,,$$

where

$$i_1 < i_2 < \dots i_a\,, \quad k_1 < k_2 < \dots < k_b\,,$$

is equal to the involutive subalgebra of E generated by $A \cup \operatorname{Re} E$, which is therefore finie–dimensional. ■

Proposition 2.3.1.50 (0) *Let E be an involutive algebra and $\overset{\leftrightarrow}{E}$ the involutive algebra obtained from E by reversing the multiplication, i.e. by endowing E with the multiplication*

$$E \times E \longrightarrow E\,, \quad (x, y) \longmapsto yx =: x \updownarrow y\,.$$

Let $\mathcal{U}$ the be set of isomorphisms of involutive real algebras $E \to E$, which are conjugate linear and $\mathcal{V}$ the set of isomorphisms of involutive algebras $E \to \overset{\leftrightarrow}{E}$. For every $u \in \mathcal{U}$, put

$$\overset{\leftrightarrow}{u} : E \to \overset{\leftrightarrow}{E}\,, \quad x \longmapsto ux^*\,.$$

Then $\overset{\leftrightarrow}{u} \in \mathcal{V}$ for every $u \in \mathcal{U}$ and the map

$$\mathcal{U} \longrightarrow \mathcal{V}\,, \quad u \longmapsto \overset{\leftrightarrow}{u}$$

is bijective. In particular, if $\mathbb{K} = \mathbb{R}$, then the involutive real algebras E and $\overset{\leftrightarrow}{E}$ are isomorphic.

Let $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\overleftrightarrow{u}(\alpha x + \beta y) = u(\alpha x + \beta y)^* = u(\overline{\alpha} x^* + \overline{\beta} y^*) = \alpha u x^* + \beta u y^* = \alpha \overleftrightarrow{u} x + \beta \overleftrightarrow{u} y,$$

$$(\overleftrightarrow{u} x)^* = (u x^*)^* = u x^{**} = \overleftrightarrow{u} x^*,$$

$$\overleftrightarrow{u}(xy) = u(xy)^* = u(y^* x^*) = (u y^*)(u x^*) = (u x^*) \updownarrow (u y^*) = (\overleftrightarrow{u} x) \updownarrow (\overleftrightarrow{u} y),$$

so that $\overleftrightarrow{u} \in \mathcal{V}$.

Take $v \in \mathcal{V}$ and put

$$u : E \longrightarrow E, \quad x \longmapsto v x^*.$$

Let $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$. Then

$$u(\alpha x + \beta y) = v(\alpha x + \beta y)^* = v(\overline{\alpha} x^* + \overline{\beta} y^*) = \overline{\alpha} v x^* + \overline{\beta} v y^* = \overline{\alpha} u x + \overline{\beta} u y,$$

$$(u x)^* = (v x^*)^* = v x^{**} = u x^*,$$

$$u(xy) = v(xy)^* = v(y^* x^*) = (v y^*) \updownarrow (v x^*) = (v x^*)(v y^*) = (u x)(u y),$$

$$\overleftrightarrow{u} x = u x^* = v x^{**} = v x,$$

so that $u \in \mathcal{U}$ and $\overleftrightarrow{u} = v$. Hence the map

$$\mathcal{U} \longrightarrow \mathcal{V}, \quad u \longmapsto \overleftrightarrow{u}$$

is surjective. Since it is obviously injective, it is bijective.

If $\mathbb{K} = \mathbb{R}$, then $\mathcal{U}$ contains the identity map, so that $\mathcal{V}$ is nonempty. ■

2.3.2 Involutive Banach Algebras

Definition 2.3.2.1 (0) *An* ***involutive normed space*** *is an involutive vector space endowed with a norm such that*

$$\|x^*\| = \|x\|$$

for every $x \in E$. *If, in addition,* E *is a normed (unital) algebra, then it is called an* ***involutive normed (unital) algebra.*** *If, furthermore,* E *is complete, we speak of an* ***involutive Banach space*** *and an* ***involutive Banach (unital) algebra.*** *Given an involutive normed space* E, *put*

$$\operatorname{Re} E^{\#} := (\operatorname{Re} E)^{\#}.$$

If E, F are involutive normed spaces, then $\mathcal{L}(E,F)$ is an involutive normed space with respect to the canonical involution.

Example 2.3.2.2 $\mathbb{H}$ *is an involutive real unital Banach algebra with respect to the involution*

$$\mathbb{H} \longrightarrow \mathbb{H}, \quad \alpha + i\beta + j\gamma + k\delta \longmapsto \alpha - i\beta - j\gamma - k\delta$$

and to the Euclidean norm (Corollary 2.1.4.17). ■

Example 2.3.2.3 (0) *If* $(T, \mathfrak{T}, \mu)$ *is a measure space and* $p \in [1, \infty[$, *then* $L^p(\mu)$ *is an invoutive Banach space with respect to the involution*

$$L^p(\mu) \longrightarrow L^p(\mu), \quad x \longmapsto \overline{x}$$

and $L^\infty(\mu)$ *is an involutive commutative unital Banach algebra with respect to the involution*

$$L^\infty(\mu) \longrightarrow L^\infty(\mu), \quad x \longmapsto \overline{x}$$

(Example 2.2.2.2). If q *is the conjugate exponent of* p *(and if* $L^\infty(\mu)$ *is order complete when* $p = 1$*), then the canonical map*

$$L^q(\mu) \longrightarrow L^p(\mu)',$$

is an isomorphism of involutive Banach spaces. ■

Example 2.3.2.4 (0) *Let $(E_t)_{t\in T}$ be a family of involutive normed (unital) algebras and*

$$E := \{x \in \prod_{t\in T} E_t \mid (\|x(t)\|)_{t\in T} \in \ell^\infty(T)\}.$$

With

$$E \longrightarrow \mathbb{R}_+, \quad x \longmapsto \sup_{t\in T} \|x(t)\|$$

as a norm and with

$$E \longrightarrow E, \quad (x(t))_{t\in T} \longmapsto (x(t)^*)_{t\in T}$$

as involution, E is an involutive normed (unital) algebra and

$$\{x \in E \mid (\|x(t)\|)_{t\in T} \in c_0(T)\}$$

is an involutive closed ideal of it. If all E_t $(t \in T)$ are complete, then E is complete too (Example 2.2.2.3). In particular, $\ell^\infty(T)$ is an involutive commutative unital Banach algebra and $c_0(T)$ is an involutive closed ideal of it. If T is a topological space and all E_t $(t \in T)$ are equal, then

$$\{x \in E \mid x \text{ is continuous}\}$$

is a closed involutive (unital) subalgebra of E. If in addition $E_t = \mathbb{K}$ for all $t \in T$, then $C(T)$ is a closed involutive unital subalgebra of $\ell^\infty(T)$ and if T is locally compact, then $C_0(T)$ is a closed involutive ideal of $C(T)$. In this case, every closed involutive subalgebra of $C_0(T)$ is quasiunital.

Only the final assertion requires proof. Let $\mathcal{F}$ be a closed involutive subalgebra of $C_0(T)$. Put

$$S := \bigcap_{x\in\mathcal{F}} \overset{-1}{x}(0).$$

Let K be a compact set of $T\backslash S$ and let ε be a strictly positive real number. By Theorem 1.3.5.14, there is an $x \in \mathcal{F}$ with

$$|x - 1| < \varepsilon$$

on K. Define

$$f : \mathbb{K} \longrightarrow \mathbb{K}, \quad \alpha \longmapsto \begin{cases} 1 & \text{if } |\alpha| \geq 1 \\ |\alpha| & \text{if } |\alpha| < 1 \end{cases}.$$

By Corollary 1.3.5.17, $f \circ x \in \mathcal{F}$.

Given a compact set K of $T\backslash S$ and $\varepsilon > 0$, put

$$\mathcal{F}(K,\varepsilon) := (x \in \operatorname{Re}\mathcal{F} \mid e_K - \varepsilon \leq x \leq 1\}.$$

The considerations above show that $\mathcal{F}(K,\varepsilon)$ is not empty. Let $\mathfrak{F}$ denote the filter on $\mathcal{F}$ generated by the filter base

$$\{\mathcal{F}(K,\varepsilon) \mid K \text{ compact set of } T\backslash S, \varepsilon > 0\}.$$

It is easy to see that $\mathfrak{F}$ is an approximate unit of $\mathcal{F}$. ■

Example 2.3.2.5 (1) *Let T be a set. For $k \in \ell^{2,2}(T,T)$ define*

$$k^* : T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto \overline{k(t,s)}.$$

Then $k^* \in \ell^{2,2}(T,T)$ for every $k \in \ell^{2,2}(T,T)$ and the Banach algebra $\ell^{2,2}(T,T)$ (Example 2.2.2.5 a)) is an involutive Banach algebra with respect to the involution

$$\ell^{2,2}(T,T) \longrightarrow \ell^{2,2}(T,T), \quad k \longmapsto k^*$$ ■

Example 2.3.2.6 (0) *For $n \in \mathbb{N}$, consider $\mathbb{K}^n$ with the euclidean norm and the unital Banach algebra $\mathcal{L}(\mathbb{K}^n)$ (Example 2.2.2.4) with the involution obtained by the identification with the involutive algebra $\mathbb{K}_{n,n}$ of $n \times n$-matrices (Example 2.3.1.32). Then $\mathcal{L}(\mathbb{K}^n)$ is a strongly symmetric involutive unital Banach algebra.*

Take $u \in \mathcal{L}(\mathbb{K}^n)$. By Example 1.3.4.12, the transpose of the matrix associated to u is the matrix associated to u', and by Theorem 1.3.4.2 b).

$$\|u\| = \|u'\|.$$

By Example 2.3.1.32, $\mathcal{L}(\mathbb{K}^n)$ is a strongly symmetric involutive unital Banach algebra. ■

Proposition 2.3.2.7 (0) *If E is an involutive normed space, then*

$$\|\operatorname{re} x\| \leq \|x\|$$

for every $x \in E$ and the map

$$E \longrightarrow \operatorname{Re} E, \quad x \longmapsto \operatorname{re} x$$

is both $\mathbb{R}$-linear and continuous. If $\mathbb{K} = \mathbb{C}$, then

$$\|\operatorname{im} x\| \leq \|x\|$$

for every $x \in E$ *and the map*

$$E \longrightarrow \operatorname{Re} E, \quad x \longmapsto \operatorname{im} x$$

is both $\mathbb{R}$*-linear and continuous.*

Given $x \in E$,

$$\|\operatorname{re} x\| = \|\frac{1}{2}(x + x^*)\| \leq \frac{1}{2}(\|x\| + \|x^*\|) = \|x\|,$$

$$\|\operatorname{im} x\| = \|\frac{1}{2i}(x - x^*)\| \leq \frac{1}{2}(\|x\| + \|x^*\|) = \|x\|. \qquad \blacksquare$$

Proposition 2.3.2.8 (0) *Let* E *be an involutive normed (Banach) algebra. Then the normed (Banach) unital algebra* $\mathbb{K} \times E$ *defined in Proposition 2.2.1.3 is an involutive normed (Banach) unital algebra with respect to the involution*

$$\mathbb{K} \times E \longrightarrow \mathbb{K} \times E, \quad (\alpha, x) \longmapsto (\overline{\alpha}, x^*).$$

The assertion follows from Proposition 2.2.1.3 and Corollary 2.3.1.9. $\blacksquare$

Definition 2.3.2.9 (0) *Let* E *be an involutive normed (Banach) algebra. The associated involutive unital algebra* F *of* E *(Proposition 2.3.1.9) endowed with an extension of the norm of* E *which renders* F *an involutive normed (Banach) unital algebra is called an* ***involutive normed (Banach) unital algebra associated to*** E *(Proposition 2.3.2.8).*

Proposition 2.3.2.10 (0) *Let* E *be an involutive normed space (normed algebra, normed unital algebra) and* F *a closed involutive subspace (ideal, proper ideal) of* E. *Then* E/F *is an involutive normed space (normed algebra, normed unital algebra) (Proposition 2.2.1.14, Proposition 2.3.1.21).*

Let $X \in E/F$ and $x \in X$. Then $x^* \in X^*$, so

$$\|X^*\| \leq \|x^*\| = \|x\|.$$

We get

$$\|X^*\| \leq \|X\|,$$

$$\|X\| = \|X^{**}\| \leq \|X^*\|,$$

$$\|X\| = \|X^*\|. \qquad \blacksquare$$

Proposition 2.3.2.11 (0) *If E is an involutive normed space, then its involution is an isometry of real normed spaces and* $\mathrm{Re}\, E$ *is a closed real vector subspace of E.* ■

Corollary 2.3.2.12 (0) *If E is an involutive normed space, then*

$$\overline{A^*} = (\overline{A})^*$$

for every $A \subset E$. Hence, if A is an involutive set of E, then $\overline{A}$ is also an involutive set of E. ■

Corollary 2.3.2.13 (0) *Let E be an involutive normed algebra and F an involutive subalgebra (ideal) of E. Then $\overline{F}$ is an involutive subalgebra (ideal) of E.*

The assertion follows immediately from Corollary 2.2.1.8 and Corollary 2.3.2.12.

■

Corollary 2.3.2.14 (0) *Let E be an involutive normed (unital) algebra, A a subset of E and F the involutive (unital) subalgebra of E generated by A.*

a) *$\overline{F}$ is the smallest closed involutive (unital) subalgebra of E containing A. It is called* ***the closed involutive (unital) subalgebra of E generated by A.***

b) *If $A \cup A^*$ is commutative, then $\overline{F}$ is also commutative.*

c) *If A is countable, then $\overline{F}$ is separable.*

a) follows from Corollary 2.3.2.13.
b) follows from Proposition 2.3.1.19 and Corollary 2.2.1.10.
c) follows from Proposition 2.3.1.19 and Corollary 2.2.1.9. ■

Definition 2.3.2.15 (0) *Let E be an involutive normed (unital) algebra and x an element of E. The closed involutive (unital) subalgebra of E generated by $\{x\}$ is called* ***the closed involutive (unital) subalgebra of E generated by x.*** *It is denoted by $E(x)$ (by $E(x,1)$).*

Corollary 2.3.2.16 (0) *If E is an involutive normed (unital) algebra and x a normal element of E, then $E(x)$ ($E(x,1)$) is commutative and separable.*

■

Corollary 2.3.2.17 *Let $\widetilde{E}$ be the completion of the involutive (unital) normed algebra E (Corollary 2.2.1.13). If we endow $\widetilde{E}$ with the continuous extention of the involution of E, then $\widetilde{E}$ becomes an involutive (unital) Banach algebra. If u is a continuous involutive (unital) algebra homomorphism of E into an involutive(unital) Banach algebra, then the continuous extention of u to $\widetilde{E}$ is an involutive (unital) algebra homomorphism.* ■

Corollary 2.3.2.18 (0) *If E is an involutive normed (unital) algebra, then* $\mathrm{No}\, E$ ($\mathrm{Un}\, E$) *is a closed set of E.*

The assertion follows from Proposition 2.3.2.11 and Proposition 2.2.1.6. ■

Corollary 2.3.2.19 (0) *If E is an involutive unital Banach algebra and $x \in \mathrm{Re}\, E$, then:*

a) $e^x \in \mathrm{Re}\, E$.

b) $\lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} < 1$ *and* $\alpha \in \mathbb{R} \Rightarrow [1+x]^\alpha \in \mathrm{Re}\, E$,

The assertions follow immediately from Proposition 2.3.2.11. ■

Corollary 2.3.2.20 *Let E be a complex involutive algebra. If the interior of* $\mathrm{No}\, E$ *is nonempty, then E is commutative.*

Let x be an interior point of $\mathrm{No}\, E$ and take $y \in E$. Then $x+\alpha y$ is normal for all sufficient small $\alpha \in \mathbb{R}$ so that

$$0 = (x+\alpha y)^*(x+\alpha y) - (x+\alpha y)(x+\alpha y)^* =$$

$$= (x^* + \alpha y^*)(x+\alpha y) - (x+\alpha y)(x^*+\alpha y^*) =$$

$$= x^*x + \alpha x^*y + \alpha y^*x + \alpha^2 y^*y - xx^* - \alpha xy^* - \alpha yx^* - \alpha^2 yy^* =$$

$$= \alpha((x^*y + y^*x - xy^* - yx^*) + \alpha(y^*y - yy^*)).$$

It follows that

$$y^*y - yy^* = 0,$$

i.e. that y is normal. By Corollary 2.3.1.25, E is commutative. ■

Proposition 2.3.2.21 (0) *Let E be an involutive Banach algebra. Then the following are equivalent:*

a) *E is symmetric.*

b) *Every closed involutive subalgebra of E is symmetric.*

c) *Every closed involutive commutative subalgebra of E is symmetric.*

a $\Rightarrow$ b. Let F be a closed involutive subalgebra of E and take $x \in \mathrm{No}\, F$. Then

$$\sigma_E(x^*x) \subset \mathbb{R}_+ .$$

By Corollary 2.2.4.27,

$$\sigma_F(x^*x) \subset \mathbb{R}_+ .$$

Thus F is symmetric.

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. Take $x \in \mathrm{No}\, E$. By Corollary 2.3.2.16, $F := E(x)$ is commutative. By c), it is therefore symmetric. Then

$$\sigma_E(x^*x) \subset \sigma_F(x^*x) \subset \mathbb{R}_+$$

(Corollary 2.1.3.14 a)). Hence E is symmetric. ■

Proposition 2.3.2.22 (0) *Let E, F be involutive normed spaces.*

a) *$\mathcal{L}(E,F)$ is an involutive normed space (Example 2.3.1.6).*

b) *E' is an involutive Banach space.*

c) *$\mathrm{Re}\, \mathcal{L}(E,F)$ is closed with respect to the topology of pointwise convergence.*

d) *$u^{*\prime} = u'^*$ for every $u \in \mathcal{L}(E,F)$.*

e) *The evaluation $j_E : E \to E''$ is involutive.*

f) *If G (resp. H) is an involutive vector subspace of E (resp. of E'), then G° (resp. ${}^\circ H$) is an involutive set of E' (resp. of E), the map*

$$E_H \longrightarrow E_H , \quad x \longmapsto x^*$$

is continuous, and $\mathrm{Re}\, E$ is a closed set of E_H.

g) *Let G be a closed involutive subspace of E, $q : E \to E/G$ the quotient map, and $i : G \to E$ the inclusion map. Then the map*

$$(E/G)' \longrightarrow G^\circ, \quad x' \longmapsto q'x'$$

and the factorization $E'/G^\circ \to G'$ of i' through E'/G° are involutive isometries (b), f) and Proposition 2.3.2.10).

h) *If G is a closed involutive subspace of E'_E and $q : E \to E/{}^\circ G$ is the quotient map, then the map*

$$(E/{}^\circ G)' \longrightarrow G, \quad x' \longmapsto q'x'$$

is an involutive isometry (b), f), and Proposition 2.3.2.10).

i) *If G is an involutive normed space, then*

$$(v \circ u)^* = v^* \circ u^*$$

for every $u \in \mathcal{L}(E,F)$ and $v \in \mathcal{L}(F,G)$.

j) *For every $x' \in \operatorname{Re} E'$,*

$$\|x'\| = \sup\{|x'(x)| \mid x \in \operatorname{Re} E^\#\}.$$

k) *$\operatorname{Re} E'^\#$ is a compact set of E'_E.*

l) *For every $x \in \operatorname{Re} E \backslash \{0\}$ there is an $x' \in \operatorname{Re} E'$ such that*

$$\|x'\| = 1, \quad x'(x) = \|x\|.$$

In particular,

$$\|x\| = \sup\{|x'(x)| \mid x' \in \operatorname{Re} E'^\#\}.$$

m) *Take $\mathbb{K} = \mathbb{C}$ and $x \in E$ such that $x'(x) \in \mathbb{R}$ for every $x' \in \operatorname{Re} E'$. Then x is selfadjoint.*

n) *If u is a projection in E and $\operatorname{Im} u$ is an involutive set of E, then u^* and $\operatorname{re} u$ are projections of E onto $\operatorname{Im} u$. If in addition $\operatorname{Ker} u$ is also an involutive set of E, then u is selfadjoint.*

o) *If $u \in \operatorname{Re} \mathcal{L}(E,F)$, then the algebraic isomorphism $\overline{u} : E/\operatorname{Ker} u \to \operatorname{Im} u$ associated to u is involutive.*

a) Take $u \in \mathcal{L}(E,F)$. Then

$$\|u^*x\| = \|(ux^*)^*\| = \|ux^*\| \leq \|u\|\,\|x^*\| = \|u\|\,\|x\|$$

for every $x \in E$ and so

$$\|u^*\| \leq \|u\|.$$

It follows that

$$\|u^*\| = \|u\|.$$

b) follows from a).

c) follows from

$$\operatorname{Re}\mathcal{L}(E,F) = \bigcap_{x\in E}\{u \in \mathcal{L}(E,F) \mid ux^* = (ux)^*\}.$$

d) Given $(x,y') \in E \times F'$,

$$\langle u^*x, y'\rangle = \langle (ux^*)^*, y'\rangle = \overline{\langle ux^*, y'^*\rangle} = \overline{\langle x^*, u'y'^*\rangle} = \overline{\langle x^*, (u'^*y')^*\rangle} = \langle x, u'^*y'\rangle$$

and so

$$u^{*\prime} = u'^*$$

by Theorem 1.3.4.2 a).

e) Given $(x^*,x') \in E \times E'$,

$$\langle (j_E x)^*, x'\rangle = \overline{\langle j_E x, x'^*\rangle} = \overline{\langle x, x'^*\rangle} = \langle x^*, x'\rangle = \langle j_E x^*.x'\rangle,$$

so that

$$(j_E x)^* = j_E x^*$$

for every $x \in E$.

f) Take $x' \in G^\circ$. Then

$$\langle x, x'^*\rangle = \overline{\langle x^*, x'\rangle} = 0$$

for every $x \in G$. Thus $x'^* \in G^\circ$ and G° is an involutive set of E'.

Take $x \in {}^\circ H$. Then

$$\langle x^*, x'\rangle = \overline{\langle x, x'^*\rangle} = 0$$

for every $x' \in H$. Hence $x^* \in {}^\circ H$ and ${}^\circ H$ is an involutive set of E.

We have

$$\langle x^*, x' \rangle = \overline{\langle x, x'^* \rangle}$$

for all $x \in E$ and $x' \in H$. Hence the map

$$E_H \longrightarrow E_H, \quad x \longmapsto x^*$$

is continuous and so $\operatorname{Re} E$ is a closed set of E_H.

g) The first assertion follows from d) and Proposition 1.3.5.2. The second assertion follows from the first one, d), and Theorem 1.3.5.12 c).

h) follows from d) and Corollary 1.3.5.5.

i) Since

$$(v \circ u)^* x = (vux^*)^* = v^*(ux^*)^* = v^* u^* x$$

for every $x \in E$,

$$(v \circ u)^* = v^* \circ u^*.$$

j) First take $\mathbb{K} = \mathbb{R}$. Given $x \in E^\#$,

$$x'(x) = \operatorname{re}(x'(x)) = x'(\operatorname{re} x)$$

(Proposition 2.3.1.7). Since

$$\|\operatorname{re} x\| \leq \|x\| \leq 1$$

(Proposition 2.3.2.7),

$$|x'(x)| \leq \sup\{|x'(y)| \mid y \in \operatorname{Re} E^\#\},$$

$$\|x'\| \leq \sup\{|x'(x)| \mid x \in \operatorname{Re} E^\#\} \leq \sup\{|x'(x)| \mid x \in E^\#\} = \|x'\|,$$

$$\|x'\| = \sup\{|x'(x)| \mid x \in \operatorname{Re} E^\#\}.$$

Now consider $\mathbb{K} = \mathbb{C}$. Since

$$\operatorname{re}(x'(x)) = x'(\operatorname{re} x)$$

for every $x \in E$ (Proposition 2.3.1.7),

$$\|x'\| = \|\operatorname{re} \circ x'\| = \sup\{|x'(\operatorname{re} x)| \mid x \in E^\#\} \leq$$

$$\leq \sup\{|x'(x)| \mid x \in \operatorname{Re} E^{\#}\} \leq \sup\{|x'(x)| \mid x \in E^{\#}\} = \|x'\|$$

(Corollary 1.2.1.6).

k) follows from c) and the Alaoglu–Bourbaki Theorem.

l) The image of x with respect to the evaluation $E \to E''$ belongs to $\operatorname{Re} E''$ (e) and Proposition 2.3.1.7). By j),

$$\|x\| = \sup\{|x'(x)| \mid x' \in \operatorname{Re} E'^{\#}\}.$$

The additional assertion follows from k).

m) Since

$$\langle j_E x, x' \rangle = \langle x, x' \rangle \in \mathbb{R}$$

for every $x' \in \operatorname{Re} E'$, $j_E x$ is involutive by Corollary 2.3.1.23 b $\Rightarrow$ a. By e), j_E is involutive and so

$$j_E(\operatorname{im} x) = \operatorname{im} j_E x = 0$$

(Corollary 2.3.1.23 a $\Rightarrow$ c, Corollary 2.3.1.24 a)), and hence

$$\operatorname{im} x = 0,$$

$$x \in \operatorname{Re} E$$

(Corollary 2.3.1.24 a)).

n) By i), u^* is a projection. We have

$$u^* x = (u x^*)^* \in (\operatorname{Im} u)^* = \operatorname{Im} u$$

for every $x \in E$ so $\operatorname{Im} u^* \subset \operatorname{Im} u$. If $x \in \operatorname{Im} u$ then

$$u^* x = (u x^*)^* = x^{**} = x,$$

so u^* is a projection of E on $\operatorname{Im} u$. It follows that $\operatorname{re} u$ is also a projection of E on $\operatorname{Im} u$.

Assume now in addition that $\operatorname{Ker} u$ is also an involutive set of E. Then for every $x \in E$

$$u^* x = (u x^*)^* = 0$$

iff $x \in \operatorname{Ker} u$, so $\operatorname{Ker} u = \operatorname{Ker} u^*$ and $u = u^*$.

o) Let $q : E \to E/\mathrm{Ker}\, u$ be the quotient map. By Propositions 2.3.1.21 b) and 2.3.2.10, $E/\mathrm{Ker}\, u$ is an involutive normed space and q is involutive. We get for every $x \in E$

$$\overline{u}(qx)^* = \overline{u}qx^* = ux^* = (ux)^* = (\overline{u}(qx))^* .$$

Since q is surjective, $\overline{u}$ is involutive. ■

Remark. $\mathcal{L}(E)$ is a unital Banach algebra, but by g), it is not an involutive algebra.

Corollary 2.3.2.23 (0) *Let E be an involutive normed space and F an involutive subspace of E. Take $x' \in \mathrm{Re}\, F'$ (Proposition 2.3.2.22 b)). Then there is a $y' \in \mathrm{Re}\, E'$ with*

$$y'|F = x' \text{ and } \|y'\| = \|x'\| .$$

By the Hahn–Banach Theorem, there is a $z' \in E'$, such that

$$z'|F = x', \quad \|z'\| = \|x'\| .$$

Put

$$y' := \mathrm{re}\, z' \in \mathrm{Re}\, E' .$$

Then

$$y'|F = x'$$

and

$$\|x'\| \leq \|y'\| \leq \|z'\| = \|x'\|$$

(Proposition 2.3.2.7), so that

$$\|x'\| = \|y'\| .$$ ■

Proposition 2.3.2.24 (0) *Let E be an involutive normed space, A a nonempty closed convex set of $\mathrm{Re}\, E$ and take $x \in (\mathrm{Re}\, E)\backslash A$. Then there is an $x' \in \mathrm{Re}\, E'$ (Proposition 2.3.2.22 b)), such that*

$$\sup_{y \in A} x'(y) < x'(x) .$$

By Corollary 1.3.1.7, there is an $y' \in E'$ such that

$$\sup_{y \in A} \operatorname{re}(y'(y)) < \operatorname{re}(y'(x)) .$$

Put

$$x' := \operatorname{re} y' \in \operatorname{Re} E' .$$

Then

$$x'(y) = \frac{1}{2}(y'(y) + y'^*(y)) = \frac{1}{2}(y'(y) + \overline{y'(y)}) = \operatorname{re} y'(y)$$

for every $y \in \operatorname{Re} E$, so that

$$\sup_{y \in A} x'(y) = \sup_{y \in A} \operatorname{re}(y'(y)) < \operatorname{re}(y'(x)) = x'(x) . \qquad \blacksquare$$

Proposition 2.3.2.25 (0) *Let E be an involutive normed space, A' a nonempty convex set (cone) of $\operatorname{Re} E'$ (Proposition 2.3.2.22 b)) which is closed in E'_E. Take $x' \in (\operatorname{Re} E') \backslash A'$. Then there is an $x \in \operatorname{Re} E$ such that*

$$\sup_{y' \in A'} y'(x) < x'(x) \quad \left(\sup_{y' \in A'} y'(x) = 0 < x'(x) \right) .$$

By Proposition 1.3.1.8, there is a $y \in E$ such that

$$\sup_{y' \in A'} \operatorname{re}(y'(y)) < \operatorname{re}(x'(y)) .$$

Put

$$x := \operatorname{re} y \in \operatorname{Re} E .$$

Then

$$\sup_{y' \in A'} y'(x) = \sup_{y' \in A'} \operatorname{re}(y'(y)) < \operatorname{re}(x'(y)) = x'(x)$$

(Proposition 2.3.1.7). If A' is a cone, then

$$\sup_{y' \in A'} y'(x) = 0 < x'(x) . \qquad \blacksquare$$

Proposition 2.3.2.26 (4) *Let E, F be involutive Banach algebras such that*

$$\|x^* x\| = \|x\|^2$$

for every $x \in E$ and $u : F \to E$ an injective involutive algebra homomorphism. For $\mathbb{K} = \mathbb{R}$, we assume in addition that

$$\sigma_{\overset{\circ}{E}}((x^* x, 0)) \subset \mathbb{R}$$

for every $x \in E$. Then F is semi-simple.

Take $x \in E$. By induction,

$$\|(x^*x)^{2^n}\| = \|x^*x\|^{2^n}$$

for every $n \in \mathbb{N}$. Thus

$$r_E(x^*x) = \lim_{n\to\infty} \|(x^*x)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(x^*x)^{2^n}\|^{2^{-n}} = \|x^*x\|$$

by Theorem 2.2.5.4.

Let y be an element of the radical of F. Then y^*y also belongs to the radical of F. By Corollary 2.1.3.12 and Corollary 2.1.3.24,

$$r_E((uy)^*(uy)) = r_E(u(y^*y)) \leq r_F(y^*y) = 0.$$

By the above,

$$\|uy\|^2 = \|(uy)^*(uy)\| = r_E((uy)^*(uy)) = 0,$$

$$uy = 0, \quad y = 0.$$

Hence the radical of F is equal to $\{0\}$ and F is semi-simple. ■

Remark. An involutive complex Banach algebra such that $\|x^*x\| = \|x\|^2$ for all its elements x is called a complex C^*-algebra. They will be studied in Chapter 4 (also in the real case). The above map u is continuous (even if it is not injective), but the full story will be told in the above mentioned Chapter 4. We anticipated it a bit here in order to be able to prove that some important involutive Banach algebras are semi-simple.

Example 2.3.2.27 (0) *Let $(T, \mathfrak{T}, \mu)$ be a measure space. We identify $L^2(\mu)'$ canonically with $L^2(\mu)$ and define*

$$\langle x|y\rangle := \int x\overline{y}d\mu$$

for $x, y \in L^2(\mu)$ and

$$u^* : L^2(\mu) \longrightarrow L^2(\mu), \quad x \longmapsto \overline{u'\overline{x}}$$

for $u \in \mathcal{L}(L^2(\mu))$. (This definition of u^* differs from the definition of u^* obtained by combining Definition 2.3.1.1 with Example 2.3.2.3).

a) *$u^* \in \mathcal{L}(L^2(\mu))$ and $\|u^*\| = \|u\|$ for every $u \in \mathcal{L}(L^2(\mu))$.*

b) $\langle ux|y\rangle = \langle x|u^*y\rangle$ *for every* $u \in \mathcal{L}(L^2(\mu))$ *and* $x, y \in L^2(\mu)$.

c) *The involution*

$$\mathcal{L}(L^2(\mu)) \longrightarrow \mathcal{L}(L^2(\mu)), \quad u \longmapsto u^*$$

renders $\mathcal{L}(L^2(\mu))$ *a unital involutive Banach algebra such that*

$$\|u^* \circ u\| = \|u\|^2$$

for every $u \in \mathcal{L}(L^2(\mu))$.

d) *Let* $L^2(\mu, \mathbb{R})$, $L^2(\mu,\mathbb{C})$ *denote the real and complex vector space* $L^2(\mu)$, *respectively. Let* $\overset{\circ}{\widehat{L^2}}(\mu, \mathbb{R})$ *and* $\overset{\circ}{\mathcal{L}}(L^2(\mu, \mathbb{R}))$ *denote the complexification of* $L^2(\mu, \mathbb{R})$ *and* $\mathcal{L}(L^2(\mu, \mathbb{R}))$, *respectively. Given* $(u, v) \in \overset{\circ}{\mathcal{L}}(L^2(\mu, \mathbb{R}))$, *define*

$$\widetilde{(u, v)} : \overset{\circ}{\widehat{L^2}}(\mu, \mathbb{R}) \longrightarrow \overset{\circ}{\widehat{L^2}}(\mu, \mathbb{R}), \quad (x, y) \longmapsto (ux - vy, uy + vx)$$

and identify $\overset{\circ}{\widehat{L^2}}(\mu, \mathbb{R})$ *with* $L^2(\mu,\mathbb{C})$ *via the map*

$$\overset{\circ}{\widehat{L^2}}(\mu, \mathbb{R}) \longmapsto L^2(\mu,\mathbb{C}), \quad (x, y) \longmapsto x + iy$$

(Example 2.1.5.2). Then the map

$$\overset{\circ}{\mathcal{L}}(L^2(\mu, \mathbb{R})) \longrightarrow \mathcal{L}(L^2(\mu,\mathbb{C})), \quad (u, v) \longmapsto \widetilde{(u, v)}$$

(Lemma 2.1.5.3) is a isomorphism of involutive unital algebras.

a) is easy to check (Theorem 1.3.4.2).

b) We have

$$\langle ux|y\rangle = \langle ux, \overline{y}\rangle = \langle x, u'\overline{y}\rangle = \langle x|\overline{u'\overline{y}}\rangle = \langle x|u^*y\rangle.$$

c) By b),

$$\langle u^{**}x|y\rangle = \overline{\langle y|u^{**}x\rangle} = \overline{\langle u^*y|x\rangle} = \langle x|u^*y\rangle = \langle ux|y\rangle$$

for all $x, y \in \ell^2(T)$, so that

$$u^{**} = u.$$

Hence the map

$$\mathcal{L}(L^2(\mu)) \longrightarrow \mathcal{L}(L^2(\mu)), \quad u \longmapsto u^*$$

is an involution. This map is obviously conjugate linear and $1^* = 1$.

Take $u, v \in \mathcal{L}(L^2(\mu))$. By Corollary 1.3.4.5

$$(uv)^* x = \overline{(uv)'\overline{x}} = \overline{v'u'\overline{x}} = \overline{v'\overline{u^* x}} = v^* u^* x$$

for every $x \in L^2(\mu)$, so that

$$(uv)^* = v^* u^*.$$

Hence $\mathcal{L}(L^2(\mu))$ is an involutive Banach algebra.

Take $u \in \mathcal{L}(L^2(\mu))$. By a),

$$\|ux\|^2 = \langle ux|ux\rangle = \langle u^* ux|x\rangle \leq \|u^* ux\|\,\|x\| \leq \|u^* u\|\,\|x\|^2$$

for every $x \in L^2(\mu)$. Hence

$$\|u\|^2 \leq \|u^* u\|.$$

The reverse inequality follows from a) and b).

d) By Example 2.1.5.8, the above map is an isomorphism of unitai algebras.

Take $(u, v) \in \overset{\circ}{\mathcal{L}}(L^2(\mu, \mathbb{R}))$ and $(x, y) \in \overset{\circ}{\widetilde{L^2}}(\mu, \mathbb{R})$. By Proposition 2.1.5.6,

$$\left(\widetilde{(u,v)}\right)^* (x,y) = \left(\overline{\widetilde{(u,v)}}^{\,\prime}\right)\overline{(x,y)} = \overline{\widetilde{(u',v')}(x,-y)} =$$

$$= \overline{(u'x + v'y, -u'y + v'x)} = (u'x + v'y, u'y - v'x) =$$

$$= (u^* x + v^* y, u^* y - v^* x) = \widetilde{(u^*, -v^*)}(x,y) = \widetilde{(u,v)^*}(x,y).$$

Hence

$$\left(\widetilde{(u,v)}\right)^* = \widetilde{(u,v)^*}$$

and the above map is involutive. ∎

Proposition 2.3.2.28 (4) *Let T be a locally compact group and $\mathcal{M}_b(T)$ the unital Banach algebra defined in Theorem 2.2.2.10 b). For $\mu \in \mathcal{M}_b(T)$ let μ^* denote the element of $\mathcal{M}_b(T)$ with*

$$\int x d\mu^* = \int x(t^{-1}) d\overline{\mu}(t)$$

for every $x \in C_0(T)$.

a) *$\mathcal{M}_b(T)$ is an involutive unital Banach algebra with respect to the involution*

$$\mathcal{M}_b(T) \longrightarrow \mathcal{M}_b(T), \quad \mu \longmapsto \mu^*.$$

b) *$\ell^1(T)$, canonically identified with a subset of $\mathcal{M}_b(T)$ (Theorem 2.2.2.10 d)), is a closed involutive subalgebra of $\mathcal{M}_b(T)$ and e_t is unitary for every $t \in T$.*

c) *If λ is a left invariant Haar measure on T, then the injective unital continuous algebra homomorphism*

$$\mathcal{M}_b(T) \longrightarrow \mathcal{L}(L^2(\lambda)), \quad \mu \longmapsto \widetilde{\mu}$$

defined in Proposition 2.2.2.14 e) is involutive (Example 2.3.2.27 c)).

d) *Let $\mathcal{M}_b(T, \mathbb{K})$ be the involutive unital algebra $\mathcal{M}_b(T)$ with respect to $\mathbb{K}$ and $\overset{\circ}{\mathcal{M}}_b(T, \mathbb{R})$ the complexification of $\mathcal{M}_b(T, \mathbb{R})$. Then the map*

$$\overset{\circ}{\mathcal{M}}_b(T, \mathbb{R}) \longrightarrow \mathcal{M}_b(T, \mathbb{C}), \quad (\mu, \nu) \longmapsto \mu + i\nu$$

is an isomorphism of involutive unital complex algebras.

e) *$\mathcal{M}_b(T)$ is semi-simple.*

f) *The map*

$$\mathcal{M}_b(T) \longrightarrow \mathbb{K}, \quad \mu \longmapsto \mu(T)$$

is an involutive unital algebra homomorphism.

g) *Every algebra homomorphism $\ell^1(T) \to \mathbb{K}$ is involutive.*

a) Take $\mu, \nu \in \mathcal{M}_b(T)$. Then

$$\langle x, \nu^* * \mu^* \rangle = \int x(st) d(\nu^* \otimes \mu^*)(s,t) = \int (\int x(st) d\nu^*(s)) d\mu^*(t) =$$

$$= \int \left(\int x(s^{-1}t) d\overline{\nu}(s) \right) d\mu^*(t) = \int \left(\int x(s^{-1}t^{-1}) d\overline{\nu}(s) \right) d\overline{\mu}(t) =$$

$$= \int \left(\int x(s^{-1}t^{-1}) d\overline{\mu}(t) \right) d\overline{\nu}(s) = \int \left(\int x(t^{-1}s^{-1}) d\overline{\mu}(s) \right) d\overline{\nu}(t) =$$

$$= \int x((st)^{-1}) d\overline{(\mu \otimes \nu)}(s,t) = \int x d(\mu * \nu)^* = \langle x, (\mu * \nu)^* \rangle ,$$

$$\langle x, \mu^{**} \rangle = \int x(t^{-1}) d\overline{\mu^*}(t) = \overline{\int \overline{x(t^{-1})} d\mu^*(t)} =$$

$$= \overline{\int \overline{x(t)} d\overline{\mu}(t)} = \int x(t) d\mu(t) = \langle x, \mu \rangle$$

for every $x \in C_0(T)$, so that

$$(\mu * \nu)^* = \nu^* * \mu^* , \quad \mu^{**} = \mu .$$

The other conditions are easy to verify.

b) Take $t \in T$. Then

$$\langle x, e_t^* \rangle = \int x(t^{-1}) d\overline{\delta}_t = x(t^{-1}) = \langle x, e_{t^{-1}} \rangle$$

for every $x \in C_0(T)$, so that

$$e_t^* = e_{t^{-1}} .$$

c) Take $x, y \in C_0(T)$ and $\mu \in \mathcal{M}_b(T)$. By Example 2.3.2.27 b) and Fubini's Theorem,

$$\langle x | \overline{\mu}^* y \rangle = \langle \overline{\mu} x | y \rangle = \langle \mu * x | y \rangle =$$

$$= \int (\mu * x)(t) \overline{y(t)} d\lambda(t) = \int \left(\int x(s^{-1}t) d\mu(s) \right) \overline{y(t)} d\lambda(t) =$$

$$= \int \left(\int x(s^{-1}t) \overline{y(t)} d\lambda(t) \right) d\mu(s) =$$

$$= \int \left(\int x(t) \overline{y(st)} d\lambda(t) \right) d\mu(s) = \int x(t) \left(\int \overline{y(st)} d\mu(s) \right) d\lambda(t) =$$

$$= \int x(t)\Big(\int \overline{y(s^{-1}t)}\,d\overline{\mu^*}(s)\Big)d\lambda(t) = \int x(t)\overline{(\mu^* * y)(t)}\,d\lambda(t) =$$

$$= \langle x|\mu^* * y\rangle = \langle x|\widetilde{\mu^*}y\rangle\,.$$

Hence

$$\widetilde{\mu}^* = \widetilde{\mu^*}$$

and the map is involutive.

d) This requires a long verification.

e) By d) and Proposition 2.1.5.15, we may assume $\mathbb{K} = \mathbb{C}$. In this case, however, the assertion follows from c), Example 2.3.2.27 c), and Proposition 2.3.2.26.

f) follows from Theorem 2.2.2.10 g).

g) Let $x' : \ell^1(T) \to \mathbb{K}$ be an algebra homomorphism. By Proposition 2.2.4.19, x' is continuous with $\|x'\| = 0$ or $\|x'\| = 1$. We may assume $\|x'\| = 1$. Take $t \in T$. By b),

$$1 = x'(1) = x'(e_t^* e_t) = x'(e_t^*)x'(e_t)\,.$$

Since

$$|x'(e_t)| \leq \|x'\|\,\|e_t\| = 1\,,$$

$$|x'(e_t^*)| \leq \|x'\|\,\|e_t^*\| = \|x'\|\,\|e_t\| = 1$$

it follows that

$$x'(e_t^*) = \overline{x'(e_t)}\,.$$

Hence x' is involutive. ■

Proposition 2.3.2.29 (4) *We use the notation of Proposition 2.3.2.28. In addition, we denote by λ a left and right invariant Haar measure on T.*

a) *$L^1(\lambda)$, canonically identified with a subset of $\mathcal{M}_b(T)$ (Theorem 2.2.2.10 e)), is a closed involutive subalgebra of $\mathcal{M}_b(T)$.*

b) $x \in L^1(\lambda)\,,\ t \in T \Rightarrow x^*(t) = \overline{x(t^{-1})}\,.$

c) *If we identify* $L^\infty(\lambda)$ *with* $L^1(\lambda)'$ *canonically, then*

$$x'(x^* * x) = \iint x'(s^{-1}t)\overline{x(s)}x(t)d\lambda(s)d\lambda(t)$$

for every $x \in L^1(T)$ *and* $x' \in L^\infty(T)$.

d) *Let* $L^1(T, \mathbb{K})$ *be the involutive algebra* $L^1(T)$ *with respect to* $\mathbb{K}$ *and* $\overset{\circ}{\overbrace{L^1}}(T, \mathbb{R})$ *the complexification of* $L^1(T, \mathbb{R})$. *Then the map*

$$\overset{\circ}{\overbrace{L^1}}(T, \mathbb{R}) \longrightarrow L^1(T,\mathbb{C}), \quad (x, y) \longmapsto x + iy$$

is an isomorphism of involutive complex algebras.

e) $L^1(\lambda)$ *is semi-simple.*

a & b. For $y \in C_0(T)$,

$$\int yd(x \cdot \lambda)^* = \int y(t^{-1})d\overline{x \cdot \lambda}(t) = \int y(t^{-1})\overline{x(t)}d\lambda(t) =$$

$$= \int y(t)\overline{x(t^{-1})}d\lambda(t).$$

Hence

$$x^*(t) = \overline{x(t^{-1})}$$

for every $t \in T$. Thus $x^* \in L^1(T)$ and the assertion follows from Theorem 2.2.2.10 e).

c) By b) and Theorem 2.2.2.10 e),

$$(x^* * x)(t) = \int x^*(s)x(s^{-1}t)d\lambda(s) =$$

$$= \int x^*(s^{-1})x(st)d\lambda(s) = \int \overline{x(s)}x(st)d\lambda(s)$$

so that

$$x'(x^* * x) = \int x'(t)(x^* * x)(t)d\lambda(t) = \int x'(t)(\int \overline{x(s)}x(st)d\lambda(s))d\lambda(t) =$$

$$= \int (\int x'(t)x(st)d\lambda(t))\overline{x(s)}d\lambda(s) = \int (\int x'(s^{-1}t)x(t)d\lambda(t))\overline{x(s)}d\lambda(s) =$$

$$= \iint x'(s^{-1}t)\overline{x(s)}x(t)d\lambda(s)d\lambda(t)\,.$$

d) follows from a) and Proposition 2.3.2.28 d).

e) By d) and Proposition 2.1.5.15, we may assume that $\mathbb{K} = \mathbb{C}$. By Proposition 2.3.2.28 c), there is an injective involutive algebra homomorphism of $L^1(\lambda)$ in $\mathcal{L}(L^2(\lambda))$. By Example 2.3.2.27 c) and Proposition 2.3.2.26, $L^1(T)$ is semi- simple. ■

Example 2.3.2.30 (4) *Let T be a finite group, $\ell^1(T)$ the involutive unital Banach algebra defined in Proposition 2.3.2.28 b), and E the involutive unital algebra of $T \times T$-matrices (Example 2.3.1.32). For $x \in \ell^1(T)$, let ux denote the matrix $[x(st^{-1})]_{s,t\in T}$.*

a) *$u : \ell^1(T) \to E$ is an injective involutive unital algebra homomorphism.*

b) *$\sigma_E(ux) = \sigma_{\ell^1(T)}(x)$ for every $x \in \ell^1(T)$.*

c) *$\ell^1(T)$ is strongly symmetric.*

a) u is obviously linear and injective. Given $x, y \in \ell^1(T)$ and $s, t \in T$

$$(x * y)(st^{-1}) = \sum_{r\in T} x(st^{-1}r^{-1})y(r) = \sum_{r\in T} x(sr^{-1})y(rt^{-1})\,,$$

$$x^*(st^{-1}) = \overline{x(ts^{-1})}\,,$$

$$\delta_1(st^{-1}) = \delta_{s,t}\,.$$

Hence u is an injective involutive unital algebra homomorphism.

b) If x is invertible in $\ell^1(T)$, then, by a), ux is invertible in E. Assume that ux is invertible in E. Then the determinant of ux does not vanish. Thus there is a unique $y \in \ell^1(T)$ such that

$$\sum_{t\in T} x(st^{-1})y(t) = \delta_{s1}$$

for every $s \in T$. Hence

$$x * y = \delta_1\,.$$

$$(ux)(uy) = u(x * y) = u(\delta_1) = 1\,,$$

$$u(y * x) = (uy)(ux) = 1\,,$$

$$y * x = \delta_1$$

(since u is injective), i.e. x is invertible. Hence

$$\sigma_E(ux) = \sigma_{\ell^1(T)}(x)\,.$$

c) follows from a), b), and Example 2.3.1.32. ■

Remark. If T is a compact group, then the involutive Banach algebra $L^1(T)$ is symmetric (Theorem 3.1.6.11). This is an generalization of c) above.

Example 2.3.2.31 *The map*

$$\ell^{2,2}(T,T) \longrightarrow \mathcal{L}(\ell^2(T))\,, \quad k \longmapsto \overset{\cap}{k}$$

is an injective involutive continuous algebra homomorphism (Example 2.3.2.5, Example 2.3.2.27 c)).

By Example 2.2.2.5 a), the map

$$\ell^{2,2}(T,T) \longrightarrow \mathcal{L}(\ell^2(T))\,, \quad k \longmapsto \overset{\cap}{k}$$

is an injective continuous algebra homomorphism. Let $k \in \ell^{2,2}(T,T)$. Then

$$(\overset{\cap}{k}^* x)(s) = \overline{(\overset{\cap}{k'}\overline{x})(s)} = \overline{\sum_{t\in T} k'(s,t)\overline{x}(t)} = \sum_{t\in T} \overline{k'(s,t)}x(t) =$$

$$= \sum_{t\in T} k^*(s,t)x(t) = (\overset{\cap}{k^*}x)(s)$$

for every $x \in \ell^2(T)$ and $s \in T$, so

$$\overset{\cap}{k}^* x = \overset{\cap}{k^*}x$$

for every $x \in \ell^2(T)$ and

$$\overset{\cap}{k}^* = \overset{\cap}{k^*}.$$

Hence the above map is involutive. ■

Example 2.3.2.32 (0) *Let $(T, \mathfrak{T}, \mu)$ be a measure space. Given $x \in L^\infty(\mu)$, define*

$$\widetilde{x} : L^2(\mu) \longrightarrow L^2(\mu), \quad y \longmapsto xy.$$

Then the map

$$L^\infty(\mu) \longrightarrow \mathcal{L}(L^2(\mu)), \quad x \longmapsto \widetilde{x}$$

(Example 2.2.2.22 a)) is a continuous homomorphism of involutive unital algebras (Example 2.3.2.3, Example 2.3.2.27 c)) and has the property that $\widetilde{x}$ is normal for every $x \in L^\infty(\mu)$. If $x \in L^\infty(\mu)$, then $\widetilde{x}$ is selfadjoint (unitary) iff

$$x(t) \in \mathbb{R} \quad (|x(t)| = 1)$$

for μ-almost all $t \in T$.

By Example 2.2.2.22 b), the map

$$L^\infty(\mu) \longrightarrow \mathcal{L}(L^2(\mu)), \quad x \longmapsto \widetilde{x}$$

is a continuous unital algebra homomorphism. Take $x \in L^\infty(\mu)$. Then

$$\langle y|\widetilde{x}^* z\rangle = \langle \widetilde{x}y|z\rangle = \int xy\bar{z}d\mu = \int y\overline{\bar{x}z}d\mu = \langle y|\bar{x}z\rangle = \langle y|\widetilde{\bar{x}}z\rangle = \langle y|\widetilde{x^*}z\rangle$$

for all $y, z \in L^2(\mu)$ (Example 2.3.2.27 b)), so

$$\widetilde{x}^* = \widetilde{x^*}$$

and the above map is involutive.

The last assertion follows immediately from the first one. ■

Proposition 2.3.2.33 (0) *Let E be a unital algebra. Take $x \in E$ with $\sigma(x) \subset \mathbb{R}$. Then the following are equivalent:*

a) $\sigma(x) \subset \mathbb{R}_+$.

b) $r(\alpha 1 - x) \leq \alpha$ *for every* $\alpha \in \mathbb{R}$ *with* $r(x) \leq \alpha$.

c) $r(\alpha 1 - x) \leq \alpha$ *for some* $\alpha \in \mathbb{R}$.

a $\Rightarrow$ b. We have that

$$\sigma(x) \subset [0, \alpha],$$

$$\sigma(\alpha 1 - x) = \alpha - \sigma(x) \subset [0, \alpha],$$

$$r(\alpha 1 - x) \leq \alpha.$$

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. We have that

$$\alpha - \sigma(x) = \sigma(\alpha 1 - x) \subset [-\alpha, \alpha],$$

so that

$$\sigma(x) \subset \alpha - [-\alpha, \alpha] = [0, 2\alpha] \subset \mathbb{R}_+. \qquad \blacksquare$$

Proposition 2.3.2.34 (0) *Let E be a symmetric involutive unital Banach algebra such that*

$$\|x\| = r(x)$$

for every $x \in \operatorname{Re} E$.

a) $\{x \in \operatorname{Re} E | \sigma(x) \subset \mathbb{R}_+\}$ *is a closed sharp convex cone of E.*

b) *E admits a unique order relation rendering it an ordered Banach space such that*

$$E_+ = \{x \in \operatorname{Re} E | \sigma(x) \subset \mathbb{R}_+\}.$$

c) *If $\mathbb{K} = \mathbb{C}$ and $n \in \mathbb{N}$, then*

$$x \in E_+ \Longrightarrow x^n \in E_+,$$
$$x \in \operatorname{Re} E \Longrightarrow x^{2n} \in E_+,$$
$$x \in E \Longrightarrow x^* x + x x^* \in E_+.$$

a) Put

$$C := \{x \in \operatorname{Re} E | \sigma(x) \subset \mathbb{R}_+\}.$$

Take $x, y \in C$. Define

$$\alpha := \|x\|, \quad \beta := \|y\|.$$

By Proposition 2.3.2.33 a $\Rightarrow$ b,

$$\|(\alpha+\beta)1-(x+y)\| = \|(\alpha 1-x)+(\beta 1-y)\| \leq$$

$$\leq \|\alpha 1-x\| + \|\beta 1-y\| \leq \alpha+\beta\,,$$

so that

$$\sigma(x+y) \subset \mathbb{R}_+$$

by Proposition 2.3.2.33 c $\Rightarrow$ a (and Proposition 2.3.1.29). Hence $x+y \in C$. Since obviously

$$\sigma(\gamma x) \subset \mathbb{R}_+$$

for every $\gamma \in \mathbb{R}_+$, C is a convex cone of E.

Take $x \in C \cap (-C)$. Then $-x \in C$, so that

$$-\sigma(x) \subset \sigma(-x) \subset \mathbb{R}_+\,.$$

Since

$$\sigma(x) \subset \mathbb{R}_+\,,$$

as well, we see that

$$\sigma(x) \subset \{0\}\,,$$

$$r(x) = 0\,,$$

$$x = 0\,.$$

Hence C is sharp.

By Proposition 2.3.2.33 c $\Rightarrow$ a $\Rightarrow$ b,

$$C = \{x \in \operatorname{Re} E \mid \|x\|1 - x\| \leq \|x\|\}\,.$$

Hence C is closed.

b) follows from a) and Lemma 1.7.1.3.

c) If $x \in E_+$, then, by Theorem 2.1.3.4 b)

$$\sigma(x^n) = \sigma(x)^n \subset \mathbb{R}_+\,,$$

so $x^n \in E_+$.

If $x \in \operatorname{Re} E$, then

$$\sigma(x^{2n}) = \sigma(x)^{2n} \subset \mathbb{R}_+,$$

by Theorem 2.1.3.4 b). Thus $x^{2n} \in E_+$.

Take $x \in E$. Define

$$a := \operatorname{re} x, \quad b := \operatorname{im} x.$$

Then

$$x^*x + xx^* = (a - ib)(a + ib) + (a + ib)(a - ib) =$$

$$= a^2 + iab - iba + b^2 + a^2 - iab + iba + b^2 = 2a^2 + 2b^2 \in E_+$$

by the above considerations and b). ■

Example 2.3.2.35 (0) *Let E be an involutive complex Banach (unital) algebra, $\overset{\circ}{E}$ the complexification of the underlying involutive real algebra of E (Proposition 2.3.1.40) and $\overset{\circ}{\overbrace{E'}}$ the complexification of the underlying involutive real vector space of E' (Proposition 2.3.1.38). For all $x, y \in E$, define*

$$x \updownarrow y := yx$$

and let $\overset{\leftrightarrow}{E}$ denote the involutive (unital) Banach algebra obtained from E by replacing the multiplication with

$$E \times E \longrightarrow E, \quad (x, y) \longmapsto x \updownarrow y.$$

a) *$\overset{\circ}{E}$ endowed with the norm*

$$\overset{\circ}{E} \longrightarrow \mathbb{R}_+, \quad (x, y) \longmapsto \sup\{\|x + iy\|, \|x - iy\|\}$$

is an involutive complex Banach (unital) algebra.

b) *If we endow $E \times \overset{\leftrightarrow}{E}$ with pointwise multiplication, pointwise involution, and supremum norm (Example 2.3.2.4), then the map*

$$\overset{\circ}{E} \longrightarrow E \times \overset{\leftrightarrow}{E}, \quad (x, y) \longmapsto ((x + iy), (x - iy)^*)$$

is an isometry of involutive Banach (unital) algebras.

c) $\overset{\circ}{\widehat{E'}}$ *endowed with the norm*

$$\overset{\circ}{\widehat{E'}} \longrightarrow \mathbb{R}_+, \quad (x', y') \longmapsto \|x' + iy'\| + \|x' - iy'\|$$

and involutive Banach space.

d) *For each* $(x', y') \in \overset{\circ}{\widehat{E'}}$, *define*

$$\widetilde{(x', y')} : \overset{\circ}{E} \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto \langle x + iy, x' + iy' \rangle + \overline{\langle x - iy, x' - iy' \rangle}.$$

Then $\widetilde{(x', y')} \in (\overset{\circ}{E})'$ *and the map*

$$\overset{\circ}{\widehat{E'}} \longrightarrow (\overset{\circ}{E})', \quad (x', y') \longmapsto \widetilde{(x', y')}$$

is an isometry of involutive Banach spaces.

e) *The map*

$$E \longrightarrow \overset{\leftrightarrow}{E}, \quad x \longmapsto x^*$$

is an isometry of involutive real (unital) Banach algebras which is conjugate linear.

f) *If* u *is a conjugate involution of* E *preserving the norms, then*

$$E \longrightarrow \overset{\leftrightarrow}{E}, \quad x \longmapsto (ux)^*$$

and

$$\overset{\circ}{E} \longrightarrow E \times E, \quad (x, y) \longmapsto (x + iy, ux + iuy)$$

are isometries of involutive complex (unital) Banach algebras, where $E \times E$ *is endowed with pointwise multiplication and involution and with supremum norm.*

a) Take $(x, y) \in \overset{\circ}{E}$ and $\alpha, \beta \in \mathbb{R}$. Then

$$(\alpha + i\beta)(x, y) = \alpha(x, y) + i\beta(x, y) = (\alpha x, \alpha y) + (-\beta y, \beta x) = (\alpha x - \beta y, \beta x + \alpha y),$$

$$\|(\alpha + i\beta)(x, y)\| = \sup\{\|(\alpha x - \beta y) + i(\beta x + \alpha y)\|, \|(\alpha x - \beta y) - i(\beta x + \alpha y)\|\} =$$

$$= \sup\{\|(\alpha + i\beta)(x + iy)\|, \|(\alpha - i\beta)(x - iy)\|\} =$$

$$= \sup\{|\alpha + i\beta| \, \|x + iy\|, |\alpha - i\beta| \, \|x - iy\|\} = |\alpha + i\beta| \, \|(x, y)\| ,$$

$$\|(x, y)^*\| = \|(x^*, -y^*)\| = \sup\{\|x^* - iy^*\|, \|x^* + iy^*\|\} =$$

$$= \sup\{\|(x + iy)^*\|, \|(x - iy)^*\|\} = \sup\{\|x + iy\|, \|x - iy\|\} = \|(x, y)\| .$$

Now take $(x_1, y_1), (x_2, y_2) \in \mathring{E}$. Then

$$\|(x_1 + x_2) \pm i(y_1 + y_2)\| \leq \|x_1 \pm iy_1\| + \|x_2 \pm iy_2\| \leq$$

$$\leq \|(x_1, y_1)\| + \|x_2, y_2)\| ,$$

$$\|(x_1x_2 - y_1y_2) \pm i(x_1y_2 + y_1x_2)\| = \|(x_1 \pm iy_1)(x_2 \pm iy_2)\| \leq$$

$$\leq \|x_1 \pm iy_1\| \, \|x_2 \pm iy_2\| \leq \|(x_1, y_1)\| \, \|(x_2, y_2)\| .$$

Hence

$$\|(x_1, y_1) + (x_2, y_2)\| \leq \|(x_1, y_1)\| + \|(x_2, y_2)\| ,$$

$$\|(x_1, y_1)(x_2, y_2)\| \leq \|(x_1, y_1)\| \, \|(x_2, y_2)\| .$$

b) Denote the map by u and take $(x, y) \in \mathring{E}$. Then

$$u(i(x, y)) = u((-y, x)) = ((-y + ix), (-y - ix)^*) =$$

$$= i((x + iy) , (x - iy)^*) = iu((x, y)) ,$$

$$u((x, y)^*) = u((x^*, -y^*)) = ((x^* - iy^*), (x^* + iy^*)^*) =$$

$$= ((x + iy)^* , (x - iy)) = u((x, y))^* .$$

Hence u is linear and involutive.

Take now $(x_1, y_1), (x_2, y_2) \in \mathring{E}$. Then

$$u((x_1, y_1)(x_2, y_2)) = u((x_1x_2 - y_1y_2, x_1y_2 + y_1x_2)) =$$

$$((x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)\,,\ ((x_1x_2 - y_1y_2) - i(x_1y_2 + y_1x_2))^*) =$$

$$= ((x_1 + iy_1)(x_2 + iy_2)\,,\ ((x_1 - iy_1)(x_2 - iy_2))^*) =$$

$$= ((x_1 + iy_1)(x_2 + iy_2)\,,\ (x_2 - iy_2)^*(x_1 - iy_1)^*) =$$

$$= ((x_1 + iy_1)(x_2 + iy_2)\,,\ (x_1 - iy_1)^* \updownarrow (x_2 - iy_2)^*) =$$

$$= u((x_1, y_1))u((x_2, y_2))\,.$$

Hence u is an isometry of involutive Banach (unital) algebras.

c) Take $(x', y') \in \overset{\circ}{\widetilde{E'}}$ and $\alpha, \beta \in \mathbb{R}$. Then

$$\|(\alpha + i\beta)(x', y')\| = \|(\alpha x' - \beta y', \beta x' + \alpha y')\| =$$

$$= \|(\alpha x' - \beta y') + i(\beta x' + \alpha y')\| + \|(\alpha x' - \beta y') - i(\beta x' + \alpha y')\| =$$

$$= \|(\alpha + i\beta)(x' + iy')\| + \|(\alpha - \imath\beta)(x' - iy')\| =$$

$$= |\alpha + \imath\beta|\,\|x' + \imath y'\| + |\alpha - i\beta|\,\|\|x' - iy'\| = |\alpha + \imath\beta|\,\|(x', y')\|\,,$$

and

$$\|(x', y')^*\| = \|(x'^*, -y'^*)\| = \|x'^* - \imath y'^*\| + \|x'^* + iy'^*\| =$$

$$= \|(x' + iy')^*\| + \|(x' - iy')^*\| = \|x' + iy'\| + \|x' - \imath y'\| = \|(x', y')\|\,.$$

d) $\widetilde{(x', y')}$ is obviously $\mathbb{R}$–linear. Take $(x, y) \in \overset{\circ}{E}$. Then

$$\langle \imath(x, y), \widetilde{(x', y')}\rangle = \langle (-y, x), \widetilde{(x', y')}\rangle =$$

$$= \langle -y + ix, x' + iy'\rangle + \overline{\langle -y - ix, x' - iy'\rangle} =$$

$$= i\langle x + iy, x' + iy'\rangle + i\,\overline{\langle x - iy, x' - iy'\rangle} = i\langle (x, y), \widetilde{(x', y')}\rangle\,.$$

Hence $\widetilde{(x', y')}$ is $\mathbb{C}$–linear and thus belongs to $(\overset{\circ}{E})'$.

We now prove the final assertion and denote for that the map by u. It is easy to see that u is $\mathbb{R}$–linear. Take $(x', y') \in \overset{\circ}{\widehat{E'}}$ and $(x, y) \in \overset{\circ}{E}$. Then

$$\langle (x,y), u(i(x',y')) \rangle = \langle (x,y), u((-y',x')) \rangle =$$

$$= \langle x+iy, -y'+ix' \rangle + \overline{\langle x-iy, -y'-ix' \rangle} =$$

$$= i\langle x+iy, x'+iy' \rangle + i\,\overline{\langle x-iy, x'-iy' \rangle} = \langle (x,y), iu((x',y')) \rangle .$$

Hence u is $\mathbb{C}$–linear.

We have

$$\langle (x,y), u^*((x',y')) \rangle = \langle (x,y), u((x',y')^*)^* \rangle = \langle (x,y), u((x'^*, -y'^*))^* \rangle =$$

$$= \overline{\langle (x,y)^*, u((x'^*, -y'^*)) \rangle} = \overline{\langle (x^*, -y^*), u((x'^*, -y'^*)) \rangle} =$$

$$= \overline{\langle x^* - iy^*, x'^* - iy'^* \rangle} + \langle x^* + iy^*, x'^* + iy'^* \rangle =$$

$$= \overline{\langle (x+iy)^*, (x'+iy')^* \rangle} + \langle (x-iy)^*, (x'-iy')^* \rangle =$$

$$= \langle x+iy, x'+iy' \rangle + \overline{\langle x-iy, x'-iy' \rangle} = \langle (x,y), u((x',y')) \rangle .$$

Hence $u^* = u$ and u is involutive.

Moreover,

$$|\langle (x,y), u((x',y')) \rangle| = |\langle x+iy, x'+iy' \rangle + \overline{\langle x-iy, x'-iy' \rangle}| \leq$$

$$\leq |\langle x+iy, x'+iy' \rangle| + |\langle x-iy, x'-iy' \rangle| \leq$$

$$\leq \|x+iy\|\,\|x'+iy'\| + \|x-iy\|\,\|x'-iy'\| \leq \|(x,y)\|(\|x'+iy'\| + \|x'-iy'\|) =$$

$$= \|(x,y)\|\,\|(x',y')\| .$$

Hence

$$\|u((x',y'))\| \leq \|(x',y')\| .$$

Take $\varepsilon > 0$. There are $x_0, y_0 \in E^{\#}$ such that

$$\langle x_0, x' + iy' \rangle > \|x' + iy'\| - \varepsilon\,, \quad \langle y_0, x' - iy' \rangle > \|x' - iy'\| - \varepsilon\,.$$

Put

$$x := \frac{1}{2}(x_0 + y_0)\,, \quad y := \frac{1}{2i}(x_0 - y_0)\,.$$

Then

$$x + iy = x_0\,, \qquad x - iy = y_0\,,$$

so that

$$\|(x, y)\| \leq 1$$

and

$$\|x' + iy'\| - \varepsilon + \|x' - iy'\| - \varepsilon < \langle x_0, x' + iy' \rangle + \overline{\langle y_0, x' - iy' \rangle} =$$

$$= \langle x + iy, x' + iy' \rangle + \overline{\langle x - iy, x' - iy' \rangle} = \langle (x, y), u((x', y')) \rangle \leq \|u((x', y'))\|\,.$$

Since ε is arbitrary,

$$\|(x', y')\| = \|x' + iy'\| + \|x' - iy'\| \leq \|u((x'y'))\|\,,$$

$$\|u((x', y'))\| = \|(x', y')\|\,.$$

Hence u' preserves the norms.

Finally, we show that u is surjective. Take $z' \in (\overset{\circ}{E})'$. Define

$$\begin{cases} a' : E \longrightarrow \mathbb{C}\,, & x \longmapsto \frac{1}{2} z'((x, -ix))\,, \\ b' : E \longrightarrow \mathbb{C}\,, & x \longmapsto \frac{1}{2}\overline{z'((x, ix))}\,. \end{cases}$$

a' and b' are obviously $\mathbb{R}$–linear. For any $x \in E$ we have

$$\begin{cases} a'(ix) = \frac{1}{2} z'((ix, x)) = \frac{1}{2} z'(i(x, -ix)) = \frac{i}{2} z'((x, -ix)) = ia'(x)\,, \\ b'(ix) = \frac{1}{2}\overline{z'((ix, -x))} = \frac{1}{2}\overline{z'(-i(x, ix))} = i(\frac{1}{2}\overline{z'((x, ix))}) = ib'(x)\,. \end{cases}$$

Hence a' and b' are $\mathbb{C}$–linear and therefore belong to E'. Put

$$\begin{cases} x' = \frac{1}{2}(a' + b')\,, \\ y' = \frac{1}{2i}(a' - b')\,. \end{cases}$$

Then

$$\begin{cases} x' + iy' = a', \\ x' - iy' = b'. \end{cases}$$

Thus for any $(x, y) \in \overset{\circ}{E}$,

$$\langle (x,y), \widetilde{(x',y')} \rangle = \langle x + iy, x' + iy' \rangle + \overline{\langle x - iy, x' - iy' \rangle} =$$

$$= a'(x + iy) + \overline{b'(x - iy)} = \frac{1}{2} z'((x + iy, -ix + y)) + \frac{1}{2} z'((x - iy, ix + y)) =$$

$$= \frac{1}{2} z'((2x, 2y)) = z'((x, y)).$$

Hence

$$z' = \widetilde{(x', y')} = u((x', y'))$$

and u is surjective.

e) is easy to see.

f) follows from e) and b). ■

Proposition 2.3.2.36 *Let E be a unital Banach algebra, U (resp. $\mathcal{U}$) the group of invertible elements of E (resp. $\mathcal{L}(E)$) and $\mathcal{F}$ be the set of elements of $\mathcal{U}$ which are unital algebra homomorphisms. Given $a \in U$, put*

$$\tilde{a} : E \longrightarrow E, \quad x \longmapsto axa^{-1}.$$

Then $\mathcal{F}$ is a subgroup of $\mathcal{U}$ closed in the topology of pointwise convergence. For each $a \in U$, $\tilde{a} \in \mathcal{F}$, the map

$$U \longrightarrow \mathcal{F}, \quad a \longmapsto \tilde{a}$$

is a group homomorphism and $\{\tilde{a} | a \in U\}$ is a normal subgroup of $\mathcal{F}$. If E is an involutive unital Banach algebra, then $\tilde{a}$ is involutive for every unitary $a \in E$.

It is obvious that $\mathcal{F}$ is a subgroup of $\mathcal{U}$. It follows from

$$\mathcal{F} := \left(\bigcap_{x,y \in E} \{u \in \mathcal{U} | u(xy) = (ux)(uy)\} \right) \cap \{u \in \mathcal{U} | u1 = 1\}$$

that $\mathcal{F}$ is a closed set of $\mathcal{U}$ in the topology of pointwise convergence.

Take $a \in U$. Then $\tilde{a} \in \mathcal{L}(E)$ and

$$\tilde{a}(xy) = axya^{-1} = axa^{-1}aya^{-1} = (\tilde{a}x)(\tilde{a}y)\,,$$

$$\tilde{a}1 = a1a^{-1} = 1$$

for every $x, y \in E$. Hence $\tilde{a}$ is a unital algebra homomorphism, i.e. $\tilde{a} \in \mathcal{F}$.

Take $a, b \in U$. Then

$$\widetilde{ab}x = (ab)x(ab)^{-1} = abxb^{-1}a^{-1} = \tilde{a}\tilde{b}x$$

$$\tilde{1}x = 1x1^{-1} = x$$

for every $x \in E$, so that

$$\widetilde{ab} = \tilde{a}\tilde{b}, \quad \tilde{1} = 1_{\mathcal{U}}\,.$$

Hence the map

$$U \longrightarrow \mathcal{F}, \quad a \longmapsto \tilde{a}$$

is a group homomorphism.

Take $a \in U$ and $u \in \mathcal{F}$. Then

$$(u^{-1}\tilde{a}u)x = u^{-1}(a(ux)a^{-1}) = (u^{-1}a)x(u^{-1}a^{-1}) = (u^{-1}a)x(u^{-1}a)^{-1} = \widetilde{u^{-1}a}x$$

for every $x \in E$, so that

$$u^{-1}\tilde{a}u = \widetilde{u^{-1}a}$$

and $\{\tilde{a} | a \in U\}$ is a normal subgroup of $\mathcal{F}$.

Now suppose that E is involutive and let a be a unitary element of E. Then

$$\tilde{a}^*x = (\tilde{a}x^*)^* = (ax^*a^*)^* = axa^* = \tilde{a}x$$

for every $x \in E$. Thus

$$\tilde{a}^* = \tilde{a}$$

and so $\tilde{a}$ is involutive. ■

Proposition 2.3.2.37 (6) *Let E be a complex involutive unital Banach algebra and x a normal (selfadjoint) element of E. Take $\alpha_0 \in \mathbb{C}$ ($\alpha_0 \in \mathbb{R}$), $r_1, r_2 \in \mathbb{R}_+$ such that*

$$U := \{\alpha \in \mathbb{C} | r_1 < |\alpha - \alpha_0| < r_2\} \subset \mathbb{C} \backslash \sigma(x)$$

and let

$$\sum_{n=-\infty}^{\infty} (t - \alpha_0)^n x_n$$

be the Laurent series of the resolvent of x *in* U *(Theorem 2.2.5.1 and 1.3.10.8). Then* x_n *is normal (selfadjoint) for every* $n \in \mathbb{Z}$ *and*

$$x_m x_n^* = x_n^* x_m$$

for $m, n \in \mathbb{Z}$.

Now

$$V := \{\alpha \in \mathbb{C} | r_1 < |\alpha - \overline{\alpha}_0| < r_2\} \subset \mathbb{C} \backslash \sigma(x^*)$$

and

$$\sum_{n=-\infty}^{\infty} (t - \overline{\alpha}_0)^n x_n^*$$

in the Laurent series of the resolvent of x^* in V (Proposition 2.3.1.14, Proposition 2.3.2.11).

If x is selfadjoint and $\alpha_0 \in \mathbb{R}$, then x_n is selfadjoint for every $n \in \mathbb{Z}$ (Theorem 1.3.10.8). Now suppose that x is normal. Then

$$x_m x_n^* = x_n^* x_m$$

for every $m, n \in \mathbb{Z}$, by Corollary 2.2.6.3. In particular, x_n is normal for every $n \in \mathbb{Z}$. ■

Proposition 2.3.2.38 *If* E *is an involutive normed space, then* $\operatorname{Re} E''^{\#}$ *is the closure of* $j_E(\operatorname{Re} E^{\#})$ *in* $E''_{E'}$.

By Proposition 2.3.2.22 e),k), the closure of $j_E(\operatorname{Re} E^{\#})$ in $E''_{E'}$ is contained in $\operatorname{Re} E''^{\#}$. Let $x'' \in \operatorname{Re} E''^{\#}$. By Corollary 1.3.6.8, there is a filter $\mathfrak{F}$ on $E^{\#}$ such that $j_E(\mathfrak{F})$ converges to x'' in $E''_{E'}$. By Proposition 2.3.2.7, $\operatorname{re}(\mathfrak{F})$ is a filter on $E^{\#}$ such that

$$x'' = \operatorname{re} x'' = \lim \operatorname{re}(j_E(\mathfrak{F})) = \lim j_E(\operatorname{re}(\mathfrak{F}))$$

(Proposition 2.3.2.22 e)). ■

2.3.3 Sesquilinear Forms

Definition 2.3.3.1 (0) *Let E, F, G be vector spaces. A map*

$$f : E \times F \longrightarrow G$$

is called ***sesquilinear*** *if for every $(x, y) \in E \times F$ the map $f(\cdot, y)$ is linear and the map $f(x, \cdot)$ is conjugate linear. If $E = F$, then we define*

$$\widetilde{f} : E \longrightarrow G, \quad x \longmapsto f(x, x)$$

and call $\widetilde{f}$ ***the associated quadratic map of*** *f. If, in addition, G is an involutive vector space, then we define*

$$f^* : E \times E \longrightarrow G, \quad (x, y) \longmapsto f(y, x)^*$$

and call f^* ***the adjoint of*** *f.*

If $G = \mathbb{K}$ then f is called ***sesquilinear form on*** *$E \times F$. A* ***sesquilinear form on*** *E is a sesquilinear form on $E \times E$.*

Every sesquilinear map is bilinear over $\mathbb{R}$.

Proposition 2.3.3.2 (0) *Let E, F be vector spaces, $f : E \times E \to F$ a sesquilinear map, $x, y \in E$, and $\alpha, \beta \in \mathbb{K}$.*

a) $\widetilde{f}(\alpha x + \beta y) = |\alpha|^2 \widetilde{f}(x) + \alpha\overline{\beta} f(x, y) + \overline{\alpha}\beta f(y, x) + |\beta|^2 \widetilde{f}(y)$.

b) $\widetilde{f}(x + y) + \widetilde{f}(x - y) = 2(\widetilde{f}(x) + \widetilde{f}(y))$ ***(Parallelogram Law).***

c) $\widetilde{f}(x + y) - \widetilde{f}(x - y) = 2(f(x, y) + f(y, x))$.

d) *If $\mathbb{K} = \mathbb{C}$, then*

$$f(x, y) = \frac{1}{4}(\widetilde{f}(x + y) - \widetilde{f}(x - y)) + \frac{i}{4}(\widetilde{f}(x + iy) - \widetilde{f}(x - iy)) =$$

$$= \frac{1}{4}\sum_{k=1}^{4} i^k \widetilde{f}(x + i^k y) \quad \textbf{(Polarization Identity)}.$$

a) $\widetilde{f}(\alpha x + \beta y) = f(\alpha x + \beta y, \alpha x + \beta y) = \alpha f(x, \alpha x + \beta y) + \beta f(y, \alpha x + \beta y) =$

$$= \alpha(\overline{\alpha} f(x, x) + \overline{\beta} f(x, y)) + \beta(\overline{\alpha} f(y, x) + \overline{\beta} f(y, y)) =$$

$$= |\alpha|^2 \tilde{f}(x) + \alpha\overline{\beta} f(x,y) + \overline{\alpha}\beta f(y,x) + |\beta|^2 \tilde{f}(y) .$$

b) By a),

$$\tilde{f}(x+y) + \tilde{f}(x-y) =$$

$$= \tilde{f}(x) + f(x,y) + f(y,x) + \tilde{f}(y) + \tilde{f}(x) - f(x,y) - f(y,x) + \tilde{f}(y) =$$

$$= 2(\tilde{f}(x) + \tilde{f}(y)) .$$

c) By a),

$$\tilde{f}(x+y) - \tilde{f}(x-y) =$$

$$= \tilde{f}(x) + f(x,y) + f(y,x) + \tilde{f}(y) - \tilde{f}(x) + f(x,y) + f(y,x) - \tilde{f}(y) =$$

$$= 2(f(x,y) + f(y,x)) .$$

d) By c),

$$\tilde{f}(x+iy) - \tilde{f}(x-iy) = 2(f(x,iy) + f(iy,x)) = 2(-if(x,y) + if(y,x)) ,$$

so

$$\frac{1}{4}(\tilde{f}(x+y) - \tilde{f}(x-y)) + \frac{i}{4}(\tilde{f}(x+iy) - \tilde{f}(x-iy)) =$$

$$= \frac{1}{2}(f(x,y) + f(y,x)) + \frac{i}{2}(-if(x,y) + if(y,x)) = f(x,y) . \quad \blacksquare$$

Remark. If f is sesquilinear and $\mathbb{K} = \mathbb{C}$, then, by the polarization identity,

$$f = 0 \iff \tilde{f} = 0 .$$

Proposition 2.3.3.3 (0) *Let E be a vector space, F an involutive vector space, and $\mathcal{S}$ the set of sesquilinear maps $E \times E \to F$.*

a) *$\mathcal{S}$ is a vector subspace of $F^{E \times E}$.*

b) *$f^* \in \mathcal{S}$ for every $f \in \mathcal{S}$.*

c) *$\mathcal{S}$ is an involutive vector space with respect to then involution*

$$\mathcal{S} \longrightarrow \mathcal{S}, \quad f \longmapsto f^*.$$

The selfadjoint elements of $\mathcal{S}$ are called ***Hermitian.***

d) *If $F = \mathbb{K}$, then the set*

$$\{f \in \mathrm{Re}\,\mathcal{S} | \widetilde{f}(E) \subset \mathbb{R}_+\}$$

is a sharp convex cone of $\mathcal{S}$.

e) *If $F = \mathbb{K}$, then there is a unique order on $\mathcal{S}$, with respect to which $\mathcal{S}$ is an ordered vector space such that*

$$\mathcal{S}_+ = \{f \in \mathrm{Re}\,\mathcal{S} | \widetilde{f}(E) \subset \mathbb{R}_+\}.$$

a), b), and c) are easy to verify.

d) Put

$$\mathcal{K} := (f \in \mathrm{Re}\,\mathcal{S} | \widetilde{f}(E) \subset \mathbb{R}_+\}.$$

It is easy to see that $\mathcal{K}$ is a convex cone. Let

$$f \in \mathcal{K} \cap (-\mathcal{K}).$$

Then

$$\widetilde{f}(x) = 0$$

for every $x \in E$. If $\mathbb{K} = \mathbb{R}$ (resp. $\mathbb{K} = \mathbb{C}$), then

$$f(x, y) = 0$$

for every $x, y \in E$, by Proposition 2.3.3.2 c) (resp. d)). Hence $f = 0$ and $\mathcal{K}$ is sharp.

e) follows from d) and Proposition 1.7.1.3. ■

Corollary 2 3.3.4 (∩) *Let E be a vector space, F an involutive vector space, and $f : E \times E \to F$ a Hermitian sesquilinear map. Then*

$$\widetilde{f}(\alpha x + \beta y) = |\alpha|^2 \widetilde{f}(x) + 2\mathrm{re}(\alpha\overline{\beta} f(x, y)) + |\beta|^2 \widetilde{f}(y)$$

for all $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$.

The assertion follows from Proposition 2.3.3.2 a). ■

Example 2.3.3.5 (0) *If T is a set, then the function*

$$\ell^2(T) \times \ell^2(T) \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \sum_{t \in T} x(t)\overline{y(t)}$$

is a positive sesquilinear form. If $(T, \mathfrak{T}, \mu)$ is a measure space, then the map

$$L^2(\mu) \times L^2(\mu) \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \int x\overline{y}d\mu$$

is a positive sesquilinear form. ■

Example 2.3.3.6 (0) *Let $n \in \mathbb{N}$, S the involutive vector space of sesquilinear forms on $\mathbb{K}^n$ (Proposition 2.3.3.3 c)), and $\mathbb{K}_{n,n}$ the involutive unital algebra of $n \times n$-matrices (Example 2.3.1.32). Given $a := [a_{ij}] \in \mathbb{K}_{n,n}$, put*

$$\widetilde{a} : \mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \sum_{i,j=1} a_{ij} x_i \overline{y}_j .$$

Then $\widetilde{a} \in S$ for every $a \in \mathbb{K}_{n,n}$ and the map

$$E \longrightarrow S, \quad a \longmapsto \widetilde{a}$$

is an isomorphism of involutive vector spaces. If a is a selfadjoint element of $\mathbb{K}_{n,n}$, then $\widetilde{a}$ is positive iff $\sigma(a) \subset \mathbb{R}_+$.

It is easy to see that $\widetilde{a} \in S$ for every $a \in \mathbb{K}_{n,n}$ and that the map

$$\mathbb{K}_{n,n} \longrightarrow S, \quad a \longmapsto \widetilde{a}$$

is linear and bijective. Take $a \in \mathbb{K}_{n,n}$. Then

$$\widetilde{a^*}(x,y) = \overline{\widetilde{a}(y,x)} = \sum_{i,j=1}^{n} \overline{a_{ij}}\, \overline{y}_i x_j = \sum_{i,j=1}^{n} (a^*)_{ji} x_j \overline{y}_i = \widetilde{a^*}(x,y)$$

for every $(x,y) \in \mathbb{K}^n \times \mathbb{K}^n$. Hence

$$\widetilde{a}^* = \widetilde{a^*}$$

and the above map is involutive.

Now take $a \in \mathbb{K}_{n,n}$ such that $\widetilde{a}$ is positive and $\alpha \in \sigma(a)$. Then there is an $x \in \mathbb{K}^n \backslash \{0\}$ with

$$\sum_{j=1}^{n} a_{ij} x_j = \alpha x_i$$

for every $i \in \mathbb{N}_n$. It follows

$$\alpha \sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} (\alpha x_i)\overline{x}_i = \sum_{i,j=1}^{n} a_{ij} x_j \overline{x}_i \in \mathbb{R}_+$$

and so $\sigma(a) \subset \mathbb{R}_+$.

Let now a be a selfadjoint element of $\mathbb{K}_{n,n}$ with $\sigma(a) \subset \mathbb{R}_+$ and

$$f : \mathbb{K}^n \longrightarrow \mathbb{K}, \quad x \longmapsto \widetilde{a}(x, x).$$

We have

$$\overline{f(x)} = \overline{\widetilde{a}(x,x)} = \sum_{i,j=1}^{n} = \overline{a}_{ij} \overline{x}_i x_j = \sum_{i,j=1}^{n} a_{ji} \overline{x}_i x_j =$$

$$= \sum_{i,j=1}^{n} a_{ij} x_i \overline{x}_j = \widetilde{a}(x, x) = f(x),$$

for every $x \in \mathbb{K}^n$, so f is real. Let x be a point of $\{\|x\| = 1\}$, where f takes its infimum. By the Lagrange Theorem, there is an α, so that

$$\sum_{j=1}^{n} a_{ij} \overline{x}_j = \alpha \overline{x}_i$$

for every $i \in \mathbb{N}_n$. We get $\alpha \in \sigma(a)$, so $\alpha \in \mathbb{R}_+$, and

$$f(x) = \sum_{i,j=1}^{n} a_{ij} x_i \overline{x}_j = \alpha \sum_{i=1}^{n} x_i \overline{x}_i = \alpha.$$

Hence

$$\widetilde{a}(y, y) \geq \|y\|^2 \widetilde{a}(x, x) = \alpha \|y\|^2 \geq 0$$

for every $y \in \mathbb{K}^n$ and $\widetilde{a}$ is positive. ■

Proposition 2.3.3.7 (0) *Let E be a real vector space, F an involutive real vector space, and*

$$f : E \times E \longrightarrow F$$

a sesquilinear map. Then the followmg are equivalent:

a) *f is Hermitian.*

b) *Given* $x, y \in E$,

$$f(x,y) = \frac{1}{4}(\tilde{f}(x+y) - \tilde{f}(x-y)).$$

a $\Rightarrow$ b. By Proposition 2.3.3.2 c),

$$\tilde{f}(x+y) - \tilde{f}(x-y) = 2(f(x,y) + f(y,x)) = 4f(x,y).$$

b $\Rightarrow$ a. Given $x, y \in E$,

$$f(x,y) = \frac{1}{4}(\tilde{f}(x+y) - \tilde{f}(x-y)) = \frac{1}{4}(\tilde{f}(y+x) - \tilde{f}(y-x)) = f(y,x)$$

(Proposition 2.3.3.2 a)), so that

$$f^* = f. \qquad \blacksquare$$

Proposition 2.3.3.8 (0) *Let E be a complex vector space, F an involutive complex vector space, and*

$$f : E \times E \longrightarrow F$$

a sesquilinear map. Then the following are equivalent:

a) *f is Hermitian.*

b) $\tilde{f}(E) \subset \operatorname{Re} F$.

c) $\operatorname{re} f(x,y) = \frac{1}{4}(\tilde{f}(x+y) - \tilde{f}(x-y))$ *for every* $x, y \in E$.

d) $\operatorname{im} f(x,y) = \frac{1}{4}(\tilde{f}(x+iy) - \tilde{f}(x-iy))$ *for every* $x, y \in E$.

In particular, if $F = \mathbb{C}$, f is positive (Proposition 2.3.3.3 e)) iff $\tilde{f}(E) \subset \mathbb{R}_+$.

a $\Rightarrow$ b. Since

$$\tilde{f}(x) = f(x,x) = f(x,x)^* = \tilde{f}(x)^*,$$

$$\tilde{f}(x) \in \operatorname{Re} F,$$

for every $x \in E$.

b $\Rightarrow$ c & d follows from the polarization identity (Proposition 2.3.3.2 d)).

c $\Rightarrow$ d. Given $x, y \in E$,

$$\operatorname{im} f(x,y) = \operatorname{re}(-\imath f(x,y)) = \operatorname{re} f(x,\imath y) = \frac{1}{4}(\tilde{f}(x+\imath y) - \tilde{f}(x-\imath y))$$

(Proposition 2.3.1.22).

d $\Rightarrow$ c. Given $x, y \in E$,

$$\operatorname{re} f(x,y) = \operatorname{im}(if(x,y)) = \operatorname{im} f(x,-iy) = \frac{1}{4}(\tilde{f}(x+y) - \tilde{f}(x-y))$$

(Proposition 2.3.1.22).

c & d $\Rightarrow$ a. Given $x, y \in E$,

$$\operatorname{re} f(x,y) = \frac{1}{4}(\tilde{f}(x+y) - \tilde{f}(x-y)) = \frac{1}{4}(\tilde{f}(y+x) - \tilde{f}(y-x)) = \operatorname{re} f(y,x)$$

$$\operatorname{im} f(x,y) = \frac{1}{4}(\tilde{f}(x+iy) - \tilde{f}(x-iy)) =$$

$$= \frac{1}{4}(\tilde{f}(y-ix) - \tilde{f}(y+ix)) = -\operatorname{im} f(y,x),$$

$$f(x,y) = f(y,x)^* = f^*(x,y)$$

(Proposition 2.3.3.2 a)). Thus

$$f = f^*.$$

The last assertion follows from a $\Leftrightarrow$ b (and Proposition 2.3.3.3 e)). ■

Proposition 2.3.3.9 (0) *Let f be a positive sesquilinear form on the vector space E (Proposition 2.3.3.3 e)). Then*

$$|f(x,y)| \leq \sqrt{\tilde{f}(x)\tilde{f}(y)} \quad \textit{(Schwarz Inequality)},$$

$$\sqrt{\tilde{f}(x+y)} \leq \sqrt{\tilde{f}(x)} + \sqrt{\tilde{f}(y)}$$

for every $x, y \in E$.

Take $\alpha \in \mathbb{R}$. Then

$$0 \leq \tilde{f}(\alpha x + f(x,y)y) = \alpha^2 \tilde{f}(x) + 2\alpha |f(x,y)|^2 + |f(x,y)|^2 \tilde{f}(y)$$

(Corollary 2.3.3.4). Thus

$$|f(x,y)|^4 \leq |f(x,y)|^2 \tilde{f}(x)\tilde{f}(y),$$

$$|f(x,y)| \leq \sqrt{\widetilde{f}(x)\widetilde{f}(y)}\,.$$

Hence, by Corollary 2.3.3.4,

$$\widetilde{f}(x+y) = \widetilde{f}(x) + 2\mathrm{re}\, f(x,y) + \widetilde{f}(y) \leq \widetilde{f}(x) + 2|f(x,y)| + \widetilde{f}(y) \leq$$

$$\leq \widetilde{f}(x) + 2\sqrt{\widetilde{f}(x)\widetilde{f}(y)} + \widetilde{f}(y) = \left(\sqrt{\widetilde{f}(x)} + \sqrt{\widetilde{f}(y)}\right)^2,$$

$$\sqrt{\widetilde{f}(x+y)} \leq \sqrt{\widetilde{f}(x)} + \sqrt{\widetilde{f}(y)}\,. \qquad \blacksquare$$

Example 2.3.3.10 *Let T be a locally compact group and $\mathcal{M}_b(T)$ the involutive unital Banach algebra defined in Proposition 2.3.2.28 a). Let λ be a left invariant Haar measure on T and $\mathcal{S}$ the involutive vector space of sesquilinear forms on $L^2(\lambda)$ (Proposition 2.3.3.3 c)). Define*

$$\langle x|y\rangle = \int x\bar{y}d\lambda$$

for $x, y \in L^2(\lambda)$ and

$$f_\mu : L^2(\lambda) \times L^2(\lambda) \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \langle \mu * x|y\rangle\,,$$

for $\mu \in \mathcal{M}_b(T)$ (Proposition 2.2.2.14 a)). Then $f_\mu \in \mathcal{S}$ whenever $\mu \in \mathcal{M}_b(T)$ and the map

$$u : \mathcal{M}_b(T) \longrightarrow \mathcal{S}\,, \quad \mu \longmapsto f_\mu$$

is an injective involutive linear map such that f_μ is Hermitian iff μ is selfadjoint and $f_{\mu^* * \mu}$ is positive for every $\mu \in \mathcal{M}_b(T)$ (Proposition 2.3.3.3 e)).

It is easy to see that $f_\mu \in \mathcal{S}$ whenever $\mu \in \mathcal{M}_b$ and that u is linear. By Proposition 2.3.2.28 c) and Example 2.3.2.27 b),

$$f_\mu^*(x,y) = \overline{f_\mu(y,x)} = \overline{\langle \mu * y|x\rangle} = \overline{\langle y|\mu^* * x\rangle} = \langle \mu^* * x|y\rangle = f_{\mu^*}(x,y)$$

for every $x, y \in L^2(\mu)$ and $\mu \in \mathcal{M}_b(T)$. Hence

$$f_\mu^* = f_{\mu^*}$$

for every $\mu \in \mathcal{M}_b(T)$. Thus u is involutive.

Take $\mu \in \mathcal{M}_b(T)$ with $f_\mu = 0$. Then

$$\langle \mu * x|y\rangle = f_\mu(x,y) = 0$$

for every $x, y \in L^2(\mu)$. Hence

$$\mu * x = 0$$

for every $x \in L^2(\mu)$. By Proposition 2.2.2.14 e), $\mu = 0$ and u is injective.

Let $\mu \in \mathcal{M}_b(T)$. If μ is selfadjoint, then f_μ is Hermitian, since u is involutive. If f_μ is Hermitian, then

$$f_{\mu-\mu^*} = f_\mu - f_{\mu^*} = f_\mu - f_\mu^* = 0\,,$$

$$\mu - \mu^* = 0$$

since u is injective and μ is selfadjoint.

By Propositions 2.2.2.14 e), 2.3.2.28 c), and Example 2.3.2.27 b),

$$\langle (\mu^* * \mu)x | x \rangle = \langle \mu^* * (\mu * x) | x \rangle = \langle \mu * x | \mu * x \rangle \in \mathbb{R}_+$$

for every $x \in L^2(\lambda)$. Hence $f_{\mu^* * \mu}$ is positive. ■

Proposition 2.3.3.11 (1) *Let T be a set, $\ell^{2,2}(T,T)$ the involutive Banach algebra defined in Example 2.3.2.5, and $\mathcal{S}$ the involutive vector space of sesquilinear forms on $\ell^2(T)$ (Proposition 2.3.3.3 c)). For $x, y \in \ell^2(T)$ and $k \in \ell^{2,2}(T,T)$, define*

$$\langle x | y \rangle := \sum_{t \in T} x(t)\overline{y(t)}$$

and

$$f_k : \ell^2(T) \times \ell^2(T) \longrightarrow \mathbb{K}\,, \quad (x,y) \longmapsto \langle \overset{\cap}{k} x | y \rangle\,.$$

a) $\langle \overset{\cap}{k} x | y \rangle = \langle x | \overset{\cap}{k^*} y \rangle$ *for every $x, y \in \ell^2(T)$ and $k \in \ell^{2,2}(T,T)$.*

b) $f_k \in \mathcal{S}$ *for every $k \in \ell^{2,2}(T,T)$ and the map*

$$\ell^{2,2}(T,T) \longrightarrow \mathcal{S}\,, \quad k \longmapsto f_k$$

is injective, linear, and involutive.

c) *For every $k \in \ell^{2,2}(T,T)$ the sesquilinear form f_k is Hermitian iff k is selfadjoint.*

d) *The sesquilinear form $f_{k^* \circ k}$ is positive whenever $k \in \ell^{2,2}(T,T)$.*

e) *The map*

$$\ell^{2,2}(T,T) \times \ell^{2,2}(T,T) \longrightarrow \mathbb{K}, \quad (h,k) \longmapsto \sum_{t\in T}(h \circ k^*)(t,t)$$

is a positive sesquilinear form with the property that

$$\sum_{t\in T}(k \circ k^*)(t,t) = \|k\|^2$$

for every $k \in \ell^{2,2}(T,T)$.

a) Now

$$\langle \overset{\cap}{k}x|y\rangle = \sum_{t\in T}(\overset{\cap}{k}x)(t)\overline{y(t)} = \sum_{t\in T}\overline{y(t)}\sum_{s\in T}k(t,s)x(s) =$$

$$= \sum_{s\in T}x(s)\sum_{t\in T}k(t,s)\overline{y(t)} = \sum_{s\in T}x(s)\sum_{t\in T}\overline{k^*(s,t)}\,\overline{y(t)} =$$

$$= \sum_{s\in T}x(s)\overline{\sum_{t\in T}k^*(s,t)y(t)} = \sum_{s\in T}x(s)\overline{(\overset{\cap}{k^*}y)(s)} = \langle x|\overset{\cap}{k^*}y\rangle .$$

b) It is obvious that $f_k \in \mathcal{S}$ whenever $k \in \ell^{2,2}(T,T)$ and that the map

$$\ell^{2,2}(T,T) \longrightarrow \mathcal{S}, \quad k \longmapsto f_k$$

is linear. It is injective since if $k \in \ell^{2,2}(T,T)$ and $f_k = 0$, then

$$0 = f_k(e_t,e_s) = \langle \overset{\cap}{k}e_t|e_s\rangle = (\overset{\cap}{k}e_t)(s) = \sum_{r\in T}k(s,r)e_t(r) = k(s,t)$$

for every $s,t \in T$.

Take $k \in \ell^{2,2}(T,T)$. Then, by a),

$$f_k^*(x,y) = \overline{f_k(y,x)} = \overline{\langle \overset{\cap}{k}y|x\rangle} = \overline{\langle y|\overset{\cap}{k^*}x\rangle} = \langle \overset{\cap}{k^*}x|y\rangle = f_{k^*}(x,y)$$

for every $x,y \in \ell^2(T)$, so that

$$f_k^* = f_{k^*} .$$

Hence the above map is involutive.

c) follows from b) and Proposition 2.3.1.7.

d) For $x \in \ell^2(T)$,

$$f_{k^*\circ k}(x,x) = \langle \overbrace{k^* \circ kx}^{\cap} | x\rangle = \langle \overset{\cap}{k^*}\overset{\cap}{k}x | x\rangle = \langle \overset{\cap}{k}x | \overset{\cap}{k}x\rangle \in \mathbb{R}_+$$

by a) and Example 2.2.2.5 a). Hence, by c), $f_{k^*\circ k}$ is positive.

e) Given $h, k \in \ell^{2,2}(T,T)$,

$$\sum_{t\in T}(h\circ k^*)(t,t) = \sum_{t\in T}\sum_{s\in T} h(t,s)k^*(s,t) = \sum_{s,t\in T} h(t,s)\overline{k(t,s)}$$

and the assertions now follow. ∎

Remark. This proposition will be generalized in Proposition 3.1.6.18

Proposition 2.3.3.12 (M.H. Stone, 1932) *Let f, g be sesquilinear forms on a vector space E such that g is positive. Then the set*

$$\{\tilde{f}(x) | x \in E, \quad \tilde{g}(x) = 1\}$$

is convex.

Take $x, y \in E$ with

$$\tilde{g}(x) = \tilde{g}(y) = 1\,.$$

$$\tilde{f}(x) \neq \tilde{f}(y)\,.$$

Define

$$f' : \mathbb{K}\times\mathbb{K} \longrightarrow \mathbb{K}, \quad (\alpha,\beta) \longmapsto \tilde{f}(\alpha x + \beta y)\,,$$

$$g' : \mathbb{K}\times\mathbb{K} \longrightarrow \mathbb{K}, \quad (\alpha,\beta) \longmapsto \tilde{g}(\alpha x + \beta y)\,,$$

$$h := \frac{f' - \tilde{f}(y)g'}{\tilde{f}(x) - \tilde{f}(y)}\,.$$

Take $a, b \in \mathbb{K}$ such that

$$h(\alpha,\beta) = |\alpha|^2 + a\alpha\overline{\beta} + \overline{b}\overline{\alpha}\beta$$

for all $\alpha, \beta \in \mathbb{K}$ (Proposition 2.3.3.2 a)). Define

$$\gamma := \begin{cases} 1 & \text{if} \quad a = b\,, \\ \frac{b-a}{|b-a|} & \text{if} \quad a \neq b\,, \end{cases}$$

$$\delta := a\overline{\gamma} + \overline{b}\gamma \in \mathbb{R}\,,$$

$$\varepsilon := \operatorname{re}\overline{\gamma}g(x,y) \in [-1,1]$$

(Proposition 2.3.3.9) and

$$\beta_\alpha := -\varepsilon\alpha + (1-(1-\varepsilon^2)\alpha^2)^{\frac{1}{2}} \in \mathbb{R}$$

for every $\alpha \in [0,1]$. Then

$$g'(\alpha,\beta_\alpha\gamma) = \tilde{g}(\alpha x + \beta_\alpha\gamma y) = \alpha^2 + 2\varepsilon\alpha\beta_\alpha + \beta_\alpha^2 = 1\,,$$

$$h(\alpha,\beta_\alpha\gamma) = \alpha^2 + a\alpha\beta_\alpha\overline{\gamma} + \overline{b}\alpha\beta_\alpha\gamma = \alpha^2 + \delta\alpha\beta_\alpha \in \mathbb{R}$$

for every $\alpha \in [0,1]$ (Corollary 2.3.3.4). We get

$$h(0,\beta_0\gamma) = 0\,, \quad h(1,\beta_1\gamma) = 1\,.$$

Take $\lambda \in [0,1]$. Then there is an $\alpha \in [0,1]$ with

$$h(\alpha,\beta_\alpha\gamma) = \lambda\,.$$

It follows that

$$\tilde{f}(\alpha x + \beta_\alpha\gamma y) = f'(\alpha,\beta_\alpha\gamma) = (\tilde{f}(x) - \tilde{f}(y))h(\alpha,\beta_\alpha\gamma) + \tilde{f}(g)g'(\alpha,\beta_\alpha\gamma) =$$

$$= \lambda\tilde{f}(x) + (1-\lambda)\tilde{f}(y)\,.$$

Hence

$$\{\tilde{f}(x) \mid x \in E\,,\ \tilde{g}(x) = 1\}$$

is convex. ■

2.3.4 Positive Linear Forms

Definition 2.3.4.1 (0) *Let E be an involutive algebra. The involutive linear form x' on E is called* **positive** *if*

$$x'(x^*x) \in \mathbb{R}_+$$

for every $x \in E$. In this case, we set

$$N_{x'} := \{x \in E | x'(x^*x) = 0\}.$$

Care should be taken with the word "positive" above, since, in general, F, the vector space of linear forms on E, does not admit an order relation rending it an ordered vector space for which F_+ is the set of positive linear forms – for example take the case when $x^*x = 0$ for every $x \in E$. If, however, E is an involutive quasiunital Banach algebra, then E' **does** admit such an order (Proposition 2.3.4.13 b)) and this is the only case which we pursue in this book. We find it convenient to avail oursleves of the expression "positive" in the more general setting defined above, rather than using more elliptic expressions.

The expression "positive linerar form" was already introduced in Proposition 1.7.1.8 in another context. This will produce no confusion in this book, since in the case in which both expressions are defined they will coincide.

Example 2.3.4.2 *Let T be a unimodular locally compact group and λ the Haar measure on T. Identify canonically $L^\infty(\lambda)$ with $L^1(\lambda)'$ and take $x' \in \mathrm{Re}\, L^1(\lambda)'$. Then the following are equivalent:*

a) *x' is positive (Proposition 2.3.2.29 a)).*

b) $x \in L^1(\lambda) \Rightarrow \int\int x'(s^{-1}t)\overline{x(s)}x(t)d\lambda(s)d\lambda(t) \in \mathbb{R}_+$.

The assertion follows immediately from Proposition 2.3.2.29 c). ■

Example 2.3.4.3 *Take $n \in \mathbb{N}$ and $[a_{ij}] \in \mathbb{K}_{n,n}$. Then the following are equivalent:*

a) *The sesquilinear form*

$$\mathbb{K}^n \times \mathbb{K}^n \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \sum_{i,j=1}^{n} a_{ij}x_i\overline{y}_j$$

(Example 2.3.3.6) is positive.

b) *The linear form*

$$\mathbb{K}_{n,n} \longrightarrow \mathbb{K}, \quad [\alpha_{ij}] \longmapsto \sum_{i,j=1}^{n} \alpha_{ij} a_{ji}$$

is positive (Example 2.3.1.32).

If these conditions are fulfilled, then $\sigma([a_{ij}]) \subset \mathbb{R}_+$.

a $\Rightarrow$ b.

$$\sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik}^* \alpha_{kj} \right) a_{ji} = \sum_{i,j=1}^{n} \left(\sum_{k=1}^{n} \overline{\alpha}_{ki} \alpha_{kj} \right) a_{ji} =$$

$$= \sum_{k=1}^{n} \sum_{i,j=1}^{n} a_{ji} \alpha_{kj} \overline{\alpha}_{ki} \in \mathbb{R}_+ .$$

b $\Rightarrow$ a. Take $x \in \mathbb{K}^n$ and put

$$\alpha_{ij} := x_j \delta_{i1}$$

for $i, j \in \mathbb{N}_n$. Then

$$(\alpha^* \alpha)_{ij} = \sum_{k=1}^{n} \alpha_{ik}^* \alpha_{kj} = \sum_{k=1}^{n} \overline{\alpha_{ki}} \alpha_{kj} = \sum_{k=1}^{n} \overline{x}_i \delta_{k1} x_j \delta_{k1} = \overline{x_i} x_j ,$$

so that

$$\sum_{i,j=1}^{n} a_{ij} x_i \overline{x_j} = \sum_{i,j=1}^{n} a_{ij} (\alpha^* \alpha)_{ji} \in \mathbb{R}_+ .$$

The last assertion follows from Example 2.3.3.6. ■

Example 2.3.4.4 *Given*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{K}_{2,2} ,$$

the following are equivalent:

a) *The sesquilinear form*

$$\mathbb{K}^2 \times \mathbb{K}^2 \longrightarrow \mathbb{K}, \quad ((x,y),(X,Y)) \longmapsto ax\overline{X} + bx\overline{Y} + cy\overline{X} + dy\overline{Y}$$

is positive.

b) *The linear form*

$$\mathbb{K}_{2,2} \longrightarrow \mathbb{K}, \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \longmapsto \alpha a + \gamma b + \beta c + \delta d$$

is positive (Example 2.3.1.32).

c) $a, d \in \mathbb{R}_+, \quad b = \bar{c}, \quad |b|^2 \leq ad.$

a $\Leftrightarrow$ b follows from Example 2.3.4.3.
a $\Leftrightarrow$ c is easy to see. ■

Proposition 2.3.4.5 (0) *If E is an involutive algebra, then every involutive algebra homomorphism of E in $\mathbb{K}$ is positive.*

Let

$$x' : E \longrightarrow \mathbb{K}$$

be an involutive algebra homomorphism. Then

$$x'(x^*x) = x'(x^*)x'(x) = \overline{x'(x)}x'(x) = |x'(x)|^2 \geq 0$$

for every $x \in E$ and x' is therefore positive. ■

Proposition 2.3.4.6 (0) *Let E be an involutive algebra, x' a linear form on E, and*

$$f : E \times E \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto x'(y^*x).$$

a) *f is sesquilinear.*

b) *f is Hermitian (positive) if x' is involutive (positive); the converse implication also holds if E is unital.*

c) *If x' is positive, then*

$$|x'(x^*y)|^2 \leq x'(x^*x)x'(y^*y)$$

for every $x, y \in E$ and $N_{x'}$ is a left ideal of E.

d) *If x' is involutive, then*

$$\operatorname{re} x'(x) = x'\Big(\frac{1}{2}(x + x^*)\Big)$$

for every $x \in E$.

e) *If* $\mathbb{K} = \mathbb{C}$ *and* E *is unital, then* x' *is positive iff* $x'(x^*x) \in \mathbb{R}_+$ *for every* $x \in E$.

a) is trivial.

b) For $x, y \in E$,

$$f(x,y) = x'(y^*x), \quad f(y,x) = x'(x^*y) = \overline{x'^*(y^*x)}, \quad \tilde{f}(x) = x'(x^*x)$$

and the assertions now follow.

c) The inequality and the fact that $N_{x'}$ is a vector subspace of E follow from b) and Proposition 2.3.3.9. The inequality implies that $N_{x'}$ is a left ideal of E.

d) The complex case follows from Proposition 2.3.1.22 and Corollary 2.3.1.23 a $\Rightarrow$ d. Assume that $\mathbb{K} = \mathbb{R}$. Then

$$x'(x^*) = x'(x),$$

so that

$$x'(\frac{1}{2}(x + x^*)) = \frac{1}{2}(x'(x) + x'(x^*)) = x'(x)$$

e) follows from b) and the last assertion of Proposition 2.3.3.8. ■

Corollary 2.3.4.7 (0) *Let* E *be an involutive unital Banach algebra and* x' *a positive linear form on* E. *Then* x' *is continuous and*

$$\|x'\| = x'(1).$$

Now

$$x'(1) = x'(1^*1) \geq 0.$$

Take $x \in E$ with $\|x\| < 1$. Then

$$\|x^*x\| \leq \|x^*\| \, \|x\| = \|x\|^2 < 1.$$

Hence, by Propositions 2.3.2.19 b) and 2.2.3.13 c), there is a selfadjoint element y of E such that

$$y^2 = 1 - x^*x.$$

Then

$$x'(1) - x'(x^*x) = x'(y^2) \geq 0,$$

$$|x'(x)|^2 \leq x'(1^*1)x'(x^*x) \leq x'(1)^2$$

(Proposition 2.3.4.6 c)). Thus x' is continuous and

$$\|x'\| \leq x'(1)\,.$$

Since

$$x'(1) \leq \|x'\|\,\|1\| = \|x'\|\,,$$

$$\|x'\| = x'(1)\,. \qquad \blacksquare$$

Remark. We cannot replace "Banach" by "normed" in the above corollary, as the following example shows. Let E be the involutive unital Banach algebra consisting of the unital Banach algebra $\ell^1(\mathbb{Z})$ (Example 2.2.2.7) together with the involution

$$E \longrightarrow E\,, \quad x \longmapsto \overline{x}$$

and set

$$x' : \mathbb{K}^{(\mathbb{Z})} \longrightarrow \mathbb{K}\,, \quad x \longmapsto \sum_{n\in\mathbb{Z}} 2^n x(n)\,.$$

$\mathbb{K}^{(\mathbb{Z})}$ is an involutive unital subalgebra of E and x' is an involutive linear form on it, but it is not continuous. For $x \in \mathbb{K}^{(\mathbb{Z})}$,

$$x'(x^*x) = \sum_{n\in\mathbb{Z}} 2^n (x^*x)(n) = \sum_{n\in\mathbb{Z}} 2^n \sum_{p\in\mathbb{Z}} \overline{x(n-p)}\,x(p) =$$

$$= \sum_{p\in\mathbb{Z}} x(p) \sum_{n\in\mathbb{Z}} 2^n \overline{x(n-p)} = \sum_{p\in\mathbb{Z}} x(p) \sum_{n\in\mathbb{Z}} 2^{n+p} \overline{x(n)} = \Big| \sum_{n\in\mathbb{Z}} 2^n x(n) \Big|^2 \geq 0$$

and so x' is positive.

Corollary 2.3.4.8 (0) *Let x' be a positive linear form on the involutive unital Banach algebra E. Take $x, y \in E$. Then*

$$|x'(y^*xy)| \leq \|x\| x'(y^*y)\,.$$

Let F be a unital involutive Banach algebra associated to E and take

$$y' : F \longrightarrow \mathbb{K}\,, \quad z \longmapsto x'(y^*zy)\,.$$

Then

$$y'(z^*z) = x'(y^*z^*zy) = x'((zy)^*(zy)) \in \mathbf{R}_+ .$$

Hence y' is positive. Then

$$|x'(y^*xy)| = |y'(x)| \leq \|x\| \, \|y'\| = \|x\| y'(1) = \|x\| x'(y^*y) ,$$

by Corollary 2.3.4.7. ∎

Proposition 2.3.4.9 (0) *Let E be an involutive Banach algebra and $\mathfrak{F}$ an approximate unit of E. Define*

$$f : E \longrightarrow E, \quad x \longmapsto x^* ,$$
$$g : E \longrightarrow E, \quad x \longmapsto \operatorname{re} x ,$$
$$h : E \longrightarrow E, \quad x \longmapsto x^*x .$$

Then $f(\mathfrak{F})$, $g(\mathfrak{F})$, and $h(\mathfrak{F})$ are approximate units of E and

$$\lim_{x,\mathfrak{F}} x^*yx = y$$

for every $y \in E$. In particular, E possesses an approximate unit $\mathfrak{G}$ such that $\operatorname{Re} E^{\#} \in \mathfrak{G}$.

Since

$$0 = \lim_{x,\mathfrak{F}} \|y^*x - y^*\| = \lim_{x,\mathfrak{F}} \|x^*y - y\| = \lim_{x,f(\mathfrak{F})} \|xy - y\| ,$$

$$0 = \lim_{x,\mathfrak{F}} \|xy^* - y^*\| = \lim_{x,\mathfrak{F}} \|yx^* - y\| = \lim_{x,f(\mathfrak{F})} \|yx - y\| ,$$

for every $y \in F$, $f(\mathfrak{F})$ is an approximate unit of E. Take $y \in E$. Then

$$\|(\operatorname{re} x)y - y\| = \|\frac{1}{2}(x + x^*)y - y\| \leq \frac{1}{2}\|xy - y\| + \frac{1}{2}\|x^*y - y\| ,$$

$$\|y(\operatorname{re} x) - y\| = \|\frac{1}{2}y(x + x^*) - y\| \leq \frac{1}{2}\|yx - y\| + \frac{1}{2}\|yx^* - y\| ,$$

$$\|x^*xy - y\| \leq \|x^*xy - x^*y\| + \|x^*y - y\| \leq \|xy - y\| \, \|x^*\| + \|x^*y - y\| ,$$

$$\|yx^*x - y\| \leq \|yx^*x - yx\| + \|yx - y\| \leq \|yx^* - y\| \, \|x\| + \|yx - y\|$$

for every $x \in E$. Thus

$$\lim_{x,g(\mathfrak{F})} \|xy - y\| = 0\,, \quad \lim_{x,g(\mathfrak{F})} \|yx - y\| = 0\,,$$

$$\lim_{x,h(\mathfrak{F})} \|xy - y\| = 0\,, \quad \lim_{x,h(\mathfrak{F})} \|yx - y\| = 0$$

by the above considerations. Since y is arbitrary, $g(\mathfrak{F})$ and $h(\mathfrak{F})$ are approximate units of E (Proposition 2.3.2.7). Moreover

$$\|x^*yx - y\| \leq \|x^*yx - yx\| + \|yx - y\| \leq \|x^*y - y\|\,\|x\| + \|yx - y\|$$

for every $x \in E$, so that

$$\lim_{x,\mathfrak{F}} \|x^*yx - y\| = 0\,. \qquad \blacksquare$$

Proposition 2.3.4.10 (0) *Let E be an involutive Banach algebra, $\mathfrak{F}$ an approximate unit of E, and x' a continuous positive linear form on E.*

a) $\lim_{x,\mathfrak{F}} x'(x^*x) = \|x'\|$.

b) $|x'(x)|^2 \leq \|x'\|x'(x^*x)$ *for every* $x \in E$.

c) $N_{x'} + N_{x'}^* \subset \operatorname{Ker} x'$.

d) $\|x'\| = \sup_{x \in E^\#} x'(x^*x)$.

a) Take $x, y \in E^\#$. Since

$$|x'(x^*y)|^2 \leq x'(x^*x)x'(y^*y) \leq x'(x^*x)\|x'\|$$

(Proposition 2.3.4.6 c)),

$$|x'(y)|^2 = \lim_{x,\mathfrak{F}} |x'(x^*y)|^2 \leq \|x'\| \liminf_{x,\mathfrak{F}} x'(x^*x)$$

for every $y \in E^\#$ (Proposition 2.3.4.9). Hence

$$\|x'\|^2 \leq \|x'\| \liminf_{x,\mathfrak{F}} x'(x^*x)\,,$$

$$\|x'\| \leq \liminf_{x,\mathfrak{F}} x'(x^*x) \leq \limsup_{x,\mathfrak{F}} x'(x^*x) \leq \|x'\|$$

(since $E^\# \in \mathfrak{F}$), and

$$\|x'\| = \lim_{x,\mathfrak{F}} x'(x^*x)\,.$$

b) By Proposition 2.3.4.6 c),

$$|x'(y^*x)|^2 \leq x'(x^*x)x'(y^*y)$$

for every $y \in E$. Hence

$$|x'(x)|^2 = \lim_{y,\mathfrak{F}} |x'(y^*x)|^2 \leq x(x^*x) \lim_{y,\mathfrak{F}} x'(y^*y) = \|x'\|x'(x^*x)$$

by a) and Proposition 2.3.4.9.

c) follows from b).

d) By b),

$$\|x'\|^2 = \sup_{x\in E^\#} |x'(x)|^2 \leq \|x'\| \sup_{x\in E^\#} x'(x^*x) \leq \|x'\|^2\,,$$

$$\sup_{x\in E^\#} x'(x^*x) = \|x'\|\,. \qquad \blacksquare$$

Proposition 2.3.4.11 (0) *Let E be an involutive quasiunital non-unital normed algebra and F an associated involutive unital Banach algebra. Take $x' \in E'$, $\alpha \in \mathbb{K}$, and let y' be the linear extension of x' to F defined by*

$$y'(1) = \alpha\,.$$

Then the following are equivalent:

a) *x' is positive, $\alpha \in \mathbb{R}_+$, $\alpha \geq \|x'\|$.*

b) *y' is positive.*

Take $(\beta, x) \in \mathbb{K} \times E$. Then

$$(\beta 1 + x)^*(\beta 1 + x) = |\beta|^2 1 + \overline{\beta}x + \beta x^* + x^*x\,,$$

$$y'((\beta 1 + x)^*(\beta 1 + x)) = \alpha|\beta|^2 + \overline{\beta}x'(x) + \beta x'(x^*) + x'(x^*x)\,.$$

a $\Rightarrow$ b. We may assume that $x' \neq 0$. By Proposition 2.3.4.10 b),

$$\|x'\|y'((\beta 1 + x)^*(\beta 1 + x)) \geq$$

$$\geq \|x'\|^2|\beta|^2 - 2|\beta|\,\|x'\|\,|x'(x)| + |x'(x)|^2 =$$

$$= (\|x'\|\,|\beta| - |x'(x)|)^2 \geq 0$$

for every $(\beta, x) \in \mathbb{K} \times E$. Hence y' is positive.

b $\Rightarrow$ a. It is easy to see that x' is positive and that $\alpha \in \mathbb{R}_+$. Now, since

$$0 \leq y'((\beta 1 + x)^*(\beta 1 + x)) \leq \alpha\beta^2 + 2\beta x'(x) + \|x'\|$$

for every $(\beta, x) \in \mathbb{R} \times \operatorname{Re} E^{\#}$,

$$x'(x)^2 \leq \alpha\|x'\|$$

for every $x \in \operatorname{Re} E^{\#}$. By Proposition 2.3.2.22 j),

$$\|x'\|^2 = \sup\{x'(x)^2 | x \in \operatorname{Re} E^{\#}\} \leq \alpha\|x'\|\,.$$

$$\|x'\| \leq \alpha\,. \qquad \blacksquare$$

Corollary 2.3.4.12 (0) *Let E be an involutive quasiunital Banach algebra and F an associated involutive unital Banach algebra. Then every continuous positive linear form x' on E has a unique positive linear extension y' to F such that (Corollary 2.3.4.7)*

$$\|y'\| = \|x'\|\,.$$

Then

$$y'(1) = \|x'\|\,.$$

We may assume E to be non-unital.

By Proposition 2.3.4.11 a $\Rightarrow$ b, the linear extension y' of x' to F defined by

$$y'(1) = \|x'\|$$

is positive and, by Corollary 2.3.4.7, $y' \in F'_+$ and

$$\|y'\| = y'(1) = \|x'\|\,.$$

Now take a positive linear extension z' of x' to F such that

$$\|z'\| = \|x'\|\,.$$

By Corollary 2.3.4.7,

$$z'(1) = \|z'\| = \|x'\|\,.$$

Hence $z' = y'$. $\blacksquare$

Proposition 2.3.4.13 (0) *Let E be an involutive quasiunital Banach algebra and let C' be the set of continuous positive linear forms on E.*

a) *C' is a sharp convex cone of E', closed in E'_E.*

b) *There is a unique order on E' which renders E' an ordered Banach space such that*

$$E'_+ = C'.$$

c) *$E'^{\#}_+$ is a compact set of E'_E, metrizable whenever E is separable.*

d) *If $\mathbb{K} = \mathbb{C}$, then*

$$E'_+ = \{x' \in E' \mid x \in E \Longrightarrow x'(x^*x) \in \mathbb{R}_+\}.$$

a) It is easy to see that C' is a convex cone of E'. Take

$$x' \in C' \cap (-C').$$

Then

$$\|x'\| = \lim_{x\,\mathfrak{F}} x'(x^*x) = 0,$$

where $\mathfrak{F}$ is an approximate unit of E (Proposition 2.3.4.10 a)). Thus $x' = 0$ and C' is sharp.

By Proposition 2.3.2.22 c), $\operatorname{Re} E'$ is closed in E'_E. Since

$$C' = \bigcap_{x \in E} \{x' \in \operatorname{Re} E' \mid x'(x^*x) \geq 0\},$$

it follows that C', too, is closed in E'_E.

b) follows from a) and Proposition 1.7.1.3.

c) follows from a), b) and the Alaoglu–Bourbaki Theorem.

d) Choose $x' \in E'$, such that

$$x'(x^*x) \in \mathbb{R}_+$$

for every $x \in E$. Define

$$f : E \times E \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto x'(y^*x).$$

By Proposition 2.3.4.6 a), f is sesquilinear. Since

$$\tilde{f}(x) = x'(x^*x) \in \mathbb{R}$$

for every $x \in E$, it follows from Proposition 2.3.3.8 b $\Rightarrow$ a that f is Hermitian. Take $x \in E$ and let $\mathfrak{F}$ be an approximate unit of E. Then

$$x'(x^*) = \lim_{y,\mathfrak{F}} x'(x^*y) = \lim_{y,\mathfrak{F}} f(y,x) = \lim_{y,\mathfrak{F}} f^*(y,x) =$$

$$= \lim_{y,\mathfrak{F}} \overline{f(x,y)} = \lim_{y,\mathfrak{F}} \overline{x'(y^*x)} = \overline{x'(x)}$$

by Proposition 2.3.4.9. Hence x' is involutive and therefore positive. ■

Remark. The result does not hold without E's being quasiunital. Let E be the unital Banach algebra defined in E 2.2.6 and F the closed subalgebra

$$\{x \in E \mid x(a) = 0\}$$

of E. Take the involution

$$F \longrightarrow F, \quad x \longmapsto \overline{x}$$

on F. Then F is an involutive Banach algebra and the map

$$F \longrightarrow \mathbb{K}, \quad x \longmapsto x^{(1)}(a)$$

belongs to $C' \cap (-C')$. Hence C' is not sharp in this case.

Corollary 2.3.4.14 (0) *If E is an involutive quasiunital Banach algebra, then*

$$\|x' + y'\| = \|x'\| + \|y'\|$$

for every $x', y' \in E'_+$.

By Proposition 2.3.4.10 a),

$$\|x'\| + \|y'\| = \lim_{x,\mathfrak{F}} x'(x^*x) + \lim_{x,\mathfrak{F}} y'(x^*x) = \lim_{x,\mathfrak{F}} \langle x^*x, x' + y' \rangle = \|x' + y'\|,$$

for any approximate unit $\mathfrak{F}$ of E. ■

Corollary 2.3.4.15 *Let E be an involutive quasiunital Banach algebra. Take $x', y', z' \in E'_+$ such that*

$$\|x' - y'\| = \|x'\| + \|y'\|,$$

$$z' \leq x', \quad z' \leq y'.$$

Then $z' = 0$ (i.e. the infimum of x' and y' in E'_+ is 0).

By Corollary 2.3.4.14,

$$\|x'\| + \|y'\| = \|x' - y'\| = \|(x' - z') - (y' - z')\| \leq$$

$$\leq \|x' - z'\| + \|y' - z'\| = \|x'\| - \|z'\| + \|y'\| - \|z'\| =$$

$$= \|x'\| + \|y'\| - 2\|z'\|,$$

$$\|z'\| = 0, \quad z' = 0. \qquad \blacksquare$$

Proposition 2.3.4.16 (0) *Let A' be an upward directed set of E_+ for the involutive quasiunital Banach algebra E. Then the following are equivalent:*

a) *A' is upper bounded.*

b) *A' is bounded (in norm).*

c) *The upper section filter of A' converges.*

d) *The upper section filter of A' has a point of adherence.*

e) *A' has a supremum.*

If these conditions are fulfilled, then the upper section filter of A' converges to the supremum of A'.

In particular, E' is order complete and every operator on E is order continuous.

By Corollary 2.3.4.14,

$$\|x' + y'\| = \|x'\| + \|y'\|$$

for every $x', y' \in E'_+$, and the assertions follow from Proposition 1.7.2.5.

$\blacksquare$

Proposition 2.3.4.17 (0) *Let E be an involutive quasiunital Banach algebra and put*

$$C := \{x \in \operatorname{Re} E \mid x' \in E'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\}.$$

a) *C is a weakly closed convex cone of E and*

$$E'_+ = \{x' \in \operatorname{Re} E' \mid x \in C \Longrightarrow x'(x) \in \mathbb{R}_+\}.$$

b) *The following are equivalent:*

b_1) $E'_+ - E'_+$ *is dense in* $\operatorname{Re} E'$ *with respect to the topology of pointwise convergence on* $\operatorname{Re} E$.

b_2) C *is sharp.*

b_3) *There is an order relation on* E *with respect to which* E *is an ordered Banach space with*

$$E_+ = C .$$

If these conditions are fulfilled, then the order relation of b_3) *is unique. If* $\mathbb{K} = \mathbb{C}$, *then*

$$C := \{x \in E \mid x' \in E'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\} .$$

a) Define

$$C' := \{x' \in \operatorname{Re} E' \mid x \in C \Longrightarrow x'(x) \in \mathbb{R}_+\} .$$

It is obvious that C is a weakly closed convex cone of E, that C' is a closed convex cone of E'_E and that

$$E'_+ \subset C' .$$

Take $x' \in C' \backslash E'_+$. By Proposition 2.3.2.25, there is an $x \in \operatorname{Re} E$ such that

$$\sup_{y' \in E'_+} y'(x) = 0 < x'(x) ,$$

i.e. $-x \in C$. This implies the contradiction

$$0 \leq x'(-x) = -x'(x) < 0 .$$

Thus

$$E'_+ = C' .$$

$b_1 \Rightarrow b_2$. Take $x \in C \cap (-C)$. Then

$$x'(x) = 0$$

for every $x' \in E'_+$. By b_1),

$$x'(x) = 0$$

for every $x' \in \operatorname{Re} E'$ and, by Proposition 2.3.2.22 l), this implies that

$$x = 0 .$$

Hence C is sharp.

$b_2 \Rightarrow b_1$. Take $x \in \operatorname{Re} E$ such that

$$x'(x) = 0$$

for every $x' \in E'_+ - E'_+$. Then $x \in C \cap (-C)$, so that, by b_2),

$$x = 0 .$$

Hence $E'_+ - E'_+$ is dense in $\operatorname{Re} E'$ in the toplogy of pointwise convergence on $\operatorname{Re} E$.

$b_2 \Leftrightarrow b_3$ and the uniqueness of the order relation on E follow from Propositions 1.7.1.2 and 1.7.1.3.

Now suppose that $\mathbb{K} = \mathbb{C}$ and take $x \in E$ such that

$$x'(x) \in \mathbb{R}_+$$

for every $x' \in E'_+$. By b_1) and Proposition 2.3.1.22, $E'_+ - E'_+$ is dense in $\operatorname{Re} E'$ in the topology of pointwise convergence on E. Then

$$x'(x) \in \mathbb{R}$$

for every $x' \in \operatorname{Re} E'$ and so $x \in \operatorname{Re} E$ by Proposition 2.3.2.22 m). Hence $x \in C$.

■

Example 2.3.4.18 (0) *If T is a locally compact space, then $C_0(T)$ is an involutive commutative quasiunital Banach algebra, which possesses a unit iff T is compact (Example 2.2.2.1, Example 2.3.1.4). If we identify the Banach space $\mathcal{M}_b(T)$ of bounded Radon measures on T canonically with $C_0(T)'$, then*

$$\mu^* = \overline{\mu}$$

for every $\mu \in \mathcal{M}_b(T)$. In particular, μ is selfadjoint iff it is real as a measure. The element $\mu \in \mathcal{M}_b(T)$ is positive in the sense of Definition 2.3.4.1 iff it is positive as a measure. For every $\mu \in \operatorname{Re} \mathcal{M}_b(T)$, there exist uniquely determined $\mu_1, \mu_2 \in \mathcal{M}_b(T)_+$ such that

$$\mu = \mu_1 - \mu_2 , \quad \|\mu\| = \|\mu_1\| + \|\mu_2\| .$$

In particular, any linear form x' on $\mathcal{M}_b(T)$ such that $x'(\mathcal{M}_b(T)_+) \subset \mathbb{R}_+$ is continuous. The order relation on $C_0(T)$ introduced in Proposition 2.3.4.17 b) coincides with the natural order relation on $C_0(T)$.

Since

$$\|\overline{x}\| = \|x\|$$

for every $x \in C_0(T)$, $C_0(T)$ is an involutive Banach algebra. Take $\mu \in \mathcal{M}_b(T)$. Since

$$\langle x, \overline{\mu} \rangle = \int x \, d\overline{\mu} = \overline{\int \overline{x} \, d\mu} = \overline{\int x^* d\mu} = \overline{\langle x^*, \mu \rangle} = \langle x, \mu^* \rangle$$

for every $x \in C_0(T)$, we deduce that

$$\mu^* = \overline{\mu}\,.$$

Moeover,

$$\langle x^* x, \mu \rangle = \int x^* x \, d\mu = \int |x|^2 d\mu$$

for every $x \in C_0(T)$. Hence μ is positive in the sense of Definition 2.3.4.1 iff it is positive as a measure. The last assertions follow (Corollary 1.7.1.8). ■

Example 2.3.4.19 (4) *Let T be a group and $\ell^1(T)$ the involutive Banach algebra defined in Proposition 2.3.2.28 b).*

a) *Given $x' \in \ell^1(T)$ such that*

$$\overline{x'(t)} = x'(t^{-1})$$

for every $t \in T$ and

$$x'(1) \geq \sum_{t \in T \setminus \{1\}} |x'(t)|\,,$$

the linear form

$$\widetilde{x'} : \ell^1(T) \longrightarrow \mathbb{K}\,, \quad x \longmapsto \sum_{t \in T} x(t) x'(t)$$

is positive.

b) *There is a unique order relation on $\ell^1(T)$ which renders $\ell^1(T)$ an ordered Banach space with*

$$\ell^1(T)_+ = \{x \in \operatorname{Re} \ell^1(T) \mid x' \in \ell^1(T)'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\}\,.$$

a) By Proposition 2.3.2.29 b), $\tilde{x}'$ is selfadjoint.

Take $x \in \ell^1(T)$. Then

$$(x^* * x)(t) = \sum_{s \in T} x^*(ts^{-1})x(s) = \sum_{s \in T} \overline{x(st^{-1})}x(s),$$

$$\tilde{x}'(x^* * x) = \sum_{t \in T} (x^* * x)(t)x'(t) =$$

$$= x'(1) \sum_{s \in T} |x(s)|^2 + \sum_{t \in T \setminus \{1\}} x'(t) \sum_{s \in T} \overline{x(st^{-1})}x(s) \geq$$

$$\geq \frac{1}{2} \sum_{t \in T \setminus \{1\}} |x'(t)| \left(\sum_{s \in T} |x(s)|^2 + \sum_{s \in T} |x(st^{-1})|^2 \right) -$$

$$- \sum_{t \in T \setminus \{1\}} |x'(t)| \sum_{x \in T} |x(st^{-1})|\,|x(s)| =$$

$$= \frac{1}{2} \sum_{t \in T \setminus \{1\}} |x'(t)| \sum_{s \in T} \left(|x(s)|^2 + |x(st^{-1})|^2 - 2|x(st^{-1})|\,|x(s)| \right) =$$

$$= \frac{1}{2} \sum_{t \in T \setminus \{1\}} |x'(t)| \sum_{s \in T} \left(|x(s)| - |x(st^{-1})| \right)^2 \geq 0.$$

Thus the linear form $\tilde{x}'$

$$\ell^1(T) \longrightarrow \mathbb{K}, \quad x \longmapsto \sum_{t \in T} x(t)x'(t)$$

is positive.

b) Identify $\ell^1(T)'$ with $\ell^\infty(T)$ canonically. By a) and Proposition 2.3.2.28 c),

$$\ell^1(T) \cap \operatorname{Re} \ell^\infty(T) \subset \ell^\infty(T)_+ - \ell^\infty(T)_+ .$$

Since $\ell^1(T) \cap \operatorname{Re} \ell^\infty(T)$ is dense in $\operatorname{Re} \ell^\infty(T)$ in the topology of pointwise convergence, b) follows from Proposition 2.3.4.17 b). ■

Proposition 2.3.4.20 (7) *Let E, F be involutive quasiunital Banach algebras, $u : E \to F$ a continuous involutive algebra homomorphism, and $y' \in F'_+$.*

a) $u'y' \in E'_+$.

b) *If* E, F, *and* u *are unital, then*

$$\|u'y'\| = \|y'\|.$$

a) Given $x \in E$,

$$\langle x^*x, u'y' \rangle = \langle u(x^*x), y' \rangle = \langle (ux)^*(ux), y' \rangle \in \mathbb{R}_+.$$

Since $u'y'$ is involutive, it is positive.

b) By a) and Corollary 2.3.4.7,

$$\|u'y'\| = \langle 1, u'y' \rangle = \langle u1, y' \rangle = \langle 1, y' \rangle = \|y'\| \qquad \blacksquare$$

Proposition 2.3.4.21 (7) *Let* E, F *be involutive unital Banach algebras such that* E *is a dense involutive unital subalgebra of* F *and such that the inclusion map* $E \to F$ *is continuous. If the set* $\mathfrak{G}$ *of finite-dimensional involutive unital subalgebras of* E *is upward directed and if its union is dense in* E, *then* $u'(F'_+) = E'_+$.

By Proposition 2.3.4.20,

$$y' \in F'_+ \Longrightarrow u'y' \in E'_+, \ \|u'y'\| = \|y'\|.$$

Put

$$H := \bigcup_{G \in \mathfrak{G}} G.$$

For every involutive unital subalgebra K of E, denote by K_0 the corresponding normed involutive unital subalgebra of F.

Let $x' \in E'_+$. For $G \in \mathfrak{G}$, put

$$x'_G : G_0 \longrightarrow \mathbb{K}, \quad y \longmapsto x'(y).$$

Then $x'_G \in G'_{0+}$ and, by the above,

$$\|x'_G\| = \|x'|G\| = \|x'\|.$$

Hence the map

$$H_0 \longrightarrow \mathbb{K}, \quad y \longmapsto x'(y)$$

is continuous. Since H_0 is dense in F, this map is the restriction to H_0 of an element $y' \in F'$. Since H_0 is dense in F, $y' \in F'_+$. $u'y'$ and x' are equal, since they coincide on the dense set H. Thus $x' \in u'(F'_+)$ and $E'_+ = u'(F'_+)$. $\blacksquare$

Example 2.3.4.22 (0) *With the notation of Example 2.3.2.35 d),*

$$\widetilde{(x',y')} \in (\overset{\circ}{E})'_+ \iff x'+iy',\quad x'-iy' \in E'_+$$

for every $(x',y') \in \overset{\circ}{\widetilde{E'}}$.

For $(x,y) \in \overset{\circ}{E}$,

$$\langle (x,y)^*(x,y), \widetilde{(x',y')}\rangle = \langle (x^*,-y^*)(x,y), \widetilde{(x',y')}\rangle =$$

$$= \langle (x^*x+y^*y, x^*y-y^*x), \widetilde{(x',y')}\rangle =$$

$$= \langle (x^*x+y^*y)+i(x^*y-y^*x), x'+iy'\rangle + \overline{\langle (x^*x+y^*y)-i(x^*y-y^*x), x'-iy'\rangle} =$$

$$= \langle (x+iy)^*(x+iy), x'+iy'\rangle + \overline{\langle (x-iy)^*(x-iy), x'-iy'\rangle}\,.$$

Hence, if $x'+iy'$, $x'-iy' \in E'_+$, then $\widetilde{(x',y')} \in (\overset{\circ}{E})'_+$. Assume that $\widetilde{(x',y')} \in (\overset{\circ}{E})'_+$ and $z \in E$. If we choose

$$x := \frac{1}{2}z,\quad y := \frac{1}{2i}z,$$

then

$$x+iy = z,\quad x-iy = 0$$

and therefore

$$\langle z^*z, x'+iy'\rangle = \langle (x,y)^*(x,y), \widetilde{(x',y')}\rangle \in \mathbb{R}_+\,.$$

I we choose

$$x := \frac{1}{2}z,\quad y := \frac{i}{2}z,$$

then

$$x+iy = 0,\quad x-iy = z$$

and therefore

$$\langle z^*z, x'-iy'\rangle = \overline{\langle (x,y)^*(x,y), \widetilde{(x',y')}\rangle} \in \mathbb{R}_+\,.$$

Hence $x'+iy'$, $x'-iy' \in E'_+$. ■

2.3.5 The State Space

Definition 2.3.5.1 (0) *Let E be a involutive quasiunital Banach algebra. A* ***state of*** *E is a continuous positive linear form x' on E (Proposition 2.3.4.13 b)) with*

$$\|x'\| = 1 .$$

The set of states of E endowed with the topology of pointwise convergence (i.e. with the topology induced by E'_E) is called ***the state space of*** *E and is denoted by $\tau(E)$. By Corollary 2.3.4.14, it is a convex set of E' . Given $x \in E$, define*

$$\overset{\vee}{x} : \tau(E) \longrightarrow \mathbb{K} , \quad x' \longmapsto x'(x) .$$

The extreme points of $\tau(E)$ are called ***pure states of*** *E . We use $\tau_0(E)$ to denote the subspace of $\tau(E)$ consisting of the pure states and call it* ***the pure state space of*** *E .*

Example 2.3.5.2 *Let T be a locally compact space and identify $\mathcal{M}_b(T)$ canonically with $C_0(T)'$. Then*

$$\tau(E) = \{\mu \in \mathcal{M}_b(T) \mid \mu \text{ positive}, \ \mu(T) = 1\} ,$$

$$\tau_0(E) = \{\delta_t \mid t \in T\} .$$

It follows from Example 2.3.4.18 that

$$\tau(E) = \{\mu \in \mathcal{M}_b(T)_+ \mid \mu(T) = 1\} .$$

Hence, by Example 1.2.7.14,

$$\tau_0(E) = \{\delta_t \mid t \in T\} . \qquad \blacksquare$$

Example 2.3.5.3 *Let $\mathcal{L}(\mathbb{K}^2)$ be the involutive unital Banach algebra defined in Example 2.3.2.6. Take a 2×2-matrix*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} .$$

The following are equivalent:

a) *The linear form*

$$\mathcal{L}(\mathbb{K}^2) \longrightarrow \mathbb{K} , \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \longmapsto \alpha a + \gamma b + \beta c + \delta d$$

is in $\tau(\mathcal{L}(\mathbb{K}^2))$ (in $\tau_0(\mathcal{L}(\mathbb{K}^2))$).

b) $a, d \in \mathbb{R}_+$, $a + d = 1$, $b = \bar{c}$, $|b|^2 \leq ad$

$$(a, d \in \mathbb{R}_+, \quad a + d = 1, \quad b = \bar{c}, \quad |b|^2 = ad).$$

By Example 2.3.4.4 b $\Leftrightarrow$ c, the above linear form is positive iff

$$a, d \in \mathbb{R}_+, \quad b = \bar{c}, \quad |b|^2 \leq ad.$$

Hence the assertion for τ follows from Corrolary 2.3.4.7.

We now consider the assertion for $\tau_0(E)$. The implication a $\Rightarrow$ b is easy to see, so we prove the implication b $\Rightarrow$ a. Take $t \in]0, 1[$ and

$$\begin{bmatrix} a_1 & b_1 \\ \overline{b_1} & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ \overline{b_2} & d_2 \end{bmatrix} \in \mathbb{K}_{2,2}$$

such that

$$a_1, d_1, a_2, d_2 \in \mathbb{R}_+, \quad a_1 + d_1 = a_2 + d_2 = 1, \quad |b_1|^2 \leq a_1 d_1, \quad |b_2|^2 \leq a_2 d_2,$$

$$t \begin{bmatrix} a_1 & b_1 \\ \overline{b_1} & d_1 \end{bmatrix} + (1 - t) \begin{bmatrix} a_2 & b_2 \\ \overline{b_2} & d_2 \end{bmatrix} = \begin{bmatrix} a & b \\ \bar{b} & d \end{bmatrix}.$$

Then

$$a = ta_1 + (1 - t)a_2,$$

$$b = tb_1 + (1 - t)b_2,$$

$$d = td_1 + (1 - t)d_2,$$

so that

$$0 = ad - |b|^2 = t^2 a_1 d_1 + (1 - t)^2 a_2 d_2 + t(1 - t)(a_1 d_2 + a_2 d_1) -$$

$$-t^2 |b_1|^2 - (1 - t)^2 |b_2|^2 - t(1 - t)(b_1 \bar{b}_2 + \bar{b}_1 b_2) =$$

$$= t^2 (a_1 d_1 - |b_1|^2) + (1 - t)^2 (a_2 d_2 - |b_2|^2) + t(1 - t)(a_1 d_2 + a_2 d_1 - b_1 \bar{b}_2 - \bar{b}_1 b_2).$$

We have that

$$a_1 d_1 - |b_1|^2 \geq 0, \quad a_2 d_2 - |b_2|^2 \geq 0$$

$$a_1 d_2 + a_2 d_1 - b_1 \bar{b}_2 - \bar{b}_1 b_2 \geq a_1(1 - a_2) + a_2(1 - a_1) - |b_1|^1 - |b_2|^2 \geq$$

$$\geq a_1 + a_2 - 2a_1 a_2 - a_1(1 - a_1) - a_2(1 - a_2) = a_1^2 + a_2^2 - 2a_1 a_2 = (a_1 - a_2)^2 \geq 0 .$$

Hence

$$|b_1|^2 = a_1 d_1 , \quad |b_2|^2 = a_2 b_2 , \quad a_1 = a_2$$

and this implies

$$a_1 = a_2 = a , \quad d_1 = d_2 = d , \quad |b_1| = |b_2| = |b| .$$

Let $\theta_1, \theta_2 \in \mathbb{R}$ such that

$$b_1 = |b| e^{i\theta_1} , \quad b_2 = |b| e^{i\theta_2} .$$

Then

$$0 = a_1 d_2 + a_2 d_1 - b_1 \bar{b}_2 - \bar{b}_1 b_2 = 2ad - 2|b|^2 \cos(\theta_1 - \theta_2) ,$$

$$\cos(\theta_1 - \theta_2) = 1 ,$$

$$b_1 = b_2 = b .$$

Therefore the linear form of a) belongs to $\tau_0(\mathcal{L}(\mathbb{K}^2))$ ■

Remark. If we identify $\mathcal{L}(\mathbb{K}^2)$ with its dual via the map defined in Example 2.3.1.33, then, by Example 2.3.1.36, every element of $\tau_0(\mathcal{L}(\mathbb{K}^2))$ is idempotent and selfadjoint. This result will be generalized in Example 4.2.3.4 b).

Proposition 2.3.5.4 (0) *Let E be an involutive quasiunital Banach algebra and take $x' \in \tau(E)$. Then the following are equivalent:*

a) $x' \in \tau_0(E)$.

b) $y' \in E'_+ , \ y' \leq x' \Rightarrow y' = \|y'\| x'$.

c) $y' \in E'_+ , \ y' \leq x' \Rightarrow \exists \alpha \in \mathbb{K} , \ y' = \alpha x'$.

a $\Rightarrow$ b. We may assume that $y' \neq x'$ and $y' \neq 0$. Set

$$z' := x' - y' .$$

Then

$$\|y'\| + \|z'\| = \|x'\| = 1$$

(Corollary 2.3.4.14),

$$\frac{1}{\|y'\|} y' , \frac{1}{\|z'\|} z' \in \tau(E) ,$$

$$\|y'\| \left(\frac{1}{\|y'\|} y' \right) + \|z'\| \left(\frac{1}{\|z'\|} z' \right) = x' ,$$

so that

$$\frac{1}{\|y'\|} y' = x' ,$$

$$y' = \|y'\| x'$$

by a).

b $\Rightarrow$ a. Take $\alpha, \beta \in]0,1[$ and $y', z' \in \tau(E)$ with

$$\alpha + \beta = 1 , \quad \alpha y' + \beta z' = x' .$$

Then $\alpha y' \in E'_+$ and $\alpha y' \leq x'$. Hence, by b),

$$\alpha y' = \|\alpha y'\| x' = \alpha x' ,$$

$$y' = x' ,$$

and thus $x' \in \tau_0(E)$.

b $\Leftrightarrow$ c is trivial. ■

Proposition 2.3.5.5 (0) *Let E be an involutive quasiunital Banach algebra which is not unital, F an associated involutive unital Banach algebra, and x'_0 the element of $\tau(F)$ which vanishes identically on E. For every $x' \in \tau(E)$, let $\widetilde{x}'$ denote the unique element of $\tau(F)$ such that*

$$\widetilde{x}' | E = x'$$

(Corollary 2.3.4.12).

a) $\tau(F)$ *is the convex hull of* $\{\widetilde{x}' \mid x' \in \tau(E)\} \cup \{x_0'\}$.

b) $\tau_0(F) = \{\widetilde{x}' \mid x' \in \tau_0(E)\} \cup \{x_0'\}$.

a) Take $y' \in \tau(F) \backslash \{x_0'\}$ and set

$$x' := \frac{1}{\|y'|E\|} y'|E .$$

Then $x' \in \tau(E)$,

$$\widetilde{x}'|E = x' = \frac{1}{\|y'|E\|} y'|E .$$

$$y' = \|y'|E\| \widetilde{x}' + (1 - \|y'|E\|) x_0' .$$

Hence y' belongs to the convex hull of $\{\widetilde{x}' \mid x' \in \tau(E)\} \cup \{x_0'\}$.

The reverse inclusion is trivial.

b) follows from a). ■

Proposition 2.3.5.6 (0) *If E is an involutive quasiunital Banach algebra, then every state of E which is an algebra homomorphism is a pure state.*

Let x' be a state of E which is an algebra homomorphism. Take $y', z' \in \tau(E)$ and $\alpha, \beta \in \mathbb{R}_+ \backslash \{0\}$ such that

$$\alpha + \beta = 1 , \quad x' = \alpha y' + \beta z' .$$

Take $x \in E$. Then

$$|y'(x)|^2 \leq y'(x^* x) , \quad |z'(x)|^2 \leq z'(x^* x)$$

(Proposition 2.3.4.10 b)). Thus

$$0 = x'(x^* x) - |x'(x)|^2 = \alpha y'(x^* x) + \beta z'(x^* x) - |\alpha y'(x) + \beta z'(x)|^2 \geq$$

$$\geq \alpha(\alpha + \beta)|y'(x)|^2 + \beta(\alpha + \beta)|z'(x)|^2 - \alpha^2 |y'(x)|^2 -$$

$$-\beta^2 |z'(x)|^2 - 2\alpha\beta \operatorname{re} y'(x)\overline{z'(x)} = \alpha\beta |y'(x) - z'(x)|^2 ,$$

$$y'(x) = z'(x) .$$

Hence $y' = z'$ and $x' \in \tau_0(E)$. ■

Proposition 2.3.5.7 (4) *Let T be a locally compact group.*

a) *The map*

$$\mathcal{M}_b(T) \longrightarrow \mathbb{K}, \quad \mu \longmapsto \mu(T)$$

is a pure state (Proposition 2.3.2.28 a)).

b) *If λ is a left and right invariant Haar measure on T, then the map*

$$L^1(\lambda) \longrightarrow \mathbb{K}, \quad x \longmapsto \int x \, d\lambda$$

is a pure state (Proposition 2.3.2.29 a)).

By Propositions 2.3.2.29 a) and 2.3.2.28 f), the given maps are involutive algebra homomorphisms. Hence they are positive (Proposition 2.3.4.5). Since they have norm 1, the assertions follow from Proposition 2.3.5.6. ■

Proposition 2.3.5.8 (0) *Let E be an involutive quasiunital Banach algebra. Then $\overset{\vee}{x} \in C(\tau(E))$ whenever $x \in E$ and the map*

$$E \longrightarrow C(\tau(E)), \quad x \longmapsto \overset{\vee}{x}$$

is linear, involutive, and continuous with norm at most 1. Moreover, for every $x \in E$, if $\|x\| = \|\overset{\vee}{x}\|$, then $\|\overbrace{x^*x}^{\vee}\| = \|x^*x\| = \|x\|^2$.

It is obvious that $\overset{\vee}{x}$ is continuous. Since

$$|\overset{\vee}{x}(x')| = |x'(x)| \leq \|x'\| \, \|x\| = \|x\|,$$

$\overset{\vee}{x} \in C(\tau(E))$ and

$$\|\overset{\vee}{x}\|_\infty \leq \|x\|.$$

The first assertion follows, since the map

$$E \longrightarrow C(\tau(E)), \quad x \longmapsto \overset{\vee}{x}$$

is obviously linear and involutive.

Now choose $x \in E$ with $\|x\| = \|\overset{\vee}{x}\|$. By Proposition 2.3.4.10 b),

$$|\overset{\vee}{x}(x')|^2 = |x'(x)|^2 \leq x'(x^*x) = \overbrace{x^*x}^{\vee}(x') \leq \|\overbrace{x^*x}^{\vee}\|$$

for every $x' \in \tau(E)$. By the above considerations,

$$\|x\|^2 = \|\overset{\vee}{x}\|^2 = \sup_{x' \in \tau(E)} |\overset{\vee}{x}(x')|^2 \leq \|\overset{\vee}{\widehat{x^*x}}\| \leq \|x^*x\| \leq \|x\|^2 ,$$

so that

$$\|\overset{\vee}{\widehat{x^*x}}\| = \|x^*x\| = \|x\|^2 . \qquad \blacksquare$$

Proposition 2.3.5.9 (0) *Let E be an involutive quasiuntial Banach algebra.*

a) *If E is unital, then $\tau(E)$ is compact.*

b) *If E is separable, then $\tau(E)$ is metrizable and separable.*

By Proposition 2.3.4.13 c), $E'^{\#}_{+}$ is a compact set of E'_E and it is metrizable whenever E is separable.

a) follows from the above remark and the fact that

$$\tau(E) = \{x' \in E'^{\#}_{+} \mid x'(1) = 1\}$$

(Corollary 2.3.4.7).

b) follows from a), Proposition 2.3.5.5, and the above remark. $\blacksquare$

Proposition 2.3.5.10 (0) *Let E be an involutive quasiunital Banach algebra.*

a) *An element of $\tau(E)$ belongs to $\tau_0(E)$ iff it is an extreme point of $E'^{\#}_{+}$.*

b) *The convex hull of $\tau_0(E)$ is dense in $\tau(E)$.*

c) *If E is unital then every nonempty closed face of $\tau(E)$ possesses a point of $\tau_0(E)$.*

d) *If E is unital, then, given $x \in E$, the extreme points of $\overset{\vee}{x}(\tau(E))$ are contained in $\overset{\vee}{x}(\tau_0(E))$.*

a) Take $x' \in \tau_0(E)$, $y', z' \in E'^{\#}_{+}$, and $\alpha, \beta \in]0,1[$ such that

$$\alpha + \beta = 1 , \quad \alpha y' + \beta z' = x' .$$

Then

$$1 = \|x'\| = \|\alpha y' + \beta z'\| = \alpha\|y'\| + \beta\|z'\|$$

(Corollary 2.3.4.14) and we deduce successively that

$$\|y'\| = \|z'\| = 1\,,$$

$$y', z' \in \tau(E)\,,$$

$$y' = z'\,.$$

Hence x' is an extreme point of $E'^{\#}_{+}$.

The converse is trivial.

b) By Proposition 2.3.4.13 b), c), $E'^{\#}_{+}$ is a compact convex set of E'_E. By a), $\tau_0(E) \cup \{0\}$ is the set of its extreme points. Let A' be the convex hull of $\tau_0(E) \cup \{0\}$ and B' the convex hull of $\tau_0(E)$. Define

$$\varphi : A' \backslash \{0\} \longrightarrow B', \quad x' \longmapsto \frac{1}{\|x'\|} x'\,.$$

Take $x' \in \tau(E)$. By the Krein–Milman Theorem (Theorem 1.3.1.10 c)), there is a filter $\mathfrak{F}$ on A' converging to x' in E'_E. Since the map

$$E'_E \longrightarrow \mathbf{R}_+\,, \quad y' \longmapsto \|y'\|$$

is lower semicontinous (Proposition 1.2.6.6),

$$1 = \|x'\| \leq \lim_{y',\mathfrak{F}} \inf \|y'\| \leq \lim_{y',\mathfrak{F}} \sup \|y'\| \leq 1\,,$$

$$\lim_{y',\mathfrak{F}} \|y'\| = 1\,.$$

Hence $\varphi(\mathfrak{F})$ is a filter on B' converging to x' in E'_E and so B' is dense in $\tau(E)$.

c) By Proposition 2.3.5.9 a), $\tau(E)$ is compact. Hence, by the Klein–Milman Theorem (Theorem 1.3.1.10 b)) every closed face of $\tau(E)$ has an extreme point.

d) Let α be an extreme point of $\overset{\vee}{x}(\tau(E))$. Then $\overset{\vee}{x}^{-1}(\alpha)$ is a closed face of $\tau(E)$ (Proposition 1.2.7.10) since $\overset{\vee}{x}$ is continuous (Proposition 2.3.5.8). By c), this face contains an element $x' \in \tau_0(E)$. Thus

$$\alpha = \overset{\vee}{x}(x') \in \overset{\vee}{x}(\tau_0(E))\,. \qquad \blacksquare$$

Proposition 2.3.5.11 (0) *If E is an involutive quasiunital Banach algebra, then the following are equivalent:*

a) *For every* $x' \in \operatorname{Re} E'$ *there are* $y', z' \in E'_+$ *such that*

$$x' = y' - z', \quad \|x'\| = \|y'\| + \|z'\|.$$

b) $\|x\| = \|\overset{\vee}{x}\|$ *for every* $x \in \operatorname{Re} E$.

c) $\operatorname{Re} E'^{\#}$ *is the closure in* E'_E *of the convex hull of* $\tau(E) \cup (-\tau(E))$.

If these conditions are fulfilled then

d) $\|\overbrace{x^2}^{\vee}\| = \|x^2\| = \|x\|^2$ *for every* $x \in \operatorname{Re} E$.

e) *There is a unique order relation which renders* E *an ordered Banach space with*

$$E_+ = \{x \in \operatorname{Re} E \mid x' \in E'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\}.$$

Moreover,

$$E'_+ = \{x' \in \operatorname{Re} E' \mid x'(E_+) \subset \mathbb{R}_+\}.$$

If $\mathbb{K} = \mathbb{C}$, *then*

$$E_+ = \{x \in E \mid x' \in E'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\},$$
$$E'_+ = \{x' \in E' \mid x'(E_+) \subset \mathbb{R}_+\}.$$

If in addition E *is unital then:*

f) *For every* $x \in E_+$ *and every* $n \in \mathbb{N}$ *there is a* $y \in \operatorname{Re} E$ *such that*

$$y^n = x.$$

g) *The following are equivalent for every* $x \in \operatorname{Re} E$*:*

g_1) $x \in E_+$.

g_2) $\exists y \in \operatorname{Re} E, \quad x = y^2$.

g_3) $\exists y \in E, \quad x = y^* y$.

g_4) $\alpha \geq \|x\| \Rightarrow \|\alpha 1 - x\| \leq \alpha$.

g_5) $\exists \alpha \in \mathbb{R}_+, \quad \|\alpha 1 - x\| \leq \alpha$.

h) *If* x'' *is a linear form on* E' *such that* $x''(E'_+) \subset \mathbb{R}_+$, *then* x'' *is continuous.*

a ⇒ b. By Proposition 2.3.2.22 l), there is an $x' \in \operatorname{Re} E'$ such that

$$\|x'\| = 1\,, \quad x'(x) = \|x\|\,.$$

By a), there are $y', z' \in E'_+$ such that

$$x' = y' - z'\,, \quad \|y'\| + \|z'\| = 1\,.$$

We get

$$\|x\| = x'(x) = y'(x) - z'(x) \leq |y'(x)| + |z'(x)| \leq$$

$$\leq \|y'\|\,\|x\| + \|z'\|\,\|x\| = \|x\|\,.$$

We may assume $z' \neq 0$. Then $\frac{1}{\|z'\|} z' \in \tau(E)$ and

$$\|x\| = \frac{1}{\|z'\|}|z'(x)| \leq \|\overset{\vee}{x}\|\,,$$

$$\|x\| = \|\overset{\vee}{x}\|\,.$$

b ⇒ c. Let K' be the closure in E'_E of the convex hull of $\tau(E) \cup (-\tau(E))$. Assume there is an $x' \in \operatorname{Re} E'^{\#} \backslash K'$. By Proposition 2.3.2.25, there is an $x \in \operatorname{Re} E$ such that

$$\sup_{y' \in K'} y'(x) < x'(x)\,.$$

By b), we deduce that

$$\|x\| = \|\overset{\vee}{x}\| = \sup_{y' \in \tau(E)} |y'(x)| \leq \sup_{y' \in K'} y'(x) < x'(x) \leq \|x\|$$

which is a contradiction. Hence

$$\operatorname{Re} E'^{\#} \subset K'\,.$$

The converse inclusion is trivial (Proposition 2.3.2.22 k)).

c ⇒ a. We may assume that $\|x'\| = 1$. We define

$$f : \mathbb{R} \times E' \times E' \longrightarrow E'\,, \quad (\alpha, y', z') \longmapsto \alpha y' - (1-\alpha) z'\,.$$

It is readily verified that $f([0,1] \times \tau(E) \times \tau(E))$ is convex. Hence it contains the convex hull of $\tau(E) \cup (\tau(E))$. Let $\mathfrak{U}$ be the neighbourhood filter of x' in E'_E. By hypothesis, for every $U \in \mathfrak{U}$, there is an element

$$(\alpha_U, y'_U, z'_U) \in [0,1] \times \tau(E) \times \tau(E)$$

such that

$$\alpha_U y'_U - (1-\alpha_U) z'_U \in U\,.$$

Then

$$\|x'\| = 1 = \alpha_U + (1-\alpha_U) = \|\alpha_U y'_U\| + \|(1-\alpha_U) z'_U\|\,.$$

Order $\mathfrak{U}$ by inclusion and let $\mathfrak{F}$ be an ultrafilter on $\mathfrak{P}(E')$ finer than the lower section filter of $\mathfrak{U}$. ${E'_+}^{\#}$ is a compact set of E'_E (Proposition 2.3.4.13 c)). Set

$$y' := \lim_{U,\mathfrak{F}} \alpha_U y'_U \in {E'_+}^{\#}\,, \quad z' := \lim_{U,\mathfrak{F}} (1-\alpha_U) z'_U \in {E'_+}^{\#}\,,$$

where the limits are taken in E'_E. Then

$$x' = \lim_{U,\mathfrak{F}} (\alpha_U y'_U - (1-\alpha_U) z'_U) = \lim_{U,\mathfrak{F}} \alpha_U y'_U - \lim_{U,\mathfrak{F}} (1-\alpha_U) z'_U = y' - z'\,,$$

$$\|y'\| + \|z'\| \geq \|y' - z'\| = \|x'\| = \lim_{U,\mathfrak{F}} (\|\alpha_U y'_U\| + \|(1-\alpha_U) z'_U\|) =$$

$$= \lim_{U,\mathfrak{F}} \|\alpha_U y'_U\| + \lim_{U,\mathfrak{F}} \|(1-\alpha_U) z'_U\| \geq \|y'\| + \|z'\|\,,$$

by Proposition 1.2.6.6. Thus

$$\|x'\| = \|y'\| + \|z'\|\,.$$

b $\Rightarrow$ d follows from Proposition 2.3.5.8.

a $\Rightarrow$ e follows from Proposition 2.3.4.17.

f) We may assume $\|x\| \leq 1$. By e),

$$0 \leq \overset{\vee}{x} \leq 1\,,$$

so, by b),

$$\|1 - x\| = \|1 - \overset{\vee}{x}\| \leq 1\,.$$

We define (Proposition 2.2.3.13 a))

$$y := [1 + (x-1)]^{\frac{1}{n}}\,.$$

By Corollary 2.3.2.19 b), $y \in \operatorname{Re} E$ and by Proposition 2.2.3.13 c), $y^n = x$.

g) $g_1 \Rightarrow g_2$ follows from f).

$g_2 \Rightarrow g_3$ is trivial.

$g_3 \Rightarrow g_4$. We have

$$0 \leq \alpha - \|x\| \leq \alpha - x'(y^*y) \leq \alpha$$

for every $x' \in \tau(E)$. By b),

$$\|\alpha 1 - x\| = \|\overbrace{\alpha 1 - x}^{\vee}\| = \sup_{x' \in \tau(E)} |\alpha - x'(y^*y)| \leq \alpha$$

$g_4 \Rightarrow g_5$ is trivial.

$g_5 \Rightarrow g_1$. By b),

$$\alpha - x'(x) \leq \|\overbrace{\alpha 1 - x}^{\vee}\| = \|\alpha 1 - x\| \leq \alpha,$$

$$x'(x) \geq 0$$

for every $x' \in \tau(E)$. By e), $x \in E_+$.

h) By c) and Proposition 2.3.5.9 a) (and Proposition 2.3.1.22, Proposition 2.3.2.7) the absolute convex hull of $E_+'^{\#}$ is a 0-neighbourhood in E', so the assertion follows from Corollary 1.7.1.8 b). ■

Remark. a) Let E be the involutive Banach algebra obtained from the vector space $\mathbb{K}^2$ using the pointwise multiplication, supremum norm, and involution

$$\mathbb{K}^2 \longrightarrow \mathbb{K}^2, \quad (\alpha, \beta) \longmapsto (\overline{\beta}, \overline{\alpha}).$$

Then $\tau(E) = \emptyset$ but $\|x^2\| = \|x\|^2$ for every $x \in \operatorname{Re} E$.

b) By Corollary 2.3.4.15, the infimum of y' and z' in E_+' is 0, where y', z' are the elements from a).

Lemma 2.3.5.12 (0) *Let T be a set, $\mathcal{F}$ a subset of $\mathbb{R}^T$, $\mathcal{G}$ the convex hull of $\mathcal{F}$, and $\overline{\mathcal{G}}$ the closure of $\mathcal{G}$ in the topology of pointwise convergence on $\mathbb{R}^T$. Then*

$$\sup_{x \in \mathcal{F}} x(t) = \sup_{x \in \overline{\mathcal{G}}} x(t)$$

for every $t \in T$.

Take $\alpha \in \mathbb{R}$ with

$$\sup_{x \in \mathcal{F}} x(t) < \alpha$$

and let

$$\mathcal{H} := \{x \in \mathbb{R}^T \mid x(t) \leq \alpha\}.$$

$\mathcal{H}$ is a convex set of $\mathbb{R}^T$ containing $\mathcal{F}$ and it is closed in the topology of pointwise convergence. Hence $\mathcal{H}$ contains $\overline{\mathcal{G}}$ and so

$$\sup_{x \in \overline{\mathcal{G}}} x(t) \leq \alpha.$$

Since α is arbitrary,

$$\sup_{x \in \overline{\mathcal{G}}} x(t) \leq \sup_{x \in \mathcal{F}} x(t).$$

The reverse inequality is trivial. ■

Proposition 2.3.5.13 (0) *Let E be a quasiunital (resp. unital) involutive Banach algebra, A a subset of $\tau(E)$, B the convex hull of A, and $\overline{B}$ the closure of B in $\tau(E)$. Then the following are equivalent:*

a) $\tau_0(E) \subset \overline{B}$.

b) $\tau(E) = \overline{B}$.

c) *(resp. d)) For every* $x \in \operatorname{Re} E$,

$$\sup_{x' \in A} x'(x) = \sup_{x' \in \tau(E)} x'(x)$$

$$(\textit{resp.} \quad \sup_{x' \in A} x'(x^2) = \sup_{x' \in \tau(E)} x'(x^2)).$$

d) *is fulfilled if*

$$\|x^2\| = \sup_{x' \in A} x'(x^2)$$

for all $x \in \operatorname{Re} E$.

a $\Rightarrow$ b follows from Proposition 2.3.5.10 b).

b $\Rightarrow$ a is trivial.

b $\Rightarrow$ c follows from Lemma 2.3.5.12.

c $\Rightarrow$ b (resp. d $\Rightarrow$ b). Let $B^{c\ell}$ be the closure of B in E'_E. By Proposition 2.3.4.13 c), $B^{c\ell}$ is a compact set of $(E'^{\#}_+)_E$. Take $y' \in \tau(E)\backslash B^{c\ell}$. By Proposition 2.3.2.25, there is a $y \in \operatorname{Re} E$ such that

$$\sup_{x' \in B^{c\ell}} x'(y) < y'(y).$$

We may assume that $\|y\| < 1$.

First suppose c) is fulfilled. Then

$$y'(y) \leq \sup_{x' \in \tau(E)} x'(y) = \sup_{x' \in A} x'(x) \leq \sup_{x' \in B^{c\ell}} x'(x) < y'(y)$$

which is a contradiction.

Now suppose d) is fulfilled (and E is unital). By Proposition 2.2.3.13 and Corollary 2.3.2.19, there is an $x \in \operatorname{Re} E$ such that

$$x^2 = 1 + y.$$

Then

$$y'(x^2) \leq \sup_{x' \in \tau(E)} x'(x^2) = \sup_{x' \in A} x'(x^2) = \sup_{x' \in \overline{B}} x'(x^2) =$$

$$= 1 + \sup_{x' \in \overline{B}} x'(y) \leq 1 + \sup_{x' \in B^{c\ell}} x'(y) < 1 + y'(y) = y'(x^2)$$

(Lemma 2.3.5.12), which is a contradiction.

Hence $\tau(E) \subset B^{c\ell}$,

$$\tau(E) = \tau(E) \cap B^{c\ell} = \overline{B}.$$

As for the last assertion, we have that

$$x'(x^2) \leq \|x'\| \, \|x^2\| = \|x^2\|$$

for every $x' \in \tau(E)$ and hence

$$\sup_{x' \in \tau(E)} x'(x^2) \leq \|x^2\| = \sup_{x' \in A} x'(x^2).$$

This implies d). ■

Proposition 2.3.5.14 (0) *Let E be an involutive unital complex (real) Banach algebra such that*

$$x' \in E' \ (x' \in \operatorname{Re} E') \,, \ x'(1) = \|x'\| \Longrightarrow x' \in E'_+ \,.$$

Then E is strongly symmetric and

$$\sigma(x) \subset \overset{\vee}{x}(\tau(E))$$

for every $x \in E$ $(x \in \operatorname{Re} E)$.

Let $\alpha \in \sigma(x)$. By Proposition 2.2.4.30, there is an $y' \in E'$, such that

$$\|y'\| = y'(1) = 1 \,, \quad y'(x) = \alpha \,.$$

Assume $\mathbb{K} = \mathbb{C}$. We set

$$x' := y' \,.$$

Then

$$x'(x) = y'(x) = \alpha \,,$$

and, by hypothesis, $x' \in \tau(E)$.

For the case $\mathbb{K} = \mathbb{R}$, set

$$x' := \operatorname{re} y' \in \operatorname{Re} E' \,.$$

Then

$$\|x'\| \geq x'(1) = \frac{1}{2}(y'(1) + y'^*(1)) = \frac{1}{2}(y'(1) + \overline{y'(1^*)}) = 1 \,,$$

$$\|x'\| \leq \|y'\| = 1$$

(Proposition 2.3.2.7), so that

$$\|x'\| = x'(1) = 1 \,.$$

By hypothesis, $x' \in \tau(E)$. Thus

$$x'(x) = \frac{1}{2}(y'(x) + y'^*(x)) = \frac{1}{2}(y'(x) + \overline{y'(x^*)}) = \alpha \,.$$

By the above,

$$\sigma(y^*y) \subset \overset{\vee}{\overbrace{y^*y}}(\tau(E)) \subset \mathbb{R}_+$$

for every $y \in E$. E is therefore strongly symmetric. ∎

Proposition 2.3.5.15 (0) *Let E be an involutive unital complex (real) Banach algebra, such that*

$$\|x\| = r(x)$$

whenever $x \in \mathrm{No}\, E$ $(x \in \mathrm{Re}\, E)$. Then the following are equivalent:

a) *E is strongly symmetric.*

b) *$x' \in E'_+$ whenever $x' \in E'$ $(x' \in \mathrm{Re}\, E')$ and $\|x'\| = x'(1)$.*

If these conditions are fulfilled, then:

c) *$x \in \mathrm{No}\, E$ $(x \in \mathrm{Re}\, E) \Rightarrow \overset{\vee}{x}(\tau(E))$ is the convex hull of $\sigma(x)$.*

d) *$\mathbb{K} = \mathbb{C}$, $x \in E \Rightarrow \sigma(x) \subset \overset{\vee}{x}(\tau(E))$.*

e) *If F is an involutive subalgebra of E, then for every $x' \in \mathrm{Re}\, F'$, there are continuous positive linear forms y', z' on F such that*

$$x' = y' - z', \quad \|x'\| = \|y'\| + \|z'\|.$$

f) *There is a unique order relation on E with respect to which E is an ordered Banach space with*

$$E_+ = \{x \in E \ \ (x \in \mathrm{Re}\, E) \mid x' \in E'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\}.$$

g) *If x'' is a linear form on E' such that $x''(E'_+) \subset \mathbb{R}_+$, then x'' is continuous.*

a $\Rightarrow$ b. We may assume that $x'(1) = 1$. Take $x \in E$. Then

$$\alpha x^* x + \beta 1 \in \mathrm{No}\, E \quad (\alpha x^* x + \beta 1 \in \mathrm{Re}\, E),$$

so, by assumption,

$$\|\alpha x^* x + \beta 1\| = r(\alpha x^* x + \beta 1)$$

for all $\alpha, \beta \in \mathbb{K}$. By a), $\sigma(x^* x) \subset \mathbb{R}_+$, so by Corollary 2.2.4.31,

$$x'(x^* x) \in \mathbb{R}_+.$$

Thus $x' \in E'_+$ (Proposition 2.3.4.13 d)).

b $\Rightarrow$ a & c & d follows from Proposition 2.3.5.14 and Corollary 2.2.4.31.

e & f & g. Assume first $F = E$. By c),

$$r(x) \leq \sup_{x' \in \tau(E)} |x'(x)| \leq \|x\|,$$

so that

$$\|x\| = \sup_{x' \in \tau(E)} |x'(x)|$$

for every $x \in \operatorname{Re} E$. e), f) and g) follow from Proposition 2.3.5.11 b $\Rightarrow$ a & e & h.

Let now F be arbitrary. By Corollary 2.3.2.23, there is an $u' \in \operatorname{Re} E'$, so that

$$u'|F = x', \quad \|u'\| = \|x'\|.$$

By the above considerations, there exist $v', w' \in E'_+$, so that

$$u' = v' - w', \quad \|u'\| = \|v'\| + \|w'\|.$$

We set

$$y' := v'|F, \quad z' := w'|F.$$

Then y', z' are continuous positive linear forms on F, so that

$$x' = y' - z'$$

and

$$\|x'\| \leq \|y'\| + \|z'\| \leq \|v'\| + \|w'\| = \|u'\| = \|x'\|,$$

so

$$\|x'\| = \|y'\| + \|z'\|. \qquad \blacksquare$$

Corollary 2.3.5.16 *If T is a Hausdorff space, then the bounded Radon measure μ on T is positive iff*

$$\|\mu\| = \mu(T).$$

Let $\mathcal{B}$ be the set of bounded Borel functions on T. It is an involutive unital Banach algebra with respect to the structure induced by $\ell^\infty(T)$ (Example 2.3.2.4). Identifying μ with the continuous linear form

$$\mathcal{B} \longrightarrow \mathbb{K}, \quad x \longmapsto \int x d\mu,$$

the assertion follows from Proposition 2.3.5.15 a $\Rightarrow$ b. $\blacksquare$

2.3.6 Involutive Modules

Throughout this subsection E is an involutive (unital) Banach algebra.

Definition 2.3.6.1 (0) *An* ***involutive (unital) E-module (E-algebra)*** *is a (unital) E-module (E-algebra) F endowed with an involution such that F is an involutive Banach space (algebra) and*

$$(xa)^* = a^*x^*, \quad (ax)^* = x^*a^*$$

for $(x, a) \in E \times F$.

If F and G are involutive (unital) E-modules (E-algebras) then a map $u : F \to G$ is called a ***homomorphism of involutive E-modules (of unital E- algebras)*** *if it is an involutive homomorphism of E-modules (of (unital) E-algebras).*

E is an involutive (unital) E-algebra.

If F is an involutive unital E-algebra then the map

$$E \longrightarrow F, \quad x \longmapsto x1$$

is a homomorphism of involutive unital E-algebras.

Proposition 2.3.6.2 (0) *If F is an involutive E-module, then F' is also an involutive E-module. In particular, E' is an involutive E-module.*

By Proposition 2.2.7.2, F' is an E-module. Take $(x, a') \in E \times F'$ and $a \in F$. Then

$$\langle a, (xa')^* \rangle = \overline{\langle a^*, xa' \rangle} = \overline{\langle a^*x, a' \rangle} = \langle x^*a, a'^* \rangle = \langle a, a'^*x^* \rangle,$$

$$\langle a, (a'x)^* \rangle = \overline{\langle a^*, a'x \rangle} = \overline{\langle xa^*, a' \rangle} = \langle ax^*, a'^* \rangle = \langle a, x^*a'^* \rangle.$$

Hence

$$(xa')^* = a'^*x^*, \quad (a'x)^* = x^*a'^*$$

which proves that F' is an involutive E-module. ■

Proposition 2.3.6.3 *Let F be an involutive E-module and A an involutive set of F. Then*

$$\{x \in E \mid a \in A \Longrightarrow xa = ax\}$$

is an involutive closed subalgebra of E.

Let $x \in E$ such that

$$xa = ax$$

for every $a \in A$. Then

$$x^*a = (a^*x)^* = (xa^*)^* = ax^*$$

for every $a \in A$. Hence

$$\{x \in E \mid a \in A \Longrightarrow xa = ax\}$$

is an involutive set of E. It is easy to see that this set is a closed subalgebra of E.

■

Proposition 2.3.6.4 (0) *Assume E quasiunital. If $x, y \in E$ and $x', y' \in E'$, then*

$$x^*x'x \leq x^*y'x,$$

whenever $x' \leq y'$,

$$x^*x'y + y^*x'x \leq x^*x'x + y^*x'y,$$

whenever $x' \geq 0$.

If x' is positive, then

$$\langle y^*y, x^*x'x\rangle = \langle xy^*yx^*, x'\rangle = \langle (yx^*)^*(yx^*), x'\rangle \in \mathbb{R}_+$$

and so $x^*x'x$ is positive. Hence

$$0 \leq (x-y)^*x'(x-y) = x^*x'x - x^*x'y - y^*x'x + y^*x'y,$$

$$x^*x'y + y^*x'x \leq x^*x'x + y^*x'y.$$

If $x' \leq y'$, then

$$0 \leq x^*(y'-x')x = x^*y'x - x^*x'x,$$

by the above considerations. Thus

$$x^*x'x \leq x^*y'x.$$

■

Proposition 2.3.6.5 *Take $x \in E$ and let x' be a positive linear form on E. If E is quasiunital, then*

$$\|x^*x'x\| = x'(xx^*),$$

and

$$x^*x' = x'x = 0,$$

whenever $x^*x'x = 0$.

If $\mathfrak{F}$ is an approximate unit of E, then

$$\|x^*x'x\| = \lim_{y,\mathfrak{F}} \langle y^*y, x^*x'x\rangle = \lim_{y,\mathfrak{F}} \langle xy^*yx^*, x'\rangle = \langle xx^*, x'\rangle$$

by Propositions 2.3.6.4 and 2.3.4.10 a) (and Proposition 2.3.4.9).

Assume that

$$x^*x'x = 0.$$

By Proposition 2.3.4.6 c),

$$|\langle y, x'x\rangle|^2 = |\langle xy, x'\rangle|^2 \leq x'(xx^*)x'(y^*y) = 0$$

for every $y \in E$. Thus

$$x'x = 0$$

and

$$x^*x' = (x'x)^* = 0$$

by Proposition 2.3.6.2. ∎

Example 2.3.6.6 *Let $(E_t)_{t\in T}$ be a family of (unital) involutive E-modules, $p \in [1,\infty] \cup \{0\}$, and F the (unital) E-module*

$$\{\xi \in \prod_{t\in T} E_t \mid (\|\xi(t)\|)_{t\in T} \in \ell^p(T)\}$$

defined in Example 2.2.7.21. Then F endowed with the involution

$$F \longrightarrow F, \quad \xi \longmapsto (\xi(t)^*)_{t\in T}$$

is an involutive (unital) E-module. ∎

Proposition 2.3.6.7 (0) *Let $n \in \mathbb{N}$ and assume the involutive vector space $E_{n,n}$ (Proposition 2.3.1.31 a)) is endowed with a norm such that*

$$\sup_{i,j \in \mathbb{N}_n} \|a_{ij}\| \leq \|a\| \leq \sum_{i,j=1}^{n} \|a_{ij}\|,$$

$$\|a\| = \|a^*\|$$

for all $a \in E_{n,n}$. We recursively define

$$E^{(0)} := E, \quad E^{(p)} := (E^{(p-1)})'$$

for every $p \in \mathbb{N}$ and put

$$\widetilde{a} : E_{n,n}^{(p-1)} \longrightarrow \mathbb{K}, \quad b \longmapsto \sum_{i,j=1}^{n} \langle b_{ij}, a_{ji} \rangle$$

for all $p \in \mathbb{N}$ and $a \in E_{n,n}^{(p)}$ and endow $E_{n,n}^{(p)}$ with the norm

$$E_{n,n}^{(p)} \longrightarrow \mathbb{R}_+, \quad a \longmapsto \|\widetilde{a}\|$$

defined inductively for all $p \in \mathbb{N}$. We have for all $p \in \mathbb{N} \cup \{0\}$:

a) $a \in E_{n,n}^{(p)} \Rightarrow \sup_{i,j \in \mathbb{N}_n} \|a_{ij}\| \leq \|a\| \leq \sum_{i,j=1}^{n} \|a_{ij}\|$.

b) *$E_{n,n}^{(p)}$ is an involutive E-module and for $p \neq 0$ the map*

$$E_{n,n}^{(p)} \longrightarrow \left(E_{n,n}^{(p-1)}\right)', \quad a \longmapsto \widetilde{a}$$

is an isometry of involutive E-modules.

c) *If we identify $E_{n,n}^{(p)}$ with $(E_{n,n})^{(p)}$ via the isometries in b), then*

$$a \in E_{n,n}^{(p)} \Longrightarrow j_{E_{n,n}^{(p)}} a = (j_{E^{(p)}} a_{ij})_{i,j \in \mathbb{N}_n},$$

$$a \in E_{n,n}^{(p+3)} \Longrightarrow (j'_{E_{n,n}^{(p)}} a) = (j'_{E^{(p)}} a_{ij})_{i,j \in \mathbb{N}_n}.$$

We prove the proposition by induction on p.

a) follows from Proposition 2.1.4.25.

b) We may assume that $p \neq 0$. Take $a \in E_{n,n}^{(p)}$. Then for each $b \in E_{n,n}^{(p-1)}$,

$$\langle b, \widetilde{a}^* \rangle = \overline{\langle b^*, \widetilde{a} \rangle} = \sum_{i,j=1}^{n} \overline{\langle (b^*)_{ij}, a_{ji} \rangle} =$$

$$= \sum_{i,j=1}^{n} \overline{\langle b_{ji}^*, a_{ji} \rangle} = \sum_{i,j=1}^{n} \langle b_{ji}, a_{ji}^* \rangle =$$

$$= \sum_{i,j=1}^{n} \langle b_{ji}, (a^*)_{ij} \rangle = \langle b, \widetilde{a^*} \rangle ,$$

so that

$$\widetilde{a}^* = \widetilde{a^*} ,$$

and

$$\|a^*\| = \|\widetilde{a^*}\| = \|\widetilde{a}^*\| = \|\widetilde{a}\| = \|a\| .$$

Let $x \in E$ and $a \in E_{n,n}^{(p)}$. Then for all $b \in E_{n,n}^{(p-1)}$,

$$\langle b, \widetilde{xa} \rangle = \sum_{i,j=1}^{n} \langle b_{ij}, x a_{ji} \rangle = \sum_{i,j=1}^{n} \langle b_{ij} x, a_{ij} \rangle = \langle bx, \widetilde{a} \rangle = \langle b, x\widetilde{a} \rangle ,$$

so that

$$\widetilde{xa} = x\widetilde{a} ,$$

$$\|xa\| = \|\widetilde{xa}\| = \|x\widetilde{a}\| \leq \|x\| \, \|\widetilde{a}\| = \|x\| \, \|a\| .$$

The relations

$$\widetilde{ax} = \widetilde{a}x , \quad \|ax\| \leq \|a\| \, \|x\|$$

are proved similarly. Finally, the relations

$$(ax)^* = x^* a^* , \quad (xa)^* = a^* x^*$$

are easy to see.

c) Assume first $a \in E_{n,n}^{(p)}$. Then for every $b \in E_{n,n}^{(p+1)}$,

$$\langle b, j_{E_{n,n}^{(p)}} a \rangle = \langle b, a \rangle = \sum_{i,j=1}^{n} \langle b_{ji}, a_{ij} \rangle =$$

$$= \sum_{i,j=1}^{n} \langle b_{ji}, j_{E^{(p)}} a_{ij} \rangle = \langle b, (j_{E^{(p)}} a_{ij})_{i,j \in \mathbb{N}_n} \rangle ,$$

so

$$j_{E_{n,n}^{(p)}} a = (j_{E^{(p)}} a_{ij})_{i,j \in \mathbb{N}_n} .$$

Assume now $a \in E_{n,n}^{(p+3)}$. Then, by the above result, for all $b \in E_{n,n}^{(p)}$,

$$\langle b, j'_{E_{n,n}^{(p)}} a \rangle = \langle j_{E_{n,n}^{(p)}} b, a \rangle = \sum_{i,j=1}^{n} \langle (j_{E_{n,n}^{p}} b)_{ji}, a_{ij} \rangle =$$

$$= \sum_{i,j=1}^{n} \langle j_{E^{(p)}} b_{ji}, a_{ij} \rangle = \sum_{i,j=1}^{n} \langle b_{ji}, j'_{E^{(p)}} a_{ij} \rangle = \langle b, (j'_{E^{(p)}} a_{ij})_{i,j \in \mathbb{N}_n} \rangle ,$$

so

$$j'_{E_{n,n}^{(p)}} a = (j'_{E^{(p)}} a_{ij})_{i,j \in \mathbb{N}_n}$$ ■

Proposition 2.3.6.8 *Let E be an involutive complex Banach algebra and F an involutive E-module. Let $\widetilde{E}$ and $\widetilde{F}$ denote the underlying involutive real Banach algebra of E and the underlying involutive $\widetilde{E}$-module of F, respectively. For every $a' \in F'$ put*

$$\widetilde{a'} : \widetilde{F} \longrightarrow \mathbb{R}, \quad a \longmapsto \operatorname{re} \langle a, a' \rangle .$$

Then $\widetilde{a'} \in (\widetilde{F})'$ and the map

$$F' \longrightarrow (\widetilde{F})', \quad a' \longmapsto \widetilde{a'}$$

is an isometry of involutive $\widetilde{E}$-modules.

Take $x \in E$. For every $a \in F$,

$$\langle a, \widetilde{xa'} \rangle = \operatorname{re} \langle a, xa' \rangle = \operatorname{re} \langle ax, a' \rangle = \langle ax, \widetilde{a'} \rangle = \langle a, x\widetilde{a'} \rangle ,$$

$$\langle a, \widetilde{a'x} \rangle = \operatorname{re} \langle a, a'x \rangle = \operatorname{re} \langle xa, a' \rangle = \langle xa, \widetilde{a'} \rangle = \langle a, \widetilde{a'}x \rangle ,$$

$$\langle a, \widetilde{a'^*} \rangle = \langle a^*, \widetilde{a'} \rangle = \operatorname{re} \langle a^*, a' \rangle = \operatorname{re} \langle a, a'^* \rangle = \langle a, \widetilde{a'^*} \rangle ,$$

so that

$$\widetilde{xa'} = x\widetilde{a'}, \quad \widetilde{a'x} = \widetilde{a'}x, \quad \widetilde{a'}^* = \widetilde{a'^*}$$

and the assertion follows from Proposition 1.3.6.26 b). ■

Exercises

E 2.3.1 Let $f : S \to T$ be a map between the sets S and T. Show that the map

$$\mathbb{K}^T \longrightarrow \mathbb{K}^S, \quad x \longmapsto x \circ f,$$

$$\ell^\infty(T) \longrightarrow \ell^\infty(S), \quad x \longmapsto x \circ f$$

are involutive unital algebra homomorphisms.

E 2.3.2 Let E be an involutive algebra. Show that the algebra $E \times E$ (Example 2.1.4.1) is an involutive algebra with respect to

$$E \times E \longrightarrow E \times E, \quad (x, y) \longmapsto (y^*, x^*)$$

as involution. If, in addition, E is an involutive normed algebra, then the above involutive algebra $E \times E$ is an involutive normed algebra with respect to the norm

$$E \times E \longrightarrow \mathbb{R}_+, \quad (x, y) \longmapsto \sup\{\|x\|, \|y\|\}.$$

E 2.3.3 Let 1 and j be the standard unit vectors of $\mathbb{K}^2$. Take $\alpha \in \{-1, 0, 1\}$, $\beta \in \mathbb{K}$ such that $|\beta| = 1$ and $\beta \in \{-1, 1\}$, whenever $\alpha \neq 0$. Let $E_{\alpha\beta}$ be the involutive algebra $\mathbb{K}^2$, such that 1 is the unit, $j^2 = \alpha 1$, and $j^* = \beta j$. Show the following.

a) If $\mathbb{K} = \mathbb{R}$, then we have:

- a_1) The involutive unital algebras $E_{\alpha\beta}$ are pairwise not isomorphic.
- a_2) Every two-dimensional involutive real unital algebra is isomorphic to one of the above involutive unital algebras.
- a_3) Every unital algebra homomorphism $E_{\alpha,\beta} \to E_{\alpha,\beta}$ is involutive.
- a_4) The set of unital algebra homomorphism $E_{\alpha,\beta} \to \mathbb{R}$ is empty if $\alpha = -1$ or if $\alpha = 1$ and $\beta = -1$. It contains exactly one element if $\alpha = 0$ and exactly two elements if $\alpha = \beta = 1$.

a_5) The involutive algebras $E_{1,1}, E_{-1,1}$ and $E_{-1,-1}$ with the norms

$$E_{1,1} \longrightarrow \mathbb{R}_+, \quad (\xi,\eta) \longmapsto \sup\{|\xi-\eta|, |\xi+\eta|\},$$

$$E_{-1,1} \longrightarrow \mathbb{R}_+, \quad (\xi,\eta) \longmapsto \sqrt{\xi^2+\eta^2},$$

$$E_{-1,-1} \longrightarrow \mathbb{R}_+, \quad (\xi,\eta) \longmapsto \sqrt{\xi^2+\eta^2},$$

are involutive unital Banach algebras such that

$$\|x^*x\| = \|x\|^2 \qquad (P)$$

for every element x. Let

$$I := \{(1,1), (1,-1), (-1,-1)\}.$$

For every tow–dimensional unital involutive real Banach algebra E possessing the above property (P), there are an $(\alpha,\beta) \in I$ and a unital involutive algebra isomorphism $E \to E_{\alpha,\beta}$. Every unital algebra homomorphism $E_{\alpha,\beta} \to E_{\alpha,\beta}$ is involutive and an isometry for every $(\alpha,\beta) \in I$. $E_{-1,-1}$ is isometric to the involutive real Banach algebra $\mathbb{C}$.

b) If $\mathbb{K} = \mathbb{C}$, then:

b_1) The involutive unital algebras $E_{\alpha\beta}$, $E_{\gamma\delta}$ are isomorphic iff one of the following conditions is fulfilled:

$$\alpha = \gamma = 0,$$

$$\alpha = \gamma \neq 0 \quad \text{and} \quad \beta = \delta,$$

$$\alpha = -\gamma \neq 0 \quad \text{and} \quad \beta = -\delta.$$

b_2) Every two–dimensional involutive complex unital algebra is isomorphic to one of the above involutive unital algebras.

b_3) Every unital algebra homomorphism $E_{1,\beta} \to E_{1,\beta}$ is involutive.

b_4) The group of isomorphisms $E_{0,\beta} \to E_{0,\beta}$ is isomorphic to the multiplicative group $\mathbb{R}\backslash\{0\}$.

b_5) The set of involutive unital algebra homomorphisms of $E_{\alpha,\beta}$ in $\mathbb{C}$ is empty if $|\alpha| = 1$ and $\beta = 1$. It contains exactly one element if $\alpha = 0$ and exactly two elements if $|\alpha| = 1$ and $\beta = -1$.

b$_6$) $E_{1,1}$ endowed with the norm

$$E_{1,1} \longrightarrow \mathbb{R}_+, \quad (\xi, \eta) \longmapsto \sup\{|\xi - \eta|, |\xi + \eta|\}$$

is an involutive unital Banach algebra, such that

$$\|x^*x\| = \|x\|^2$$

for every $x \in E_{1,1}$. Every two-dimensional involutive unital Banach algebra with the above property is isometric to $E_{1,1}$.

E 2.3.4 Let E be a symmetric involutive unital Banach algebra and define

$$C' := \{x' \in \operatorname{Re} E' \mid \|x'\| = x'(1)\},$$

$$C := \{x \in \operatorname{Re} E \mid x' \in C' \Longrightarrow x'(x) \in \mathbb{R}_+\}.$$

Prove the following:

a) C' is a closed sharp convex cone of E'.

b) For every $x \in \operatorname{Re} E$ and every $\alpha \in \sigma(x)$, there is an $x' \in C'$ such that

$$x'(x) = \alpha.$$

c) C is a closed convex cone of E and $\sigma(x) \subset \mathbb{R}_+$ for every $x \in C$.

d) If $x = 0$ whenever $x \in \operatorname{Re} E$ and $\sigma(x) = \{0\}$, then C is sharp.

E 2.3.5 Let E be an involutive complex vector space (involutive commutative complex algebra), F the underlying real structure, and $\overset{\circ}{F}$ its complexification. Show that the map

$$\overset{\circ}{F} \longrightarrow E \times E, \quad (x, y) \longmapsto (x + iy, x^* + iy^*)$$

is an isomorphism of involutive complex vector spaces (of involutive complex algebras).

E 2.3.6 Let E be an involutive commutative (normed) algebra. Show that if we endow the underlying (normed) real algebra of E with the involution

$$E \longrightarrow E, \quad x \longmapsto x,$$

then the result is an involutive commutative (normed) real algebra.

2.4 Gelfand Algebras

By the Banach–Mazur Theorem the Gelfand transform can be defined on Gelfand Banach algebras. In the special case of commutative locally compact groups this transform generalizes the Fourier transform, many properties of which are best understood in light of the Gelfand transform. But the most important role played by the Gelfand transform does not arise until we come to Gelfand C^*–algebras. For then it is an isometry and hence a functional calculus can be developed for C^*–algebras, providing the main tool for their study.

2.4.1 The Gelfand Transform

Definition 2.4.1.1 (0) *A (unital) **Gelfand algebra** is a commutative (unital) Banach algebra such that if* $\mathbb{K} = \mathbb{R}$, *then*

$$\sigma_{\overset{\circ}{E}}((x,0)) \subset \mathbb{R}$$

for every $x \in E$. *By Corollary 2.1.5.14, the above relation is equivalent to*

$$\sigma_{\overset{\circ}{E}}((x,0)) = \sigma_E(x)\,.$$

Let E *be a Gelfand algebra. We denote by* $\sigma_0(E)$ *the set of algebra homomorphisms of* E *into* $\mathbb{K}$ *endowed with the topology of pointwise convergence and by* $\sigma(E)$ *the subspace* $\sigma_0(E)\backslash\{0\}$ *of* $\sigma_0(E)$. *Given* $x \in E$, *define*

$$\widehat{x} := \overset{E}{\widehat{x}} : \sigma(E) \longrightarrow \mathbb{K}\,, \quad x' \longmapsto x'(x)\,.$$

$\sigma(E)$ *is called the **spectrum of** E and its elements are called **characters of** E.*

By Proposition 2.2.4.19, every character is continuous. $\widehat{x}$ is obviously continuous for every $x \in E$. $\sigma(E)$ may be empty (see Remark of Corollary 2.4.1.14), but if E is unital, then $\sigma(E) \neq \emptyset$ (Proposition 2.4.1.5).

Let E be a real Gelfand algebra and let $\overset{\circ}{E}$ be a complexification of E. If $\mathbb{C} \times \overset{\circ}{E}$ is a unital Banach algebra associated to $\overset{\circ}{E}$ (Proposition 2.2.1.3), then $\mathbb{C} \times \overset{\circ}{E}$ is a complexification of $\mathbb{R} \times E$. Hence every real Gelfand algebra possesses an associated unital Gelfand algebra.

Proposition 2.4.1.2 (0) *If* E *is a Gelfand algebra, then the map*

$$E \longrightarrow C(\sigma(E))\,, \quad x \longmapsto \widehat{x}$$

is a continuous algebra homomorphism of norm at most 1*; it is called the* ***Gelfand transform of*** E *(Gelfand, 1941). If* E *is unital, then its Gelfand transform is a unital algebra homomorphism.*

Let $x, y \in E$ and $\alpha, \beta \in \mathbb{K}$. Then

$$\widehat{\alpha x + \beta y}(x') = x'(\alpha x + \beta y) = \alpha x'(x) + \beta x'(y) =$$

$$= \alpha \hat{x}(x') + \beta \hat{y}(x') = (\alpha \hat{x} + \beta \hat{y})(x'),$$

$$\widehat{xy}(x') = x'(xy) = x'(x)x'(y) = \hat{x}(x')\hat{y}(x') = (\hat{x}\hat{y})(x'),$$

$$|\hat{x}(x')| = |x'(x)| \leq \|x\|$$

for every $x' \in \sigma(E)$ (Proposition 2.2.4.19). Thus $\hat{x} \in C(\sigma(E))$ and

$$\widehat{\alpha x + \beta y} = \alpha \hat{y} + \beta \hat{y}, \quad \widehat{xy} = \hat{x}\hat{y}, \quad \|\hat{x}\|_\infty \leq \|x\|,$$

proving the first assertion. If E is unital, then

$$\hat{1}(x') = x'(1) = 1$$

for every $x' \in \sigma(E)$ (Proposition 2.1.1.12), proving the second assertion. ■

Theorem 2.4.1.3 (0) *Let* E *be a Gelfand algebra.*

a) $\sigma_0(E)$ *is compact.*

b) $\sigma(E)$ *is locally compact.*

c) *If* E *is unital, then* $\sigma(E)$ *is compact.*

d) *If* E *is separable, then* $\sigma_0(E)$ *and* $\sigma(E)$ *are metrizable.*

e) *If* $\sigma(E)$ *is not compact, then* $\sigma_0(E)$ *is its Alexandroff compactification and* $\hat{x} \in C_0(\sigma(E))$ *for every* $x \in E$.

a) and d). By the Theorem of Alaoglu-Bourbaki, $E'^{\#}_{E'}$ is compact and it is metrizable whenever E is separable. By Proposition 2.2.4.19,

$$\sigma_0(E) = \{x' \in E'^{\#}_{E'} \mid x, y \in E \Longrightarrow x'(xy) = x'(x)x'(y)\}.$$

b) and e) follow from a).

c) By Proposition 2.1.1.12,

$$\sigma(E) = \{x' \in \sigma_0(E) \mid x'(1) = 1\}.$$ ■

Corollary 2.4.1.4 (0) *Let E be a Gelfand algebra which is not unital and F an associated unital Banach algebra of E. Then the map*

$$\sigma(F) \longrightarrow \sigma_0(E), \quad x' \longmapsto x'|E$$

is a homeomorphism. If $\sigma(E)$ is not compact and if we identify $\sigma(F)$ with $\sigma_0(E)$ by means of the above homeomorphism, then $\sigma(F)$ is the Alexandroff compactification of $\sigma(E)$.

The given map is obviously continuous. It is injective by Proposition 2.1.1.12 and it is surjective by Proposition 2.1.1.7. Hence the assertion follows from Theorem 2.4.1.3 a), c), e). ■

Proposition 2.4.1.5 (0) *If E is a Gelfand algebra and $\mathfrak{F}$ is the set of regular maximal proper ideals of E, then $\operatorname{Ker} x' \in \mathfrak{F}$ whenever $x' \in \sigma(E)$ and the map*

$$\sigma(E) \longrightarrow \mathfrak{F}, \quad x' \longmapsto \operatorname{Ker} x'$$

is bijective. If E is unital, then $\sigma(E) \neq \emptyset$.

Step 1 $\quad x' \in \sigma(E) \Rightarrow \operatorname{Ker} x' \in \mathfrak{F}$

If E is unital, then the assertion follows from Proposition 2.1.1.12. Hence we may assume E is not unital. Let $\mathbb{K} \times E$ be a unital Banach algebra associated to E and

$$\tilde{x}' : \mathbb{K} \times E \longrightarrow \mathbb{K}, \quad (\alpha, x) \longmapsto \alpha + x'(x).$$

Then $\tilde{x}' \in \sigma(\mathbb{K} \times E)$ (Proposition 2.1.1.7) and $\operatorname{Ker} \tilde{x}'$ is a maximal proper ideal of $\mathbb{K} \times E$ (Proposition 2.1.1.12) distinct from E. Identifying E with $\{0\} \times E$,

$$\operatorname{Ker} x' = E \cap \operatorname{Ker} \tilde{x}'.$$

Hence $\operatorname{Ker} x' \in \mathfrak{F}$.

Step 2 The map is injective

Take $x', y' \in \sigma(E)$ with

$$\operatorname{Ker} x' = \operatorname{Ker} y'.$$

Then there is an $\alpha \in \mathbb{K}$ with $y' = \alpha x'$ (Lemma 1.2.6.3). Thus

$$\alpha x'(x^2) = y'(x^2) = y'(x)^2 = \alpha^2 x'(x)^2 = \alpha^2 x'(x^2)$$

for every $x \in E$. Hence $\alpha = 1$ and $x' = y'$.

Step 3 The map is surjective

First suppose that E is unital. Take $F \in \mathfrak{F}$. Then F is closed (Corollary 2.2.4.16). Let $q : E \to E/F$ be the quotient map. E/F is a unital Banach algebra (Proposition 2.2.1.14) and a division algebra (Proposition 2.1.2.12 b)). If $\mathbb{K} = \mathbb{R}$, then by Corollary 2.1.5.19,

$$\underbrace{\sigma \circ}_{E/F}((X,0)) = \sigma_{E/F}(X).$$

By Corollary 2.2.5.5, there is an isometry of unital normed algebras $u : E/F \to \mathbb{K}$. Put

$$x' := u \circ q.$$

Then $x' \in \sigma(E)$ and

$$\operatorname{Ker} x' = \operatorname{Ker} q = F.$$

In particular, $\sigma(E) \neq \emptyset$ (Proposition 2.1.1.14 c)).

Now suppose that E has no unit and let $\mathbb{K} \times E$ be a unital Banach algebra associated to E. Take $F \in \mathfrak{F}$. Let G be a maximal proper ideal of $\mathbb{K} \times E$ containing $\{0\} \times F$ and different from $\{0\} \times E$. By the above, there is a character y' of $\mathbb{K} \times E$ such that

$$\operatorname{Ker} y' = G.$$

Let x' be the restriction of y' to E (identified with $\{0\} \times E$). Then $x' \in \sigma(E)$ and

$$\operatorname{Ker} x' = F. \qquad \blacksquare$$

Corollary 2.4.1.6 (0) *If E is a unital Gelfand algebra, then $x \in E$ is not invertible iff $x'(x) = 0$ for some $x' \in \sigma(E)$.*

x is not invertible iff it belongs to a maximal proper ideal of E (Corollary 2.1.2.11), i.e. iff $x \in \operatorname{Ker} x'$ for some $x' \in \sigma(E)$ (Proposition 2.4.1.5), which is equivalent to $x'(x) = 0$. $\blacksquare$

Corollary 2.4.1.7 (0) *Let E be a Gelfand algebra and $x, y \in E$.*

a) $\sigma(x) = \widehat{x}(\sigma(E))$ *if E is unital,*
$\sigma(x) = \widehat{x}(\sigma(E)) \cup \{0\}$ *if E is not unital.*

b) $\|\widehat{x}\|_\infty = r(x)$.

c) $\sigma(xy) \subset \sigma(x)\sigma(y)$, $\sigma(x+y) \subset \sigma(x) + \sigma(y)$,
$r(xy) \leq r(x)r(y)$, $r(x+y) \subset r(x) + r(y)$.

a) First assume that E is unital. Take $\alpha \in \mathbb{K}$. Then

$$\alpha \in \sigma(x) \Longleftrightarrow \alpha 1 - x \text{ is not invertible.}$$

Hence α belongs to $\sigma(x)$ iff there is an $x' \in \sigma(E)$ such that

$$x'(\alpha 1 - x) = 0$$

(Corollary 2.4.1.6), i.e. such that

$$\alpha = x'(x) = \widehat{x}(x').$$

Thus

$$\sigma(x) = \widehat{x}(\sigma(E)).$$

Now assume that E is not unital. Let F be a unital Banach algebra associated to E. Then

$$\sigma_E(x) = \sigma_F(x) = \widehat{x}(\sigma(F)) = \widehat{x}(\sigma(E)) \cup \{0\}$$

by the above result and Corollary 2.4.1.4.

b) and c) follow from a). ■

Corollary 2.4.1.8 *Given $A \subset \mathbb{C}$ and $\alpha > 0$, define*

$$U_\alpha(A) := \bigcup_{\beta \in A} U_\alpha^{\mathbb{C}}(\beta).$$

Given $A, B \subset \mathbb{C}$, put

$$d(A,B) := \inf\{\alpha > 0 \mid A \subset U_\alpha(B),\ B \subset U_\alpha(A)\}.$$

Let E be a complex Banach algebra and take $x, y \in E$ such that $xy = yx$, then

$$d(\sigma(x), \sigma(y)) \leq \|x - y\|.$$

We may assume E to be unital. Let F be the unital Banach subalgebra of E generated by

$$\{x, y\} \cup \{(\alpha 1 - x)^{-1} \mid \alpha \in \mathbb{C} \backslash \sigma(x)\} \cup \{(\alpha 1 - y)^{-1} \mid \alpha \in \mathbb{C} \backslash \sigma(y)\}.$$

By Proposition 2.1.2.8 and Corollary 2.2.1.10, F is commutative. Since obviously

$$\sigma_E(x) = \sigma_F(x), \quad \sigma_E(y) = \sigma_F(y),$$

we may assume that E is commutative. By Corollary 2.4.1.7 a) and Proposition 2.4.1.2,

$$\sigma(x) = \hat{x}(\sigma(E)) \subset \bigcup_{\beta \in A} U_\alpha^{\mathbb{C}}(\beta)$$

for every $\alpha > \|x - y\|$, where

$$A := \hat{y}(\sigma(E)) = \sigma(y).$$

It follows that

$$d(\sigma(x), \sigma(y)) \leq \|x - y\|. \qquad \blacksquare$$

Remark. The above corollary improves Corollary 2.2.5.2, when $xy = xy$ and $\mathbb{K} = \mathbb{C}$.

Corollary 2.4.1.9 *Let* $\sum_{n=0}^{\infty} \alpha_n t^n$ *be a power series with radius of convergence* r. *Take*

$$f : U_r^{\mathbb{C}}(0) \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \sum_{n=0}^{\infty} \alpha_n \alpha^n.$$

Let E *be a unital complex Banach algebra and take* $x \in E$ *with*

$$\lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} < r,$$

and

$$f(x) = \sum_{n=0}^{\infty} \alpha_n x^n$$

(Proposition 2.2.3.2). Then

$$\sigma(f(x)) = f(\sigma(x))$$

(Proposition 2.2.4.1).

We may assume that there is an $n_0 \in \mathbb{N}$ with $\alpha_{n_0} \neq 0$ (Theorem 2.2.5.4, Proposition 2.1.3.2). Given $m \in \mathbb{N}$, put

$$P_m := \sum_{n=0}^{m} \alpha_n t^n .$$

Then in the notation of Corollary 2.4.1.8,

$$d(P_m(\sigma(x)), \sigma(f(x))) = d(\sigma(P_m(x)), \sigma(f(x))) \leq \|P_m(x) - f(x)\|$$

for every $m \in \mathbb{N}$, $m \geq n_0$ (Corollary 2.1.3.4 b)). . Hence

$$\sigma(f(x)) = f(\sigma(x))$$

(Corollary 2.2.4.5). ∎

Corollary 2.4.1.10 *Take $\alpha, a, b \in \mathbb{R}$ with $\alpha > 0$, $[a, b] \subset]-1, 1[$. Let E be a unital complex Banach algebra and take $x \in E$ such that*

$$\sigma(x) \subset [a, b] .$$

Then

$$\sigma([1 + x]^\alpha) \subset [(1 + a)^\alpha, (1 + b)^\alpha] \subset \mathbb{R}_+ .$$

Let

$$f : U_1^{\mathbb{K}}(0) \longrightarrow \mathbb{K}, \quad \beta \longmapsto \sum_{n=0}^{\infty} \binom{\alpha}{n} \beta^n .$$

Since

$$[1 + x]^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n ,$$

it follows

$$\sigma([1 + x]^\alpha) = \sigma(f(x)) = f(\sigma(x)) \subset [(1 + a)^\alpha , (1 + b)^\alpha] \subset \mathbb{R}_+$$

by Corollary 2.4.1.9 and Lemma 2.2.3.12. ∎

Corollary 2.4.1.11 *Let E be an involutive unital complex Banach algebra. Then for every $x \in \operatorname{Re} E$ with $\sigma(x) \subset]0, \infty[$ and for every $n \in \mathbb{N}$, there is a $y \in \operatorname{Re} E$ such that $\sigma(y) \subset]0, \infty[$ and $y^n = x$.*

By Corollary 2.2.5.9, there is an $\alpha > 0$, with

$$r(1-\alpha x) = \lim_{n\to\infty} \|(1-\alpha x)^n\|^{\frac{1}{n}} < 1$$

and

$$\left(\alpha^{-\frac{1}{n}}[1-(1-\alpha x)]^{\frac{1}{n}}\right)^n = x\,.$$

Put

$$y := \alpha^{-\frac{1}{n}}[1-(1-\alpha x)]^{\frac{1}{n}}\,.$$

By Corollary 2.3.2.19, $y \in \operatorname{Re} E$. Since

$$\sigma(\alpha x - 1) \subset U_1^{\mathbb{C}}(0) \cap (\alpha\sigma(x) - 1) \subset]-1, 1[$$

it follows, by Corollary 2.4.1.10, that

$$\sigma(y) \subset]0, \infty[\,. \qquad \blacksquare$$

Corollary 2.4.1.12 *Let E be a unital Gelfand algebra. Take $x \in E$ and put*

$$F := \{P(x)Q(x)^{-1} \mid P, Q \in \mathbb{K}[t],\ Q(x) \text{ invertible}\}\,.$$

If F is dense in E, then the map

$$\sigma(E) \longrightarrow \sigma(x)\,, \quad x' \longmapsto x'(x)$$

is a homeomorphism.

By Corollary 2.4.1.7 a), the map in question is surjective. Since it is continuous and $\sigma(E)$ is compact (Theorem 2.4.1.3 c)), it is sufficient to show that this map is injective. Take $x', y' \in \sigma(E)$ with

$$x'(x) = y'(x)\,.$$

Then

$$x'|F = y'|F\,,$$

so that $x' = y'$ (Proposition 2.2.4.19). $\blacksquare$

Theorem 2.4.1.13 (0) *If E is a Gelfand algebra, then*

$$\|\widehat{x}\|_\infty = r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\|$$

for every $x \in E$ and given distinct $x', y' \in \sigma(E)$, there is an $x \in E$ with

$$\widehat{x}(x') = 0 \neq \widehat{x}(y')\,.$$

That

$$\|\widehat{x}\| = r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} \leq \|x\|$$

for every $x \in E$ follows from Corollary 2.4.1.7 b) and Theorem 2.2.5.4. By Proposition 2.4.1.5, there is an $x \in E$ such that

$$x \in \operatorname{Ker} x' \backslash \operatorname{Ker} y' .$$

Hence

$$\widehat{x}(x') = x'(x) = 0 \neq y'(x) = \widehat{x}(y') . \qquad \blacksquare$$

Corollary 2.4.1.14 (0) *If E is a Gelfand algebra, then*

$$\textit{radical of } E = \{x \in E \mid x \textit{ is quasinilpotent}\} = \{x \in E \mid \widehat{x} = 0\} .$$

If E is semi-simple, then there is a unique order relation on E with respect to which E is an ordered Banach space such that

$$E_+ = \{x \in E \mid \sigma(x) \subset \mathbb{R}_+\} = \{x \in E \mid \widehat{x} \in C_0(\sigma(E))_+\} .$$

Moreover, the Gelfand transform of E is injective iff E is semi-simple.

By Proposition 2.4.1.5 and Theorem 2.4.1.13,

$$\text{radical of } E = \bigcap_{x' \in \sigma(E)} \operatorname{Ker} x' = \{x \in E \mid \widehat{x} = 0\} = \{x \in E \mid x \text{ is quasinilpotent}\} .$$

Now suppose that E is semi-simple. By the above, the Gelfand transform is injective and so we may identify E with its image under it in $C_0(\sigma(E))$. By Corollary 2.4.1.7 a),

$$\{x \in E \mid \sigma(x) \subset \mathbb{R}_+\} = \{x \in E \mid \widehat{x} \in C_0(\sigma(E))_+\}$$

and the last assertion now follows. $\blacksquare$

Remark. If E is the Gelfand algebra of Example 2.2.4.23, then it follows from the above corollary that $\sigma(E)$ is empty.

Proposition 2.4.1.15 (0) *Let E be a (unital) Gelfand algebra and u its Gelfand transform. Then the following assertions are equivalent:*

a) *There is an $\alpha > 0$, such that*

$$r(x) \geq \alpha \|x\|$$

for every $x \in E$.

b) *E is semi-simple and* $\operatorname{Im} u$ *is closed.*

If these conditions are fulfilled, then the following assertions also hold.

c) *The map*

$$E \longrightarrow \operatorname{Im} u, \quad x \longmapsto ux$$

is an isomorphism of (unital) Banach algebras and its inverse has norm smaller than $\frac{1}{\alpha}$.

d) *There is a unique order relation on E, with respect to which E is an ordered Banach space such that*

$$E_+ = \{x \in E \mid \sigma(x) \subset \mathbf{R}_+\} = \{x \in E \mid \hat{x} \in C_0(\sigma(E))_+\}.$$

e) *Given $x \in E_+$ and $n \in \mathbf{N}$, there is a unique $y \in E_+$ such that $y^n = x$. This y is in the closed real subalgebra of E generated by x and*

$$\|y\| \leq \frac{1}{\alpha}\|x\|^{\frac{1}{n}}.$$

f) *For every $x \in E$ with $\sigma(x) \subset \mathbf{R}$, there exist uniquely $x^+, x^- \in E_+$ such that*

$$x = x^+ - x^-, \quad x^+x^- = 0,$$

x^+, x^- belong to the closed real subalgebra of E generated by x and

$$\|x^+\| \leq \frac{1}{\alpha}\|x\|, \quad \|x^-\| \leq \frac{1}{\alpha}\|x\|.$$

a $\Rightarrow$ b & c. By Theorem 2.4.1.13,

$$\alpha\|x\| \leq \|ux\| \leq \|x\|$$

for every x and so b & c follows from Proposition 1.2.1.18 b),d), Proposition 2.4.1.2, and Corollary 2.4.1.14.

b $\Rightarrow$ a. By Corollary 2.4.1.14, u is injective and the assertion follows from the Principle of Inverse Operator.

d) follows from b) and Corollary 2.4.1.14.

e) By Corollary 2.4.1.7 a), $\hat{x}$ is positive. Put

$$f : \sigma(E) \longrightarrow \mathbb{K}, \quad x' \longmapsto \hat{x}(x')^{\frac{1}{n}}.$$

By b) and Corollary 1.3.5.17 (used for $\mathbb{K} = \mathbb{R}$), $f \in \operatorname{Im} u$ and the assertion follows from a), c), and d).

f) By Corollary 2.4.1.7 a), $\widehat{x}$ is real. Define

$$g : \sigma(E) \longrightarrow \mathbb{K}, \quad x' \longmapsto \sup\{\widehat{x}(x'), 0\},$$

$$h : \sigma(E) \longrightarrow \mathbb{K}, \quad x' \longmapsto \sup\{-\widehat{x}(x'), 0\}.$$

By Corollary 1.3.5.17 (used for $\mathbb{K} = \mathbb{R}$), $g, h \in \operatorname{Im} u$ and the assertion follows from a), c), and d). ■

Corollary 2.4.1.16 (4) *If E is a finite-dimensional Gelfand algebra, then the following assertions are equivalent:*

a) *There is an $\alpha > 0$ such that*

$$r(x) \geq \alpha \|x\|$$

for every $x \in E$.

b) *E is semi-simple.*

Since E is finite dimensional, $\{\widehat{x} \mid x \in E\}$ is finite dimensional too, and hence it is closed. The assertion follows from Proposition 2.4.1.15 a $\Leftrightarrow$ b. ■

Proposition 2.4.1.17 (0) *Let E, F be unital Gelfand algebras and $u : E \to F$ a unital algebra homomorphism.*

a) *$y' \circ u \in \sigma(E)$ for every $y' \in \sigma(F)$. Define*

$$\varphi : \sigma(F) \longrightarrow \sigma(E), \quad y' \longmapsto y' \circ u.$$

b) $\widehat{ux} = \overset{E}{\widehat{x}} \circ \varphi$ *for all $x \in E$.*

c) *φ is continuous.*

d) *If $u(E)$ is dense, then φ is injective.*

e) *If F is semi-simple, then u is continuous.*

a) is trivial.

b) Given $y' \in \sigma(F)$,

$$\overset{F}{\widehat{ux}}(y') = y'(ux) = y' \circ u(x) = (\varphi(y'))(x) = \overset{E}{\widehat{x}}(\varphi(y')) = \overset{E}{\widehat{x}} \circ \varphi(y').$$

Hence

$$\overset{F}{\widehat{ux}} = \overset{E}{\widehat{x}} \circ \varphi.$$

c) Let $x \in E$. Then

$$(\varphi(y'))(x) = y' \circ u(x) = y'(ux)$$

for every $y' \in \sigma(F)$. φ is thus continuous.

d) Let $x', y' \in \sigma(F)$ such that $\varphi(x') = \varphi(y')$. Then $x' = y'$ on $u(E)$. Since $u(E)$ is dense and x', y' are continuous (Proposition 2.2.4.19) we get $x' = y'$. Hence φ is injective.

e) By the Closed Graph Theorem, it suffices to show that the graph G of u is closed. Take $(x, y) \in \overline{G}$. There is a sequence $((x_n, y_n))_{n \in \mathbb{N}}$ in G converging to $(x. y)$. By a) and Proposition 2.2.4.19,

$$y'(y - ux) = y'(y) - y' \circ u(x) = \lim_{n \to \infty} (y'(y_n) - y' \circ u(x_n)) =$$

$$= \lim_{n \to \infty} y'(y_n - ux_n) = 0$$

for every $y' \in \sigma(F)$. Since F is semi-simple,

$$y - ux = 0$$

(Corollary 2.4.1.14). Thus $(x, y) \in G$ and G is closed. ■

Corollary 2.4.1.18 *Let E be a Gelfand semi-simple unital algebra. Any two norms on E, with respect to which E is a unital Banach algebra, are equivalent.* ■

2.4.2 Involutive Gelfand Algebras

Definition 2.4.2.1 (0) *An involutive (unital) Gelfand algebra is an involutive Banach algebra which is a (unital) Gelfand algebra such that when* $\mathbb{K} = \mathbb{R}$, *every element is selfadjoint.*

Hence an involutive complex Gelfand algebra is nothing else but an involutive commutative complex Banach algebra. An involutive real Gelfand algebra is an involutive commutative real Banach algebra E such that

$$x \in E \Longrightarrow \sigma_{\overset{\circ}{E}}((x,0)) = \sigma_E(x)\,,\ x^* = x\,.$$

$\mathbb{R}$ and $\mathbb{C}$ are involutive unital real and complex Gelfand algebras, respectively.

Proposition 2.4.2.2 (0) *Let* E *be an involutive real Gelfand algebra.*

a) E *and* $\overset{\circ}{E}$ *are strongly symmetric.*

b) *Every character of* E *and the Gelfand transform of* E *are involutive.*

c) $\sigma(E) \subset E'_+$.

d) *If* E *is quasiunital then* $\sigma(E) \subset \tau_0(E)$.

a) Take $x \in E$. By Corollary 2.4.1.7 a),

$$\sigma(x^*x) = \sigma(x^2) \subset \widehat{x^2}(\sigma(E)) \cup \{0\} = \widehat{x}^2(\sigma(E)) \cup \{0\} \subset \mathbb{R}_+\,.$$

Hence E is strongly symmetric.

Let $(x,y) \in \overset{\circ}{E}$. Then

$$\sigma_{\overset{\circ}{E}}((x,y)^*(x,y)) = \sigma_{\overset{\circ}{E}}((x,-y)(x,y)) = \sigma_{\overset{\circ}{E}}((x^2+y^2,0)) =$$

$$= \sigma_E(x^2+y^2) \subset \widehat{x^2+y^2}(\sigma(E)) \cup \{0\} =$$

$$= (\widehat{x}^2 + \widehat{y}^2)\ (\sigma(E)) \cup \{0\} \subset \mathbb{R}_+$$

(Corollary 2.4.1.7 a)). Hence $\overset{\circ}{E}$ is strongly symmetric.

b) is trivial.

c) Take $x' \in \sigma(E)$ and $x \in E$. Then

$$x'(x^*x) = x'(x^2) = x'(x)^2 \in \mathbb{R}_+\,,$$

so that $x' \in E'_+$.

d) follows from Proposition 2.2.4.19 and Proposition 2.3.5.6. ■

Proposition 2.4.2.3 (0) *Let E be an involutive commutative complex Banach algebra. Then the following assertions are equivalent:*

a) *E is symmetric.*

b) *$\sigma(x) \subset \mathbb{R}$ for every $x \in \operatorname{Re} E$.*

c) *The Gelfand transform of E is involutive.*

d) *Every character of E is involutive.*

e) *$\sigma(E) \subset E'_+$.*

If E is quasiunital, then the above conditions are equivalent to each of the following ones:

f) *$\sigma(E) \subset \tau(E)$.*

g) *$\sigma(E) \subset \tau_0(E)$.*

Let F be an involutive real Banach algebra such that E is isomorphic to the complexification of F and that every element of F is selfadjoint. If E is symmetric then F is a symmetric involutive Gelfand algebra.

a $\Rightarrow$ b follows from Proposition 2.3.1.29.

b $\Rightarrow$ c. Define

$$u : E \longrightarrow \mathcal{C}(\sigma(E)), \quad x \longmapsto \widehat{x}.$$

Then

$$\widehat{\operatorname{re} x}(\sigma(E)) \subset \sigma(\operatorname{re} x) \subset \mathbb{R},$$

$$\widehat{\operatorname{re} x}(\sigma(E)) \subset \sigma(\operatorname{im} x) \subset \mathbb{R}$$

(Corollary 2.4.1.7 a)), so that

$$ux^* = u(\operatorname{re} x - i \operatorname{im} x) = \widehat{\operatorname{re} x} - i\widehat{\operatorname{im} x} = \overline{\widehat{\operatorname{re} x} + i\widehat{\operatorname{im} x}} = \overline{ux}$$

for every $x \in E$, that is to say, u is involutive.

c $\Rightarrow$ a follows from Corollary 2.3.1.7 a).

c $\Leftrightarrow$ d and g $\Rightarrow$ e $\Rightarrow$ d are trivial.

d $\Rightarrow$ e follows from Proposition 2.3.4.5.

e $\Rightarrow$ f follows from Proposition 2.2.4.19.

f $\Rightarrow$ g follows from Proposition 2.3.5.6.

Now we prove the last assertion. By a $\Rightarrow$ b,

$$\sigma_E((x,0)) \subset \mathbb{R}$$

for every $x \in F$ and F is a symmetric involutive Gelfand algebra. ■

Corollary 2.4.2.4 (0) (Shirali–Ford) *Let E be an involutive complex Banach algebra. Then the following are equivalent:*

a) *E is symmetric.*

b) *$\sigma(x) \subset \mathbb{R}$ for every $x \in \operatorname{Re} E$.*

c) *Any involutive unital Banach algebra associated to E is symmetric.*

a $\Rightarrow$ b follows from Proposition 2.3.1.29.

b $\Rightarrow$ a. Let F be a closed involutive commutative subalgebra of E and let $x \in \operatorname{Re} F$. By Corollary 2.2.4.27,

$$\sigma_F(x) = \sigma_E(x) \subset \mathbb{R},$$

and so, by Proposition 2.4.2.3 b $\Rightarrow$ a, F is symmetric. By Proposition 2.3.2.21 c $\Rightarrow$ a, E is symmetric.

b $\Rightarrow$ c. We may assume that E is not unital. Let $\mathbb{C} \times E$ be an involutive unital Banach algebra associated to E and let (α, x) be a selfadjoint element of $\mathbb{C} \times E$. Then $\alpha \in \mathbb{R}$ and $x \in \operatorname{Re} E$. Thus

$$\sigma_{\mathbb{C}\times E}((0,x)) = \sigma_E(x) \subset \mathbb{R}$$

and

$$\sigma_{\mathbb{C}\times E}((\alpha,x)) = \alpha + \sigma_{\mathbb{C}\times E}((0,x)) \subset \mathbb{R}.$$

By b $\Rightarrow$ a, $\mathbb{C} \times E$ is symmetric.

c $\Rightarrow$ a follows from Proposition 2.3.1.28. ■

Corollary 2.4.2.5 (0) *Let E be a symmetric involutive complex Banach algebra such that*

$$x \in \operatorname{Re} E,\ r(x) = 0 \Longrightarrow x = 0.$$

Take $x \in \operatorname{No} E$. Then the following are equivalent:

a) $x \in \operatorname{Re} E$.

b) $\sigma(x) \subset \mathbb{R}$.

a $\Rightarrow$ b follows from Proposition 2.3.1.29.

b $\Rightarrow$ a. By Corollary 2.3.2.16, $E(x)$ is commutative. Since the Gelfand transform of $E(x)$ is involutive (Proposition 2.4.2.3 a $\Rightarrow$ c), it follows

$$r_E(\operatorname{im} x) = r_{E(x)}(\operatorname{im} x) = 0$$

(Corollary 2.2.4.27, Corollary 2.4.1.7 a)). By hypothesis

$$\operatorname{im} x = 0 .$$

Hence $x \in \operatorname{Re} E$. ■

Corollary 2.4.2.6 (0) *If E is a symmetric involutive Gelfand algebra, then $\{\hat{x} \mid x \in E\}$ is a dense involutive subalgebra of $C_0(\sigma(E))$.*

By Proposition 2.4.2.2 b) and Proposition 2.4.2.3 a $\Rightarrow$ c, the Gelfand transform of E is involutive. Hence $\{\hat{x} \mid x \in E\}$ is an involutive subalgebra of $C_0(\sigma(E))$ (Proposition 2.3.1.16, Theorem 2.4.1.3 e)). The assertion now follows from Corollary 1.3.5.15 and Theorem 2.4.1.13. ■

Corollary 2.4.2.7 (0) *Let E be a symmetric involutive Gelfand algebra. Then the following are equivalent:*

a) *There is an $\alpha > 0$ such that*

$$r(x) \geq \alpha \|x\|$$

for every $x \in E$.

b) *The Gelfand transform is an isomorphism of involutive Banach algebras.*

c) *The Gelfand transform is bijective.*

In particular, E is unital iff $\sigma(E)$ is compact.

By Propositions 2.4.2.2 b) and 2.4.2.3 a $\Rightarrow$ c, the Gelfand transform is involutive and by Corollary 2.4.2.6 its image is dense in $C_0(\sigma(E))$.

a $\Rightarrow$ b follows from the above and Proposition 2.4.1.15 a $\Rightarrow$b & c.

b $\Rightarrow$ a & c is trivial.

c $\Rightarrow$ b follows from the above and the Principle of Inverse Operators. ■

Corollary 2.4.2.8 (4) *Let E be a finite-dimensional symmetric involutive semi-simple Gelfand algebra.*

a) *The Gelfand transform of E is an isomorphism of involutive Banach algebras.*

b) *E has a unit.*

c) *There is a unique order relation on E with respect to which E is an ordered Banach space and*

$$E_+ = \{x \in E \mid x' \in E'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\}\,.$$

d) $E_+ \subset \operatorname{Re} E$.

e) *Given $x \in E$, the following are equivalent:*

e_1) $x \in E_+$.

e_2) $x = y^*y$ *for some* $y \in E$.

e_3) $x = y^2$ *for some* $y \in E_+$.

e_4) $\sigma(x) \subset \mathbb{R}_+$.

a) By Proposition 2.4.2.2 b) and Proposition 2.4.2.3 a $\Rightarrow$ c the Gelfand transform of E is involutive. By Corollary 2.4.1.16 b $\Rightarrow$ a, there is an $\alpha > 0$ such that

$$r(x) \geq \alpha\|x\|$$

for every $x \in E$ and the assertion follows from Corollary 2.4.2.7 a $\Rightarrow$ b.

b), c), d), and e) follow from a). ■

Proposition 2.4.2.9 (0) *Let E be a symmetric involutive complex Banach algebra for which there is an $\alpha > 0$ with*

$$r(x) \geq \alpha\|x\|$$

for every $x \in \operatorname{Re} E$.

a) *For every $x \in \operatorname{No} E$ such that $\sigma(x) \subset \mathbb{R}_+$ and for every $n \in \mathbb{N}$, there is a unique $y \in \operatorname{Re} E$ such that $\sigma(y) \subset \mathbb{R}_+$ and $y^n = x$. y belongs to the closed real subalgebra of E generated by x and*

$$\|y\| \leq \frac{1}{\alpha}\|x\|^{\frac{1}{n}}\,.$$

b) *For every* $x \in \operatorname{Re} E$ *there are uniquely determined* $x^+, x^- \in \operatorname{Re} E(x)$ *such that*

$$\sigma(x^+) \subset \mathbb{R}_+, \quad \sigma(x^-) \subset \mathbb{R}_+, \quad x = x^+ - x^-, \quad x^+x^- = x^-x^+ = 0.$$

x^+ *and* x^- *belong to the closed real subalgebra of* E *generated by* x *and*

$$\|x^+\| \leq \frac{1}{\alpha}\|x\|, \quad \|x^-\| \leq \frac{1}{\alpha}\|x\|.$$

c) *Suppose* $\alpha = 1$ *and* E *is unital. Given* $x \in \operatorname{Re} E$, *the following are equivalent:*

c_1) $\sigma(x) \subset \mathbb{R}_+$.

c_2) *There is a* $y \in \operatorname{Re} E$ *such that* $\sigma(y) \subset \mathbb{R}_+$ *and* $y^2 = x$.

c_3) *There is a* $y \in E$ *such that* $y^*y = x$.

c_4) $x'(x) \in \mathbb{R}_+$ *for every positive continuous linear form* x' *on* E.

If these conditions are fulfilled then E *is strongly symmetric.*

a) Let F be a closed involutive commutative subalgebra of E containing x (e.g. $E(x)$ by Corollary 2.3.2.16). Then

$$r_F(z) = r_E(z) \geq \alpha\|z\|$$

for every $z \in F$ and

$$\sigma_F(x) = \sigma_E(x) \subset \mathbb{R}_+$$

(Corollary 2.2.4.27). By Proposition 2.4.1.15 d),e), there is a unique $y \in F$ such that $\sigma_F(y) \subset \mathbb{R}_+$ and $y^n = x$. Moreover, y belongs to the closed real algebra of E generated by x and

$$\|y\| \leq \frac{1}{\alpha}\|x\|^{\frac{1}{n}}.$$

By Corollary 2.2.4.27,

$$\sigma_E(y) = \sigma_F(y) \subset \mathbb{R}_+.$$

By Corollary 2.4.2.5 b $\Rightarrow$ a, $y \in \operatorname{Re} F \subset \operatorname{Re} E$.

Now take $z \in \operatorname{No} E$ such that $\sigma(z) \subset \mathbb{R}_+$ and $z^n = x$. Then $E(z)$ is a closed involutive commutative subalgebra of E (Corollary 2.3.2.16) containing x. By the above, $z = y$.

b) We put $F := E(x)$. By Corollary 2.3.2.16, $E(x)$ is commutative. Then

$$r_F(y) = r_E(y) \geq \alpha \|y\|$$

for every $y \in F$ and

$$\sigma_F(x) = \sigma_E(x) \subset \mathbb{R}$$

(Corollary 2.2.4.27, Proposition 2.3.1.29). The assertion now follows from Proposition 2.4.1.15 f).

$c_1 \Rightarrow c_2$ follows from a).

$c_2 \Rightarrow c_3$ is trivial.

$c_3 \Rightarrow c_1$. We endow E with the order relation defined in Proposition 2.3.2.34 b).

Step 1 $\quad x \leq 0 \Rightarrow x = 0$.

By Proposition 2.1.3.10 b),

$$\sigma(-yy^*)\backslash\{0\} = \sigma(-y^*y)\backslash\{0\} \subset \sigma(-x) \subset \mathbb{R}_+,$$

such that

$$-yy^* \geq 0.$$

By Proposition 2.3.2.34 c),

$$y^*y + yy^* \geq 0,$$

so that

$$x = y^*y \geq -yy^* \geq 0,$$

$$x = 0.$$

Step 2 $\quad x \geq 0$

By b), there are $x^+, x^- \in E_+$ with

$$x = x^+ - x^-, \quad x^+x^- = x^-x^+ = 0.$$

We get

$$(yx^-)^*(yx^-) = x^-y^*yx^- = x^-xx^- = x^-(x^+ - x^-)x^- = -(x^-)^3 \leq 0$$

by Proposition 2.3.2.34 c). By Step 1,

$$(x^-)^3 = 0\,,$$

such that

$$\sigma(x^-)^3 = \sigma((x^-)^3) = \{0\}$$

(Theorem 2.1.3.4 b)),

$$\|x^-\| = r(x^-) = 0\,,$$

$$x^- = 0\,,$$

$$x = x^+ \in E_+\,.$$

$c_3 \Rightarrow c_4$ is trivial.

$c_4 \Rightarrow c_1$ follows from Proposition 2.3.5.15 a $\Rightarrow$ c. ■

Proposition 2.4.2.10 (0) *Let E be a symmetric involutive unital Gelfand algebra and let $x \in E$ such that $E = E(x,1)$. Then the map*

$$\sigma(E) \longrightarrow \sigma(x)\,, \quad x' \longmapsto x'(x)$$

is a homeomorphism. If E has no unit then the above assertion still holds if $E = E(x)$ and if we replace $\sigma(E)$ by $\sigma_0(E)$.

Define

$$\varphi : \sigma(E) \longrightarrow \sigma(x)\,, \quad x' \longmapsto x'(x)\,.$$

φ is surjective (Corollary 2.4.1.7 a)) and continuous. Take $x', y' \in \sigma(E)$ such that

$$x'(x) = y'(x)\,.$$

Then

$$x'(x^*) = \overline{x'(x)} = \overline{y'(x)} = y'(x^*)$$

by Proposition 2.4.2.3 a $\Rightarrow$ d and Proposition 2.4.2.2 b). Let F be the involutive unital subalgebra of E generated by x and take $y \in F$. Then there is a $P \in \mathbb{K}[s,t]$ such that

$$y = P(x, x^*)$$

(Proposition 2.3.1.20). Thus

$$x'(y) = P(x'(x), x'(x^*)) = P(y'(x), y'(x^*)) = y'(y)\,.$$

Hence x' and y' coincide on F. Since x', y' are continuous (Proposition 2.2.4.19) and F is dense in E , $x' = y'$. Hence φ is injective and therefore a homeomorphism (Theorem 2.4.1.3 c)).

The final assertion has the same proof. ■

Proposition 2.4.2.11 *Let E be a symmetric involutive quasiunital Gelfand algebra and $\mathcal{M}$ the set of positive Radon measures μ on $\sigma(E)$ satisfying*

$$\mu(\sigma(E)) = 1\,.$$

Given $\mu \in \mathcal{M}$, define

$$\widetilde{\mu} : E \longrightarrow \mathbb{K}\,, \quad x \longmapsto \int \widehat{x} d\mu\,.$$

Then the Gelfand transform of E is involutive,

$$\sigma(E) \subset \tau_0(E)$$

and $\widetilde{\mu} \in \tau(E)$ for every $\mu \in \mathcal{M}$. The map

$$\mathcal{M} \longrightarrow \tau(E)\,, \quad \mu \longmapsto \widetilde{\mu}$$

is injective and $\{\widehat{x} \mid x \in E\}$ is a dense involutive subalgebra of $C_0(\sigma(E))$. Moreover, the following are equivalent:

a) $\tau_0(E) \subset \sigma(E)$.

b) $\tau_0(E) = \sigma(E)$.

c) $\|\widehat{x}\| = \|\overset{\vee}{x}\|$ *for every* $x \in E$.

d) *The map*

$$\mathcal{M} \longrightarrow \tau(E)\,, \quad \mu \longmapsto \widetilde{\mu}$$

is bijective.

e) $\tau_0(E) \subset \{\widetilde{\mu} | \mu \in \mathcal{M}\}$.

If

$$\operatorname{Re} E' = E'_+ - E'_+ ,$$

then the above conditions are equivalent to each of the following ones:

f) *The Gelfand transform of E is an isomorphism of involutive Banach algebras.*

g) *There is an $\alpha > 0$, such that*

$$r(x) \geq \alpha \|x\|$$

for all $x \in E$.

h) *The Gelfand transform of E is bijective.*

By Proposition 2.4.2.3 a $\Rightarrow$ c & g, and Proposition 2.4.2.2 b),d) the Gelfand transform of E is involutive and $\sigma(E) \subset \tau_0(E)$. By Corollary 2.4.2.6, $\{\hat{x} \mid x \in E\}$ is a dense involutive subalgebra of $C_0(\sigma(E))$. Take $\mu \in \mathcal{M}$. Then

$$\tilde{\mu}(x^*x) = \int \widehat{x^*x} d\mu = \int |\hat{x}|^2 d\mu \geq 0 ,$$

$$|\tilde{\mu}(x)| = \left| \int \hat{x} d\mu \right| \leq \int |\hat{x}| d\mu \leq \|\hat{x}\| \leq \|x\|$$

for every $x \in E$. $\tilde{\mu}$ is thus positive and

$$\|\tilde{\mu}\| \leq 1 .$$

Let K be a compact set of $\sigma(E)$ and take $\varepsilon > 0$. Since $\{\hat{x} \mid x \in E\}$ is a dense set of $C_0(\sigma(x))$, there is an $x \in E$ with $|\hat{x}| \geq 1$ on K. Let $\mathfrak{F}$ be an approximate unit of E. We set

$$f : E \longrightarrow E , \quad y \longmapsto y^*y ,$$

$$\mathfrak{G} := f(\mathfrak{F}) .$$

Then, by Proposition 2.3.4.9, $\mathfrak{G}$ is an approximate unit of E. Hence, there is a $y \in E^{\#}$ such that

$$\|x^*x - x^*xy^*y\| < \varepsilon .$$

Then

$$1 - \widehat{y^*y} \leq \widehat{x^*x}(1 - \widehat{y^*y}) \leq \|\widehat{x^*x} - \widehat{x^*x}\widehat{y^*y}\| \leq \|x^*x - x^*xy^*y\| < \varepsilon$$

on K and so

$$\|\widetilde{\mu}\| \geq \widetilde{\mu}(y^*y) = \int \widehat{y^*y}d\mu \geq \int_K (1-\varepsilon)d\mu = (1-\varepsilon)\mu(K)\,.$$

Since ε and K are arbitrary,

$$\|\widetilde{\mu}\| \geq \mu(\sigma(E)) = 1\,,$$

$$\|\widetilde{\mu}\| = 1\,, \quad \widetilde{\mu} \in \tau(E)\,.$$

Since $\{\widehat{x} \mid x \in E\}$ is dense in $\mathcal{C}_0(\sigma(E))$, the map

$$\mathcal{M} \longrightarrow \tau(E)\,, \quad \mu \longmapsto \widetilde{\mu}$$

is injective.

a $\Rightarrow$ b follows from the inclusion $\sigma(E) \subset \tau_0(E)$, which was established above.

b $\Rightarrow$ c. By Proposition 2.3.5.13 a $\Rightarrow$ c,

$$\|\widehat{x}\| \geq \sup_{x' \in \sigma(E)} x'(x) = \sup_{x' \in \tau(E)} x'(x)\,,$$

$$\|\widehat{x}\| \geq \sup_{x' \in \sigma(E)} x'(-x) = \sup_{x' \in \tau(E)} x'(-x)\,,$$

such that

$$\|\widehat{x}\| = \|\overset{\vee}{x}\|$$

for every $x \in \operatorname{Re} E$. Now take $x \in E$ and $\varepsilon > 0$. There is an $x' \in \tau(E)$ and a $\theta \in \mathbb{R}$ such that

$$|x'(x)| > \|\overset{\vee}{x}\| - \varepsilon\,, \quad x'(x) = |x'(x)|e^{i\theta}\,.$$

Put

$$y := e^{-i\theta}x\,.$$

Then

$$x'(y) = x'(e^{-i\theta}x) = e^{-i\theta}x'(x) = |x'(x)| > \|\overset{\vee}{x}\| - \epsilon,$$

$$\|\hat{x}\| = \|\hat{y}\| \geq \|\operatorname{re}\hat{y}\| = \|\widehat{\operatorname{re} y}\| = \|\overset{\vee}{\widehat{\operatorname{re} y}}\| = \|\operatorname{re}\overset{\vee}{y}\| \geq$$

$$\geq \operatorname{re}\overset{\vee}{y}(x') = \operatorname{re} x'(y) \geq \|\overset{\vee}{x}\| - \epsilon$$

(Corollary 2.3.1.23 a $\Rightarrow$ d). Thus

$$\|\hat{x}\| \geq \|\overset{\vee}{x}\|$$

since ϵ is arbitrary. The reverse inequality follows from the fact that $\sigma(E) \subset \tau_0(E)$.

c $\Rightarrow$ d. Take $x' \in \tau(E)$. By d), the map

$$\{\hat{x} \mid x \in E\} \longrightarrow \mathbb{K}, \quad \hat{x} \longmapsto x'(x)$$

is well-defined, linear, involutive, and continuous. Since $\{\hat{x} | x \in E\}$ is a dense involutive set of $\mathcal{C}_0(\sigma(E))$, there is a bounded real Radon measure μ on $\sigma(E)$ such that

$$x'(x) = \int \hat{x}\, d\mu$$

for every $x \in E$ (Example 2.3.4.18). By c),

$$\left|\int \hat{x} d\mu\right| = |x'(x)| \leq \|\overset{\vee}{x}\| = \|\hat{x}\|$$

for every $x \in E$. Since $\{\hat{x} \mid x \in E\}$ is dense in $\mathcal{C}(\sigma(E))$,

$$\|\mu\| \leq 1.$$

Let $\mathfrak{F}$ be an approximate unit of E and let μ^+ be the positive part of μ. Then

$$1 = \|x'\| = \lim_{x,\mathfrak{F}} x'(x^* x) = \lim_{x,\mathfrak{F}} \int \widehat{x^* x} d\mu \leq \lim_{x,\mathfrak{F}} \int \widehat{x^* x} d\mu^+ \leq \|\mu^+\| \leq \|\mu\|$$

(Proposition 2.3.4.10 a)), such that

$$\|\mu^+\| = \|\mu\| = 1, \quad \mu \in \mathcal{M}, \quad \tilde{\mu} = x',$$

and the map

$$\mathcal{M} \longrightarrow \tau(E), \quad \mu \longmapsto \tilde{\mu}$$

is bijective.

d $\Rightarrow$ e is trivial.

e $\Rightarrow$ a. Take $x' \in \tau_0(E)$. By f), there is a $\mu \in \mathcal{M}$, such that $\widetilde{\mu} = x'$. Assume the support of μ contains two distinct points. Then there are $\mu', \mu'' \in \mathcal{M}$ and $\alpha', \alpha'' \in \mathbb{R}_+\backslash\{0\}$ with

$$\alpha' + \alpha'' = 1, \quad \alpha'\mu' + \alpha''\mu'' = \mu, \quad \mu' \neq \mu''.$$

Thus

$$x' = \alpha'\widetilde{\mu}' + \alpha''\widetilde{\mu}'', \quad \widetilde{\mu}' \neq \widetilde{\mu}'',$$

which contradicts the fact that x' is an extreme point of $\tau(E)$. Hence the support of μ contains exactly one point which must be x'. Hence $x' \in \sigma(E)$ and

$$\tau_0(E) \subset \sigma(E).$$

d $\Rightarrow$ f. Let

$$u : E \longrightarrow \mathcal{C}_0(\sigma(E))$$

be the Gelfand transform of E. Since $\operatorname{Im} u$ is dense in $\mathcal{C}_0(\sigma(E))$, u' is injective (Corollary 1.3.5.9), and since u is involutive, u' is also involutive (Proposition 2.3.2.22 d)). By d),

$$u'(\mathcal{C}_0(\sigma(E))'_+) = E'_+$$

so that

$$u'(\operatorname{Re}\mathcal{C}_0(\sigma(E))') = E'_+ - E'_+ = \operatorname{Re} E'.$$

Thus u' is surjective (Proposition 2.3.1.22, Corollary 2.3.1.23). By the Principle of Inverse Operators, u' is an isomorphism. Hence u is an isomorphism (Corollary 1.4.2.5).

f $\Rightarrow$ e & g & h is trivial.

g $\Rightarrow$ f follows from Corollary 2.4.2.7 a $\Rightarrow$ b.

h $\Rightarrow$ f follows from the Principle of Inverse Operators. ■

Proposition 2.4.2.12 *Let E be a symmetric involutive quasiunital semisimple Gelfand algebra.*

a) *There is a unique order relation on* E *with respect to which* E *is an ordered Banach space and*

$$E_+ = \{x \in E \mid x' \in E'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\}.$$

b) $E_+ \subset \operatorname{Re} E$.

c) $\sigma(x) \subset \mathbb{R}_+$ *for every* $x \in E_+$.

d) *The following are equivalent:*

d_1) *The cone of* E' *generated by* $\sigma(E)$ *is dense in* $(E'_+)_E$.

d_2) $E_+ = \{x \in E \mid \sigma(x) \subset \mathbb{R}_+\}$.

a & b. Take $x \in E$ with

$$x'(x) = 0$$

for every $x' \in E'_+$. Since $\sigma(E) \subset E'_+$ (Proposition 2.4.2.2 c), Proposition 2.4.2.3 a $\Rightarrow$ e), $\hat{x} = 0$. By Corollary 2.4.1.14, $x = 0$ and the assertion follows from Proposition 2.3.4.17 b) (and Proposition 1.3.5.4).

c) By Corollary 2.4.1.7 a)

$$\sigma(x) \subset \hat{x}(\sigma(E)) \cup \{0\}$$

for every $x \in E$, and the assertion now follows from $\sigma(E) \subset E'_+$ (Proposition 2.4.2.2 c), Proposition 2.4.2.3 a $\Rightarrow$ e).

$d_1 \Rightarrow d_2$. Let $x \in E$ such that

$$\sigma(x) \subset \mathbb{R}_+.$$

Then $x'(x) \in \mathbb{R}_+$ whenever $x' \in \sigma(E)$. Hence, by d_1), $x'(x) \in \mathbb{R}_+$ for every $x' \in E'_+$. The reverse inclusion follows from c).

$d_2 \Rightarrow d_1$. Let C' be the closed convex cone of E'_E generated by $\sigma(E)$. Assume there is an $x' \in E'_+ \backslash C'$. By Proposition 2.3.2.25, there is an $x \in \operatorname{Re} E$ with

$$\sup_{y' \in C'} y'(x) = 0 < x'(x).$$

In particular,

$$y'(-x) \geq 0$$

for every $y' \in \sigma(E)$, i.e.

$$\sigma(-x) \subset \mathbb{R}_+$$

(Corollary 2.4.1.7 a)). By d_2), $-x \in E_+$ and we obtain the contradiction

$$x'(-x) < 0 \leq x'(-x).$$

Hence $C' = E'_+$. ■

2.4.3 Examples

Example 2.4.3.1 (0) *Let T be locally compact. For $t \in T$, define*

$$\tilde{t} : C_0(T) \longrightarrow \mathbb{K}, \quad x \longmapsto x(t).$$

Then $\tilde{t} \in \sigma(C_0(T))$ for every $t \in T$ and the map

$$T \longrightarrow \sigma(C_0(T)), \quad t \longmapsto \tilde{t}$$

is a homeomorphism.

The first assertion is trivial. Since the map

$$T \longrightarrow \mathbb{K}, \quad t \longmapsto \langle x, \tilde{t} \rangle = x(t)$$

is continuous for every $x \in C_0(T)$, the map

$$f : T \longrightarrow \sigma(C_0(T)), \quad t \longmapsto \tilde{t}$$

is also continuous. By Urysohn's Lemma, this map is injective.

Take $x' \in \sigma(C_0(T))$. By Example 1.2.2.10, there is a bounded Radon measure μ on T such that

$$x'(x) = \int x \, d\mu$$

for every $x \in C_0(T)$. Since x' is an algebraic homomorphism, it follows that the functions of $C_0(T)$ are constant on the support of μ. Hence $x' = \tilde{t}$ for some $t \in T$ and thus f is surjective. If T is compact, this map is a homeomorphism. Assume that T is not compact and let $\mathfrak{F}$ be the filter on T converging to the Alexandroff point of T. Then $x(\mathfrak{F})$ converges to 0 for every $x \in C_0(T)$. Hence the filter $f(\mathfrak{F})$ has no points of adherence in $\sigma(C_0(T))$. It follows that f is a homeomorphism. ■

Example 2.4.3.2 (7) *Let T be a set. Given $t \in T$, define*

$$\tilde{t} : c_0(T) \longrightarrow \mathbb{K}, \quad x \longmapsto x(t).$$

Then $\tilde{t} \in \sigma(c_0(T))$ for every $t \in T$ and the map

$$T \longrightarrow \sigma(c_0(T)), \quad t \longmapsto \tilde{t}$$

is a homeomorphism, where we regard T as a discrete space.

The assertion follows from Example 2.4.3.1 by taking T with the discrete opology. ■

Example 2.4.3.3 *Let βT be the Stone–Čech compactification of the comple-ely regular space T and βx the continuous extension of x to βT for every $x \in C(T)$. Given $t \in \beta T$, define*

$$\tilde{t} : C(T) \longrightarrow \mathbb{K}, \quad x \longmapsto \beta x(t).$$

Then $\tilde{t}$ is a character of $C(T)$ whenever $t \in \beta T$ and the map

$$\beta T \longrightarrow \sigma(C(T)), \quad t \longmapsto \tilde{t}$$

is a homeomorphism.

In fact, the map

$$C(T) \longrightarrow C(\beta T), \quad x \longmapsto \beta x$$

's an isometry of unital Banach algebras and the assertion follows from Example 2.4.3.1. ■

Example 2.4.3.4 *Let T be a set, βT the Stone–Čech compactification of T viewed as a discrete space. For $x \in \ell^\infty(T)$, let βx be the continuous extension of x to βT. Given $t \in \beta T$, define*

$$\tilde{t} : \ell^\infty(T) \longrightarrow \mathbb{K}, \quad x \longmapsto \beta x(t).$$

Then each $\tilde{t}$ is a character of $\ell^\infty(T)$ and the map

$$\beta T \longrightarrow \sigma(C(T)), \quad t \longmapsto \tilde{t}$$

is a homeomorphism.

The assertion follows immediately from Example 2.4.3.3, by endowing T with the discrete topology. ■

Example 2.4.3.5 (7) *Let $\widetilde{T}$ be the Alexandroff compactification of the set T viewed as a discrete space. Given $x \in c(T)$, let $\tilde{x}$ denote the continuous extension of x to $\widetilde{T}$. Given $t \in \widetilde{T}$, define*

$$\tilde{t} : c(T) \longrightarrow \mathbb{K}, \quad x \longmapsto \tilde{x}(t).$$

Then each $\tilde{t}$ is a character of $c(T)$ and the map

$$\widetilde{T} \longrightarrow \sigma(c(T)), \quad t \longmapsto \tilde{t}$$

is a homeomorphism.

In fact, the map

$$c(T) \longrightarrow C(\widetilde{T}), \quad x \longmapsto \widetilde{x}$$

is an isometry of unital Banach algebras and the assertion follows from Example 2.4.3.1. ■

Proposition 2.4.3.6 (0) *Let S, T be nonempty compact spaces and*

$$u : C(S) \longrightarrow C(T)$$

be a unital algebra homomorphism.

a) *There is a unique map $\varphi : T \to S$ such that*

$$ux = x \circ \varphi$$

for every $x \in C(S)$, and this map φ is continuous.

b) *u is continuous and $\|u\| = 1$.*

c) *φ is injective (surjective) iff u is surjective (injective).*

d) *If u is bijective, then φ is a homeomorphism.*

a) Let φ_1 (resp. φ_2) denote the homeomorphism

$$S \longrightarrow \sigma(C(S)), \quad s \longmapsto \widetilde{s}$$

$$(\text{resp. } T \longrightarrow \sigma(C(T)), \quad t \longmapsto \widetilde{t})$$

defined in Example 2.4.3.1. Define (Proposition 2.4.1.17 a))

$$\psi : \sigma(C(T)) \longrightarrow \sigma(C(S)), \quad y' \longmapsto y' \circ u,$$

$$\varphi := \varphi_1^{-1} \circ \psi \circ \varphi_2 .$$

By Proposition 2.4.1.17 c), φ is continuous. Take $x \in C(S)$. Then

$$x \circ \varphi(t) = x \circ \varphi_1^{-1} \circ \psi \circ \varphi_2(t) = x \circ \varphi_1^{-1} \circ \psi(\widetilde{t}) =$$

$$= x \circ \varphi_1^{-1}(\widetilde{t} \circ u) = \widetilde{t} \circ u(x) = \widetilde{t}(ux) = (ux)(t)$$

for every $t \in T$, so

$$u \circ \varphi = ux\,.$$

Take $\varphi' : T \to S$ such that

$$x \circ \varphi' = ux$$

for every $x \in C(S)$. Assume that $\varphi(t) \neq \varphi'(t)$ for some $t \in T$. By Urysohn's Theorem, there is an $x \in C(S)$ such that

$$x(\varphi(t)) \neq x(\varphi(t'))\,.$$

It follows that

$$(ux)(t) = x(\varphi(t)) \neq x(\varphi'(t)) = (ux)(t)\,,$$

which is a contradiction. Hence $\varphi = \varphi'$, which proves the uniqueness of φ.

b) follows from a).

c) If u is surjective, then ψ (defined in the proof of a)) is injective (Proposition 2.4.1.17 d)) and so φ is also injective. Conversely, if φ is injective, then by Tietze's Theorem, u is surjective. If φ is surjective, then u is obviously injective. If φ is not surjective, then $S\backslash\varphi(T)$ is nonempty. Since $\varphi(T)$ is compact, there is an $x \in C(S)\backslash\{0\}$ vanishing on $\varphi(T)$ (Urysohn's Theorem). Hence

$$ux = x \circ \varphi = 0\,,$$

and so u is not injective.

d) follows from c). ■

Example 2.4.3.7 *We assume* $\mathbb{K} = \mathbb{C}$. *Let* K *be a compact nonempty set of* $\mathbb{C}$. *Put*

$$\mathcal{A} := \{x \in C(K) \mid x|\overset{\circ}{K} \text{ is analytic}\}$$

and

$$\mathcal{F} := \{x \in \mathcal{A} \mid \exists y \text{ rational function on } \mathbb{C},\ y|\overset{\circ}{K} = x\}\,.$$

Assume that $\mathcal{F}$ *is dense in* $\mathcal{A}$ *and define*

$$\tilde{t} : \mathcal{A} \longrightarrow \mathbb{C}, \quad x \longmapsto x(t)$$

for $t \in K$.

a) *$\mathcal{A}$ is a closed unital subalgebra of $C(K)$.*

b) *$\tilde{t} \in \sigma(\mathcal{A})$ for every $t \in K$.*

c) *The map*

$$K \longrightarrow \sigma(\mathcal{A}), \quad t \longmapsto \tilde{t}$$

is a homeomorphism.

d) *$\mathcal{A}$ is semi-simple.*

e) *Assume that $\overline{\alpha} \in K$ whenever $\alpha \in K$ and put*

$$x^* : K \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \overline{x(\overline{\alpha})}$$

for every $x \in \mathcal{A}$. Then $\mathcal{A}$ is an involutive commutative unital Banach algebra with respect to the involution

$$\mathcal{A} \longrightarrow \mathcal{A}, \quad x \longmapsto x^*.$$

But $\mathcal{A}$ is not symmetric.

f) *If $(x_\iota)_{\iota \in I}$ is a finite family in $\mathcal{A}$ such that for every $\alpha \in K$, there is a $\iota \in I$ with $x_\iota(\alpha) \neq 0$, then there is a $(y_\iota)_{\iota \in I} \in \mathcal{A}^I$ with*

$$\sum_{\iota \in I} x_\iota y_\iota = e_K.$$

a) and b) are trivial.

c) Define

$$x : K \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \alpha.$$

Then

$$\sigma(x) = K.$$

By Corollary 2.4.1.12, the map

$$\sigma(\mathcal{A}) \longrightarrow K, \quad x' \longmapsto x'(x)$$

is a homeomorphism. It is obviously the inverse of the map

$$K \longrightarrow \sigma(\mathcal{A}), \quad t \longmapsto \tilde{t}.$$

d) Take $x \in \mathcal{A}$. By c),

$$\hat{x}(\sigma(\mathcal{A})) = x(K),$$

so that

$$\hat{x} = 0 \Longrightarrow x = 0.$$

By Corollary 2.4.1.14, $\mathcal{A}$ is semi-simple.

e) is easy to see.

f) We set

$$\mathcal{F} := \left\{ \sum_{\iota \in I} x_\iota y_\iota \;\middle|\; (y_\iota)_{\iota \in I} \in \mathcal{A}^I \right\}.$$

Then $\mathcal{F}$ is an ideal of $\mathcal{A}$. If $\mathcal{F} \neq \mathcal{A}$ then $\mathcal{F}$ is contained in a maximal proper ideal of $\mathcal{A}$. By c) and Proposition 2.4.1.5, there is an $\alpha \in K$ such that $x(\alpha) = 0$ for every $x \in \mathcal{F}$, and this is a contradiction. Hence there is a $(y_\iota)_{\iota \in I} \in \mathcal{A}^I$ with

$$\sum_{\iota \in I} x_\iota y_\iota = e_K. \qquad \blacksquare$$

Remark. Many useful conditions on K are known which ensure that $\mathcal{F}$ is dense in $\mathcal{A}$ (e.g. if $\mathbb{C}\backslash K$ has only a finite number of connected components). The Theorem of Vitushkin (1966) gives necessary and sufficient conditions for $\mathcal{F}$ to be dense in $\mathcal{A}$.

Example 2.4.3.8 *Let E be an involutive Banach algebra.*

a) *If we replace the multiplication in E by the multiplication*

$$x * y := -xy$$

for all $x, y \in E$, then we get an involutive Banach algebra, which we denote by $\widetilde{E}$.

b) *If E is unital, then -1 is the unit of $\widetilde{E}$.*

c) *If x is an invertible element of E, then x is an invertible element of $\widetilde{E}$ and x^{-1} is its inverse in $\widetilde{E}$.*

d) $x \in E \Rightarrow \sigma_{\widetilde{E}}(x) = -\sigma_E(x)$.

e) $\tau(\widetilde{E}) = -\tau(E)$.

f) *If u is an algebra homomorphism defined on E, then $-u$ is an algebra homomorphism defined on $\widetilde{E}$.*

g) *If $(x_1, y_1), (x_2, y_2) \in \overset{\circ}{\widetilde{E}}$, then*

$$(x_1, y_1) * (x_2, y_2) = -(x_1, y_1)(x_2, y_2).$$

h) *If E is a Gelfand algebra then $\widetilde{E}$ is also a Gelfand algebra and*

$$\sigma(\widetilde{E}) = -\sigma(E).$$

a) For $x, y, z \in E$,

$$(x * y) * z = -(x * y)z = (xy)z = x * (y * z),$$

$$(x * y)^* = (-xy)^* = -y^*x^* = y^* * x^*.$$

b) For $x \in E$,

$$(-1) * x = 1x = x, \quad x * (-1) = x1 = x.$$

c) We have

$$x * x^{-1} = -xx^{-1} = -1, \quad x^{-1} * x = -x^{-1}x = -1.$$

d) By b) and c), for $\alpha \in \mathbb{K}$,

$$\alpha \notin -\sigma_E(x) \iff -\alpha \notin \sigma_E(x) \iff -\alpha 1 - x \text{ is invertible in } E \iff$$

$$\iff \alpha(-1) - x \text{ is invertible in } \widetilde{E} \iff \alpha \notin \sigma_{\widetilde{E}}(x).$$

e) For $x \in E$, and $x' \in E'$,

$$\langle x^* * x, -x' \rangle = \langle x^* x, x' \rangle.$$

f) Put

$$v := -u.$$

For $x, y \in E$,

$$v(x * y) = -u(-xy) = u(xy) = (ux)(uy) = (vx)(vy).$$

g) We have

$$(x_1, y_1) * (x_2, y_2) = (x_1 * x_2 - y_1 * y_2, x_1 * y_2 + y_1 * x_2) =$$

$$= -(x_1x_2 - y_1y_2, x_1y_2 + y_1x_2) = -(x_1, y_1)(x_2, y_2).$$

h) $\widetilde{E}$ is obviously commutative. By d) and g),

$$\sigma_{\overset{\circ}{\widetilde{E}}}((x, 0)) = -\sigma_{\overset{\circ}{E}}((x, 0)) = -\sigma_E(x) = \sigma_{\widetilde{E}}(x).$$

The last assertion follows from f). ∎

2.4.4 Locally Compact Additive Groups

In this subsection we assume $\mathbb{K} = \mathbb{C}$.

One important application of the Gelfand transform is in the study of locally compact additive groups by means of the associated involutive Banach algebras L^1 (Proposition 2.3.2.29 a)). We do not pursue this theory in this book, as interesting as it is. For us, it serves only as an illuminating example.

Definition 2.4.4.1 (4) *Let* $\mathbf{T}$ *denote the subgroup* $\{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$ *of the multiplicative group* $\mathbb{C}\backslash\{0\}$ *endowed with the induced topology. Given a locally compact additive group* T, *let* $\sigma(T)$ *denote the group of continuous group homomorphisms* $T \to \mathbf{T}$ *endowed with the topology of compact convergence (i.e. the topology of uniform convergence on the compact sets of* T*). Let* λ *be the Haar measure of* T.

Given $p \in [1, \infty]$, *define*

$$L^p(T) := L^p(\lambda)$$

and identify $L^1(T)$ *with the closed involutive subalgebra*

$$\{x \cdot \lambda \mid x \in L^1(T)\}$$

of $\mathcal{M}_b(T)$ *(Theorem 2.3.2.29 a)).*

We have

$$u^{-1} = \overline{u}$$

for every $u \in \sigma(T)$.

If the topology of T is discrete then

$$\ell^1(T) = L^1(T).$$

Proposition 2.4.4.2 (4) *Let* T *be a locally compact additive group. Given* $u \in \sigma(T)$, *define*

$$\widetilde{u} : \mathcal{M}_b(T) \longrightarrow \mathbb{C}, \quad \mu \longmapsto \int \overline{u} d\mu.$$

Then $\widetilde{u} \in \sigma(\mathcal{M}_b(T))$ *for* $u \in \sigma(T)$ *and the map*

$$\sigma(T) \longrightarrow \sigma(\mathcal{M}_b(T)), \quad u \longmapsto \widetilde{u}$$

is continuous.

Take $u \in \sigma(T)$. Then

$$\widetilde{u}(\mu * \nu) = \int \overline{u} d(\mu * \nu) = \int \overline{u}(s+t) d(\mu \otimes \nu)(s,t) = \int \overline{u}(s)\overline{u}(t) d(\mu \otimes \nu)(s,t) =$$

$$= \left(\int \overline{u}(s) d\mu(s)\right)\left(\int \overline{u}(t) d\mu(t)\right) = \widetilde{u}(\mu)\widetilde{u}(\nu)$$

for every $\mu, \nu \in \mathcal{M}_b(T)$ (Theorem 2.2.2.10 f)), so that $\widetilde{u} \in \sigma(\mathcal{M}_b(T))$.

Now take $u_0 \in \sigma(T)$. Let $\mathcal{N}$ be a finite subset of $\mathcal{M}_b(T)$ and take $\varepsilon > 0$. There is a compact set K of T such that

$$|\mu|(T \backslash K) < \frac{\varepsilon}{3}$$

for every $\mu \in \mathcal{N}$. Take $u \in \sigma(T)$ with

$$|u - u_0| < \frac{\varepsilon}{3(1 + \sup_{\nu \in \mathcal{N}} |\nu|(K))}$$

on K. Then

$$|\widetilde{u}(\mu) - \widetilde{u}_0(\mu)| = \left|\int \overline{u} d\mu - \int \overline{u}_0 d\mu\right| \leq$$

$$\leq \int_K |u - u_0| d|\mu| + \int_{T \backslash K} |u| d|\mu| + \int_{T \backslash K} |u_0| d|\mu| <$$

$$< \frac{\varepsilon}{3(1 + \sup_{\nu \in \mathcal{N}} |\nu|(K))} |\mu|(K) + 2|\mu|(T \backslash K) < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

for every $\mu \in \mathcal{N}$. Hence the map

$$\sigma(T) \longrightarrow \sigma(\mathcal{M}_b(T)), \quad u \longmapsto \widetilde{u}$$

is continuous. ■

Proposition 2.4.4.3. (4) *Let T be a locally compact additive group, λ its Haar measure, and take $x \in L^1(\lambda)$. Given $t \in T$, define*

$$x_t : T \longrightarrow \mathbb{C}, \quad s \longmapsto x(s-t).$$

Then, for every $\varepsilon > 0$, there is a 0-neighbourhood U in T such that

$$\left|\int u x_s d\lambda - \int u x_t d\lambda\right| < \varepsilon$$

for every $u \in \sigma(T)$ and $s, t \in T$ with $s - t \in U$.

By Lemma 2.2.2.12, there is a 0-neighbourhood U in T such that

$$\|x_t - x\|_1 < \varepsilon$$

for every $t \in U$. Then

$$\left|\int ux_s d\lambda - \int ux_t d\lambda\right| = \left|\int u(x_s - x_t)d\lambda)\right| \leq \int |u|\,|x_s - x_t| d\lambda =$$

$$= \int |x_{s-t} - x| d\lambda = \|x_{s-t} - x\|_1 < \varepsilon$$

for every $u \in \sigma(T)$ and $s, t \in T$ with $s - t \in U$. ■

Theorem 2.4.4.4 (4) (5) *Let T be a locally compact additive group and let λ be its Haar measure. Given $u \in \sigma(T)$, define*

$$\tilde{u} : L^1(T) \longrightarrow \mathbb{C}, \quad x \longmapsto \int \bar{u} x d\lambda .$$

Then $\tilde{u} \in \sigma(L^1(T))$ for $u \in \sigma(T)$ and the map

$$\sigma(T) \longrightarrow \sigma(L^1(T)), \quad u \longmapsto \tilde{u}$$

is a homeomorphism.

By Proposition 2.4.4.2, $\tilde{u} \in \sigma(L^1(T))$ for every $u \in \sigma(T)$ and the map

$$\sigma(T) \longrightarrow \sigma(L^1(T)), \quad u \longmapsto \tilde{u}$$

is continuous. Given $x \in L^1(T)$ and $t \in T$, define

$$x_t : T \longrightarrow \mathbb{C}, \quad s \longmapsto x(s - t) .$$

Step 1 The map $\sigma(T) \to \sigma(L^1(T)), \quad u \longmapsto \tilde{u}$ is bijective

Take $x' \in \sigma(L^1(T))$. There is an $x \in L^1(T)$ with $x'(x) \neq 0$. We may even assume that x is continuous with compact carrier, since the set of these functions is dense in $L^1(\lambda)$. Take $y \in L^1(\lambda)$ and $t \in T$. Then

$$(x_t * y)(s) = \int x_t(r) y(s - r) d\lambda(r) = \int x(r - t) y(s - r) d\lambda(r) =$$

$$= \int x(r) y(s - r - t) d\lambda(r) = \int x(r) y_t(s - r) d\lambda(r) = (x * y_t)(s)$$

for every $s \in T$ (Theorem 2.2.2.10 e)), so that

$$x_t * y = x * y_t,$$

$$x'(x_t)x'(y) = x'(x_t * y) = x'(x * y_t) = x'(x)x'(y_t).$$

In particular,

$$x'(x_t)x'(x_{-t}) = x'(x)^2 \neq 0,$$

$$x'(x_t) \neq 0.$$

Hence, by complete induction,

$$\left(\frac{x'(x_t)}{x'(x)}\right)^{n-1} = \frac{x'(x_{nt})}{x'(x_t)}$$

for every $n \in \mathbb{Z}$. Thus $\left(\left(\frac{x'(x_t)}{x'(x)}\right)^n\right)_{n\in\mathbb{Z}}$ is bounded, and so $\frac{x'(x_t)}{x'(x)} \in \mathbf{T}$.

Define

$$u : T \longrightarrow \mathbf{T}, \quad t \longmapsto \frac{\overline{x'(x_t)}}{x'(x)}.$$

Then

$$u(s+t) = \frac{\overline{x'(x_{s+t})}}{\overline{x'(x)}} = \frac{\overline{x'(x_s)}\,\overline{x'(x_t)}}{\overline{x'(x)^2}} = u(s)u(t)$$

for all $s, t \in T$. Hence u is a group homomorphism. Take $t \in T$. Then

$$\lim_{s\to t} \|x_s - x_t\| = \lim_{s\to t} \int |x_s - x_t| d\lambda = 0.$$

Hence

$$u(t) = \frac{\overline{x'(x_t)}}{\overline{x'(x)}} = \lim_{s\to t} \frac{\overline{x'(x_s)}}{\overline{x'(x)}} = \lim_{s\to t} u(s)$$

and u is continuous at t. Hence $u \in \sigma(T)$.

Since $L^\infty(T)$ is isometric to the dual of $L^1(T)$, there is a $z \in L^\infty(T)$, such that

$$x'(y) = \int yz d\lambda$$

whenever $y \in L^1(\lambda)$. Thus

$$\tilde{u}(y) = \int \overline{u} y d\lambda = \int \frac{x'(x_t)}{x'(x)} y(t) d\lambda(t) =$$

$$= \frac{1}{x'(x)} \int \left(\int x_t(s) z(s) d\lambda(s) \right) y(t) d\lambda(t) =$$

$$= \frac{1}{x'(x)} \int \left(\int x(s-t) y(t) d\lambda(t) \right) z(s) d\lambda(s) =$$

$$= \frac{1}{x'(x)} \int \left(\int x(t) y(s-t) d\lambda(t) \right) z(s) d\lambda(s) =$$

$$= \frac{1}{x'(x)} \int (x * y) z d\lambda = \frac{1}{x'(x)} x'(x * y) = \frac{1}{x'(x)} x'(x) x'(y) = x'(y)$$

for every $y \in L^1(T)$ (Theorem 2.2.2.10 e)), so that

$$\tilde{u} = x' .$$

Hence the map

$$\sigma(T) \longrightarrow \sigma(L^1(T)), \quad u \longmapsto \tilde{u}$$

is surjective. It is easy to see that this map is injective, showing that it is bijective.

Step 2 The map $\sigma(L^1(T)) \times T \longrightarrow \mathbb{C}$, $(\tilde{u}, t) \longmapsto u(t)$ is continuous.

Take $(u_0, t_0) \in \sigma(T) \times T$ and let $\mathfrak{F}$ be a filter on $\sigma(T) \times T$ such that

$$\lim_{(u,t),\mathfrak{F}} \tilde{u} = \tilde{u}_0 , \quad \lim_{(u,t),\mathfrak{F}} t = t_0 .$$

Let $x : T \to \mathbb{C}$ be a continuous function with compact carrier such that $\tilde{u}_0(x) \neq 0$. Then

$$\tilde{u}(x_t) = \int \overline{u} x_t d\lambda = \int \overline{u}(s) x(s-t) d\lambda(s) =$$

$$= \int \overline{u}(s+t) x(s) d\lambda(s) = \overline{u(t)} \int \overline{u(s)} x(s) d\lambda(s) = \overline{u(t)} \tilde{u}(x)$$

for every $u \in \sigma(T)$ and $t \in T$. Thus

$$|u(t) - u_0(t_0)| = \left| \frac{\overline{u(t)}\tilde{u}(x)}{\tilde{u}(x)} - \frac{\overline{u_0(t_0)}\tilde{u}_0(x)}{\tilde{u}_0(x)} \right| =$$

$$= \left| \frac{\tilde{u}(x_t)}{\tilde{u}(x)} - \frac{\tilde{u}_0(x_{t_0})}{\tilde{u}_0(x)} \right| \leq \left| \frac{\tilde{u}(x_t) - \tilde{u}(x_{t_0})}{\tilde{u}(x)} \right| + \left| \frac{\tilde{u}(x_{t_0})}{\tilde{u}(x)} - \frac{\tilde{u}_0(x_{t_0})}{\tilde{u}_0(x)} \right|$$

for every $t \in T$ and $u \in \sigma(T)$ with $\tilde{u}(x) \neq 0$. Now

$$\lim_{(u,t),\mathfrak{F}} \left| \frac{\tilde{u}(x_{t_0})}{\tilde{u}(x)} - \frac{\tilde{u}(x_{t_0})}{\tilde{u}_0(x)} \right| = 0$$

and

$$\lim_{(u,t),\mathfrak{F}} \left| \frac{\tilde{u}(x_t) - \tilde{u}_0(x_{t_0})}{\tilde{u}(x)} \right| = 0,$$

by Proposition 2.4.4.3. Hence

$$\lim_{(u,t),\mathfrak{F}} u(t) = u_0(t_0).$$

Step 3 The map $\sigma(T) \longrightarrow \sigma(L^1(t))$, $u \longmapsto \tilde{u}$ is a homeomorphism

Take $u_0 \in \sigma(T)$ and let $\mathfrak{F}$ be an ultrafilter on $\sigma(T)$ such that

$$\lim_{u,\mathfrak{F}} \tilde{u} = \tilde{u}_0.$$

Let K be a compact set of T. For $u \in \sigma(T)$, take $t_u \in K$ such that

$$|u(t_u) - u_0(t_u)| = \sup_{s \in K} |u(s) - u_0(s)|.$$

Define

$$t_0 = \lim_{u,\mathfrak{F}} t_u.$$

By Step 2,

$$\lim_{u,\mathfrak{F}} u(t_u) = u_0(t_0)$$

so that

$$\lim_{u,\mathfrak{F}} |u(t_u) - u_0(t_u)| \leq$$

$$\leq \lim_{u,\mathfrak{F}} |u(t_u) - u_0(t_0)| + \lim_{u,\mathfrak{F}} |u_0(t_0) - u_0(t_u)| = 0,$$

$$\lim_{u,\mathfrak{F}} \sup_{s \in K} |u(s) - u_0(s)| = 0 .$$

Hence $\mathfrak{F}$ converges to u_0 uniformly on K. Since K is arbitrary, $\mathfrak{F}$ converges to u_0. It follows that the map

$$\sigma(L^1(T)) \longrightarrow \sigma(T), \quad \tilde{u} \longmapsto u$$

is continuous.

Assume $\sigma(L^1(T))$ is non-compact and let $\mathfrak{G}$ be an ultrafilter on $\sigma(L^1(T))$ converging to the Alexandroff point of $\sigma(L^1(T))$. Then

$$\lim_{\tilde{u},\mathfrak{F}} \int \tilde{u} x d\lambda = 0$$

for every $x \in L^1(T)$ (Theorem 2.4.1.3 e)). It follows that the image of $\mathfrak{G}$ with respect to the map

$$\sigma(L^1(T)) \longrightarrow \sigma(T), \quad \tilde{u} \longmapsto u$$

does not converge. Thus the map

$$\sigma(T) \longrightarrow \sigma(L^1(T)), \quad u \longmapsto \tilde{u}$$

is a homeomorphism. ■

Corollary 2.4.4.5 (4) *If T is a locally compact additive group, then $\sigma(T)$ is a locally compact abelian group.*

By Theorem 2.4.4.4 and Theorem 2.4.1.3 b), $\sigma(T)$ is locally compact. We need only to show that composition in $\sigma(T)$ is continuous. Let K be a compact set of T and take $\varepsilon > 0$. Take $u, v \in \sigma(T)$ such that

$$|u - e_T| < \frac{\varepsilon}{2}, \quad |v - e_T| < \frac{\varepsilon}{2}$$

on K. Then

$$|uv - e_T| \leq |uv - u| + |u - e_T| \leq |u|\,|v - e_T| + |u - e_T| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

on K. Hence $\sigma(T)$ is a locally compact group. ■

Corollary 2.4.4.6 (4) *If T is an additive discrete group, then $\sigma(T)$ is a compact abelian group, $\sigma(\ell^1(T))$ is compact, and*

$$\sigma(e_t) = \{u(t) \mid u \in \sigma(T)\}$$

for every $t \in T$. *Moreover, if* λ *denotes the Haar measure of* $\sigma(T)$, *then for* $t \in T$

$$\int u(t)d\lambda(u) = \begin{cases} 0 & \text{if } t \neq 0 \\ \lambda(\sigma(T)) & \text{if } t = 0. \end{cases}$$

Since $\ell^1(T)$ is a unital Banach algebra, $\sigma(\ell^1(T))$ is compact (Theorem 2.4.1.3 c)). Since $L^1(T) = \ell^1(T)$, $\sigma(T)$ is homeomorphic to $\sigma(\ell^1(T))$ (Theorem 2.4.4.4). Hence $\sigma(T)$ is a compact abelian group (Corollary 2.4.4.5). Further

$$\sigma(e_t) = \widehat{e}_t(\sigma(\ell^1(T))) = \{\widehat{e}_t(x') \mid x' \in \sigma(\ell^1(T))\} =$$

$$= \{x'(e_t) \mid x' \in \sigma(\ell^1(T))\} = \left\{\sum_{s \in T} \overline{u(s)} e_t(s) \,\middle|\, u \in \sigma(T)\right\} = \{u(t) \mid u \in \sigma(T)\}$$

(Corollary 2.4.1.7 a), Theorem 2.4.4.4).

Now we prove the last assertion. For $v \in \sigma(T)$,

$$\int u(t)d\lambda(u) = \int (uv)(t)d\lambda(u) = \int u(t)v(t)d\lambda(u) =$$

$$= v(t)\int u(t)d\lambda(u).$$

Hence if

$$\int u(t)d\lambda(u) \neq 0,$$

then

$$v(t) = 1$$

for every $v \in \sigma(T)$. By Proposition 2.3.2.28 g) and Proposition 2.4.2.3 d $\Rightarrow$ a, $\ell^1(T)$ is symmetric. By Corollary 2.4.2.6, the image of the Gelfand transform is dense. It follows that $t = 0$ (Theorem 2.4.4.4). ■

Corollary 2.4.4.7 (4) *Let* T *be a compact additive group and* λ *be its Haar measure. Then*

$$\int u d\lambda = 0$$

for every $u \in \sigma(T)\backslash\{e_T\}$ *and* $\sigma(T)$ *is discrete.*

Since $u \neq e_T$, there is a $t \in T$ with $u(t) \neq 1$. Then

$$\int u d\lambda = \int u(s-t+t) d\lambda(s) = u(t) \int u(s-t) d\lambda(s) = u(t) \int u d\lambda,$$

$$\int u d\lambda = 0.$$

Since

$$\int e_T d\lambda = \lambda(T) \neq 0,$$

it follows that

$$\{e_T\} = \{u \in \sigma(T) \mid \int u d\lambda \neq 0\}.$$

The map

$$\sigma(T) \longrightarrow \mathbb{C}, \quad u \longmapsto \int u d\lambda$$

being continuous, $\{e_T\}$ is an open set of $\sigma(T)$. Hence $\sigma(T)$ is discrete (Corollary 2.4.4.5). ■

Corollary 2.4.4.8 (4) *Let T be a locally compact additive group. The map*

$$\mathcal{M}_b(T) \longrightarrow \mathbb{C}, \quad \mu \longmapsto \int \overline{u} d\mu$$

is involutive whenever $u \in \sigma(T)$. Furthermore, $L^1(T)$ is symmetric (Proposition 2.3.2.29 a)), the Gelfand transform of $L^1(T)$ is injective and involutive, and its image is dense in $C_0(\sigma(L^1(T)))$.

Given $\mu \in \mathcal{M}_b(T)$,

$$\int \overline{u} d\mu^* = \int \overline{u}(-t) d\overline{\mu}(t) = \int u(t) d\overline{\mu}(t) = \overline{\int \overline{u} d\mu}.$$

Hence the map

$$\mathcal{M}_b(T) \longrightarrow \mathbb{C}, \quad \mu \longmapsto \int \overline{u} d\mu$$

is involutive. By Theorem 2.4.4.4, every character of $L^1(T)$ is involutive and $L^1(T)$ is therefore symmetric, the Gelfand transform of $L^1(T)$ is involutive, and its image is dense in $C_0(\sigma(L^1(T)))$ (Proposition 2.4.2.2 a),b), Proposition 2.4.2.3 d $\Rightarrow$ a & c, and Corollary 2.4.2.6).

By Proposition 2.3.2.29 e), $L^1(T)$ is semi-simple, so that its Gelfand transform is injective (Corollary 2.4.1.14). ■

Proposition 2.4.4.9 (4) *Let S,T be locally compact additive groups. Given $(u,v) \in \sigma(S) \times \sigma(T)$, define*

$$u \otimes v : S \times T \longrightarrow \mathbf{T}, \quad (s,t) \longmapsto u(s)v(t).$$

Then $u \otimes v \in \sigma(S \times T)$, whenever $(u,v) \in \sigma(S) \times \sigma(T)$ and the map

$$\sigma(S) \times \sigma(T) \longrightarrow \sigma(S \times T), \quad (u,v) \longmapsto u \otimes v$$

is a homeomorphism. Given a locally compact additive group G, let φ_G denote the homeomorphism

$$\sigma(G) \longrightarrow \sigma(L^1(G))$$

defined in Theorem 2.4.4.4. Then the map

$$\sigma(L^1(S)) \times \sigma(L^1(T)) \longrightarrow \sigma(L^1(S \times T)),$$

$$(x',y') \longmapsto \varphi_{S\times T}\Big(\varphi_S^{-1}(x') \otimes \varphi_T^{-1}(y')\Big)$$

is a homeomorphism.

Take $(u,v) \in \sigma(S) \times \sigma(T)$. Then

$$(u \otimes v)((s',t') + (s'',t'')) = (u \otimes v)(s' + s'', t' + t'') =$$

$$= u(s' + s'')v(t' + t'') = u(s')u(s'')v(t')v(t'') =$$

$$= ((u \otimes v)(s',t'))(u \otimes v)(s'',t''))$$

for every $(s',t'),(s'',t'') \in S \times T$. Hence $u \otimes v \in \sigma(S \times T)$.

Take $w \in \sigma(S \times T)$ and set

$$u := w(\cdot,0), \quad v := w(0,\cdot).$$

Then $u \in \sigma(S)$, $v \in \sigma(T)$ and

$$u \otimes v = w.$$

Hence the map

$$\sigma(S) \times \sigma(T) \longrightarrow \sigma(S \times T), \quad (u,v) \longmapsto u \otimes v$$

is surjective. It is easy to see that this map is injective and that it is a homeomorphism.

The last assertion follows from Theorem 2.4.4.4 ∎

Proposition 2.4.4.10 *Let $(T_\iota)_{\iota\in I}$ be a family of additive groups and*

$$T := \{x \in \prod_{\iota\in I} T_\iota \mid \{\iota \in I \mid x_\iota \neq 0\} \text{ is finite}\}.$$

Endow the additive groups T_ι $(\iota \in I)$ and T with the discrete topology and for each $x' \in \prod_{\iota\in I} \sigma(T_\iota)$ define

$$\widetilde{x'} : T \longrightarrow \mathbf{T}, \quad x \longmapsto \prod_{\iota\in I} x'_\iota(x_\iota) := \prod_{\substack{\iota\in I \\ x_\iota\neq 0}} x'_\iota(x_\iota).$$

Then $\widetilde{x'} \in \sigma(T)$ for every $x' \in \prod_{\iota\in I} \sigma(T_\iota)$ and the map

$$\varphi : \prod_{\iota\in I} \sigma(T_\iota) \longrightarrow \sigma(T), \quad x' \longmapsto \widetilde{x'}$$

is a homeomorphism.

It is easy to see that $\widetilde{x'} \in \sigma(T)$ for every $x' \in \prod_{\iota\in I} \sigma(T_\iota)$ and that φ is injective. Let $y' \in \sigma(T)$. We denote by u_ι the canonical embedding of T_ι in T and set

$$x'_\iota := y' \circ u_\iota$$

for every $\iota \in I$. Then $x'_\iota \in \sigma(T_\iota)$ for every $\iota \in I$ and $\widetilde{x'} = y'$, where $x' := (x'_\iota)_{\iota\in I}$. Hence φ is surjective. It is easy to see that φ^{-1} is continuous, so φ is a homeomorphism (Theorem 2.4.1.3 c)). ■

Proposition 2.4.4.11 *Let T be a set. We put*

$$\mathfrak{T} := \mathfrak{P}_f(T),$$

and

$$\widetilde{x'} : \mathfrak{T} \longrightarrow \{-1, +1\}, \quad A \longmapsto \prod_{t\in A} x'(t)$$

for every $x' \in \{-1, +1\}^T$.

a) *$\mathfrak{T}$ endowed with the composition law*

$$\mathfrak{T} \times \mathfrak{T} \longrightarrow \mathfrak{T}, \quad (A, B) \longmapsto A \triangle B$$

is a commutative group with $\emptyset$ as neutral element.

b) *If we set*

$$\mathbb{Z}_2^{(T)} := \{x \in \mathbb{Z}_2^T \mid \{t \in T \mid x(t) \neq 0\} \text{ is finite}\}$$

and

$$\tilde{A} : T \longrightarrow \mathbb{Z}_2, \quad t \longmapsto \begin{cases} 1 & \text{if } t \in A \\ 0 & \text{if } t \notin A, \end{cases}$$

then $\tilde{A} \in \mathbb{Z}_2^{(T)}$ *for every* $A \in \mathfrak{T}$ *and the map*

$$\mathfrak{T} \longrightarrow \mathbb{Z}_2^{(T)}, \quad A \longmapsto \tilde{A}$$

is an isomorphism of groups.

c) $\tilde{x'} \in \sigma(\mathfrak{T})$ *for every* $x' \in \{-1, +1\}^T$ *and the map*

$$\varphi : \{-1, +1\}^T \longrightarrow \sigma(\mathfrak{T}), \quad x' \longmapsto \tilde{x'}$$

is a homeomorphism.

d) $\sigma(\mathfrak{T})$ *is not a* σ*-Stone space.*

a & b. Let $A, B \in \mathfrak{T}$. We have

$$\widetilde{A \triangle B}(t) = \left\{ \begin{array}{l} 1 \text{ if } t \in A \triangle B \\ 0 \text{ if } t \notin A \triangle B \end{array} \right\} = \tilde{A}(t) + \tilde{B}(t)$$

for all $t \in T$, so that

$$\widetilde{A \triangle B} = \tilde{A} + \tilde{B}.$$

c) We have

$$\tilde{x'}(A \triangle B) = \prod_{t \in A \triangle B} x'(t) = \left(\prod_{t \in A} x'(t) \right) \left(\prod_{t \in B} x'(t) \right) = \tilde{x'}(A) \tilde{x'}(B)$$

for all $A, B \in \mathfrak{T}$, and so $\tilde{x'} \in \sigma(B)$. It is obvious that φ is injective. Let $y' \in \sigma(\mathfrak{T})$. Define

$$x' : T \longrightarrow \{-1, +1\}, \quad t \longmapsto y'(\{t\}).$$

Then

$$\tilde{x'}(\{t\}) = x'(t) = y'(\{t\})$$

for every $t \in T$, so that

$$\widetilde{x'} = y' .$$

Hence φ is surjective. It is easy to see that φ^{-1} is continuous. Thus φ is a homeomorphism (Theorem 2.4.1.3 c)).

d) follows from c) and Example 1.7.2.17. ■

Remark. Proposition 2.4.4.11 is a special case of Proposition 2.4.4.10.

2.4.5 Examples

Example 2.4.5.1 (4) *Take $n \in \mathbb{N}$ and let $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$. Define*

$$A := \{\alpha \in \mathbb{C} \mid \alpha^n = 1\}$$

and

$$\widetilde{\alpha} : \ell^1(\mathbb{Z}_n) \longrightarrow \mathbb{C}, \quad x \longmapsto \sum_{k=0}^{n-1} x(k)\alpha^k$$

for every $\alpha \in A$. Then $\widetilde{\alpha} \in \sigma(\ell^1(\mathbb{Z}_n))$ for every $\alpha \in A$ and the map

$$A \longrightarrow \sigma(\ell^1(\mathbb{Z}_n)), \quad \alpha \longmapsto \widetilde{\alpha}$$

is a homeomorphism (Theorem 2.2.2.7 a)). Identifying A with $\sigma(\ell^1(\mathbb{Z}_n))$ via the above homeomorphism,

$$x(k) = \frac{1}{n} \sum_{j=0}^{n-1} e^{-\frac{2\pi ijk}{n}} \widehat{x}\left(e^{\frac{2\pi ij}{n}}\right)$$

for $x \in \ell^1(\mathbb{Z}_n)$ and $k \in \mathbb{Z}_n$. The Gelfand transform of $\ell^1(\mathbb{Z}_n)$ is bijective but not an isometry if $n \neq 1$ and

$$\sigma(x) = \left\{ \sum_{j=0}^{n-1} x(j)\, e^{-\frac{2\pi jk}{n} i} \,\middle|\, k \in \mathbb{N}_n \right\}$$

for every $x \in \ell^1(\mathbb{Z}_n)$.

Given $\alpha \in A$, define

$$\overset{\vee}{\alpha} : \mathbb{Z}_n \longrightarrow \mathbb{T}, \quad k \longmapsto \overline{\alpha}^k.$$

Then $\overset{\vee}{\alpha} \in \sigma(\mathbb{Z}_n)$ for $\alpha \in A$ and the map

$$A \longrightarrow \sigma(\mathbb{Z}_n), \quad \alpha \longmapsto \overset{\vee}{\alpha}$$

is bijective. Hence we may identify A with $\sigma(\mathbb{Z}_n)$ via the above map. By Theorem 2.4.4.4, the map

$$A \longrightarrow \sigma(\ell^1(\mathbb{Z}_n)), \quad \alpha \longmapsto \widetilde{\alpha}$$

is a homeomorphism.

Take $x \in \ell^1(\mathbb{Z}_n)$ and $k \in \mathbb{Z}_n$. Then

$$\frac{1}{n}\sum_{j=0}^{n-1} e^{-\frac{2\pi ijk}{n}}\,\widehat{x}\left(e^{\frac{2\pi ij}{n}}\right) = \frac{1}{n}\sum_{j=0}^{n-1} e^{-\frac{2\pi ijk}{n}}\left(\sum_{m=0}^{n-1} x(m)e^{\frac{2\pi ijm}{n}}\right) =$$

$$= \frac{1}{n}\sum_{m=0}^{n-1} x(m)\left(\sum_{j=0}^{n-1} e^{\frac{2\pi ij(m-k)}{n}}\right) = x(k)\,.$$

It follows that the Gelfand transform is injective. Since $\ell^1(\mathbb{Z}_n)$ is finite-dimensional, the Gelfand transform is bijective.

Now suppose that $n \neq 1$. We put

$$x := e_0 + e^{i}e_1\,.$$

Then

$$|\widehat{x}(\alpha)| = |1 + e^{i}\alpha| < 2$$

for every $\alpha \in A$, so that

$$\|\widehat{x}\| < 2 = \|x\|_1\,.$$

Hence the Gelfand transform is not an isometry.

Take $x \in \ell^1(\mathbb{Z}_n)$. By Corollary 2.4.1.7 a) and the above,

$$\sigma(x) = \widehat{x}(\sigma(E)) = \{\widehat{x}(\alpha) \mid \alpha \in \sigma(E)\} = \left\{\sum_{j=0}^{n-1} x(j)e^{-\frac{2\pi ijk}{n}} \,\middle|\, k \in \mathbb{N}_n\right\}. \quad \blacksquare$$

Corollary 2.4.5.2 *Given* $p \in \mathbb{N}$,

$$\inf_{n\in\mathbb{N}}\left(\sum_{j=0}^{p-1}\left|\sum_{k=0}^{n}(-1)^{kp}\binom{n}{j+kp}\right|\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\sum_{j=0}^{p-1}\left|\sum_{k=0}^{n}(-1)^{kp}\binom{n}{j+kp}\right|\right)^{\frac{1}{n}} =$$

$$= \begin{cases} 2 & \text{if } p \text{ is even,} \\ 2\sin\frac{p-1}{p}\frac{\pi}{2} & \text{if } p \text{ is odd.} \end{cases}$$

Consider the element $\delta_0 - \delta_1$ of the commutative unital Banach algebra $\ell^1(\mathbb{Z}_p)$. By Example 2.4.5.1,

$$\|\widehat{\delta_0 - \delta_1}\|_\infty = 2\,,$$

if p is even and

$$\|\widehat{\delta_0 - \delta_1}\|_\infty = \left|1 - e^{\frac{p-1}{2p}2\pi i}\right| = \left|e^{-\frac{p-1}{2}\frac{\pi}{2}i} - e^{\frac{p-1}{2}\frac{\pi}{2}i}\right| =$$

$$= 2\sin\frac{p-1}{p}\frac{\pi}{2}$$

if p is odd. Then

$$(\delta_0 - \delta_1)^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\delta_k = \sum_{j=0}^{p-1}\left(\sum_{k=0}^{n}(-1)^{j+kp}\binom{n}{j+kp}\right)\delta_j\,,$$

$$\|(\delta_0 - \delta_1)^n\| = \sum_{j=0}^{p-1}\left|\sum_{k=0}^{n}(-1)^{j+kp}\binom{n}{j+kp}\right|$$

for every $n \in \mathbb{N}$ and the result follows from

$$\inf_{n\in\mathbb{N}}\|(\delta_0 - \delta_1)^n\|^{\frac{1}{n}} = \lim_{n\to\infty}\|(\delta_0 - \delta_1)^n\|^{\frac{1}{n}} = \|\widehat{\delta_0 - \delta_1}\|_\infty$$

(Theorem 2.4.1.13, Proposition 2.2.23.1). ■

Theorem 2.4.5.3 (4) *Let T be a finite additive group.*

a) *There is a unique order relation on $\ell^1(T)$ with respect to which $\ell^1(T)$ is an ordered Banach space with*

$$\ell^1(T)_+ = \{x \in \ell^1(T) \mid x' \in \ell^1(T)'_+ \Longrightarrow x'(x) \in \mathbb{R}_+\}\,.$$

b) *Given $x \in \ell^1(T)$, the following are equivalent:*

b_1) $x \in \ell^1(T)_+$.

b_2) $x = y^* * y$ *for some* $y \in \ell^1(T)$.

b_3) $x = y^2$ *for some* $y \in \ell^1(T)$.

b_4) $\sigma(x) \subset \mathbb{R}_+$.

c) *Identifying the vector space $\ell^1(T)$ canonically with $\ell^1(T)'$,*

$$\ell^1(T)_+ = \ell^1(T)'_+\,.$$

By Example 2.3.2.30 c), $\ell^1(T)$ is symmetric and by Proposition 2.3.2.28 e), $\ell^1(T)$ is semi-simple.

a & b follows from Corollary 2.4.2.8 c),e) and Corollary 2.2.4.27.

c) Identify $\ell^1(T)$ with $\mathbb{C}^{\sigma(\ell^1(T))}$ via the Gelfand transform. By b) and Corollary 2.4.1.7 a), $\ell^1(T)_+$ is then identified with $\mathbb{R}_+^{\sigma(\ell^1(T))}$. Let

$$\ell^1(T)' \longrightarrow \mathbb{C}^{\sigma(\ell^1(T))}, \quad x' \longmapsto \widehat{x'}$$

denote the resulting identification of $\ell^1(T)'$ with $\mathbb{C}^{\sigma(\ell^1(T))}$. Then

$$\langle x, x' \rangle = \sum_{t \in \sigma(\ell^1(T))} \widehat{x}(t)\widehat{x'}(t)$$

for $(x, x') \in \ell^1(T) \times \ell^1(T)'$. Hence $\ell^1(T)'_+$ is identified with $\mathbb{R}_+^{\sigma(\ell^1(T))}$, i.e.

$$\ell^1(T)_+ = \ell^1(T)'_+ . \qquad \blacksquare$$

Example 2.4.5.4

a) *$x \in \ell^1(\mathbb{Z}_2)$ is positive (Theorem 2.4.5.3 a)) iff*

$$x(0) \in \mathbb{R}_+, \quad x(1) \in \mathbb{R}, \quad |x(1)| \leq x(0) .$$

b) *Identifying $\ell^1(\mathbb{Z}_2)'$ canonically with $\ell^\infty(\mathbb{Z}_2)$, the element $x' \in \ell^1(\mathbb{Z}_2)'$ is positive iff*

$$x'(0) \in \mathbb{R}_+, \quad x'(1) \in \mathbb{R}, \qquad |x'(1)| \leq x'(0) .$$

c) *For every $x' \in \operatorname{Re} \ell^1(\mathbb{Z}_2)'$, there exist uniquely determined $y', z' \in \ell^1(\mathbb{Z}_2)'_+$ such that*

$$x' = y' - z', \quad \|x'\| = \|y'\| + \|z'\| .$$

a) By Example 2.2.2.8 b),

$$\sigma(x) = \{x(0) \pm x(1)\}$$

and by Theorem 2.4.5.3 $b_1 \Leftrightarrow b_4$,

$$x \in \ell^1(\mathbb{Z}_2)_+ \Longleftrightarrow \{x(0) \pm x(1)\} \subset \mathbb{R}_+ .$$

b) follows from a) and Theorem 2.4.5.3 c).

c) follows from b). $\blacksquare$

Example 2.4.5.5

a) *The element $x \in \ell^1(\mathbb{Z}_3)$ is positive (Theorem 2.4.5.3 a)) iff*

$$x(0) \in \mathbb{R}_+, \quad x(1) = \overline{x(2)},$$

$$-\frac{1}{2}x(0) \leq \operatorname{re} x(1) \leq \operatorname{re} x(1) + \sqrt{3}|\operatorname{im} x(1)| \leq x(0).$$

b) *Identifying* $\ell^1(\mathbb{Z}_3)'$ *canonically with* $\ell^\infty(\mathbb{Z}_3)$, $x' \in \ell^1(\mathbb{Z}_3)'$ *is positive iff*

$$x'(0) \in \mathbb{R}_+, \quad x'(1) = \overline{x'(2)},$$

$$-\frac{1}{2}x'(0) \leq \operatorname{re} x'(1) \leq \operatorname{re} x'(1) + \sqrt{3}|\operatorname{im} x'(1)| \leq x'(0).$$

a) By Example 2.4.5.1,

$$\sigma(x) = \left\{ \sum_{j=0}^{2} x(j) e^{-\frac{2\pi i j k}{3}} \;\middle|\; k \in \mathbb{N}_3 \right\}$$

and by Theorem 2.4.5.3 $b_1 \Leftrightarrow b_4$,

$$x \in \ell^1(\mathbb{Z}_3)_+ \Longleftrightarrow \sigma(x) \subset \mathbb{R}_+.$$

Hence x is positive iff

$$\begin{cases} x(0) + x(1) + x(2) \in \mathbb{R}_+ \\ x(0) + x(1)e^{-\frac{2\pi i}{3}} + x(2)e^{-\frac{4\pi i}{3}} \in \mathbb{R}_+ \\ x(0) + x(1)e^{-\frac{4\pi i}{3}} + x(2)e^{-\frac{2\pi i}{3}} \in \mathbb{R}_+ \end{cases}$$

and this is equivalent to the condition of a).

b) follows from a) and Theorem 2.4.5.3 c). ∎

Remark. $\ell^1(\mathbb{Z}_3)$ does not have property c) of Example 2.4.5.4. Indeed, let

$$x' : \mathbb{Z}_3 \longrightarrow \mathbb{C}, \quad t \longmapsto \begin{cases} 0 & \text{if } t = 0 \\ i & \text{if } t = 1 \\ -i & \text{if } t = 2. \end{cases}$$

Assume there are $y', z' \in \ell^1(\mathbb{Z}_3)'_+$ with

$$x' = y' - z', \quad \|y'\| + \|z'\| = \|x'\| = 1.$$

Then

$$y'(0) - z'(0) = 0,$$

$$y'(1) - z'(1) = i,$$

so that

$$y'(0) = z'(0) \leq \frac{1}{2},$$

and we get successively

$$\operatorname{re} y'(1) = \operatorname{re} z'(1) \geq -\frac{1}{4},$$

$$-\frac{1}{4} + \sqrt{3} \operatorname{im} y'(1) \leq \operatorname{re} y'(1) + \sqrt{3} \operatorname{im} y'(1) \leq \frac{1}{2},$$

$$\operatorname{im} y'(1) \leq \frac{\sqrt{3}}{4},$$

$$-\frac{1}{2} \leq \operatorname{im} z'(1) = \operatorname{im} y'(1) - 1 \leq \frac{\sqrt{3}}{4} - 1$$

and

$$\frac{1}{2} \leq \frac{\sqrt{3}}{4},$$

which is a contradiction.

Example 2.4.5.6 (4) *Given* $\alpha \in \mathbf{T}$, *define*

$$\widetilde{\alpha} : \ell^1(\mathbb{Z}) \longrightarrow \mathbb{C}, \quad x \longmapsto \sum_{n \in \mathbb{Z}} x_n \alpha^n.$$

Then $\widetilde{\alpha} \in \sigma(\ell^1(\mathbb{Z}))$ *for* $\alpha \in \mathbf{T}$ *and the map*

$$\mathbf{T} \longrightarrow \sigma(\ell^1(\mathbb{Z})), \quad \alpha \longmapsto \widetilde{\alpha}$$

is a homeomorphism. If we identify $\sigma(\ell^1(\mathbb{Z}))$ *with* $\mathbf{T}$ *via this homeomorphism, then*

$$x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \widehat{x}(\theta) d\theta$$

for $x \in \ell^1(\mathbb{Z})$ *and* $n \in \mathbb{Z}$. *The Gelfand transform of* $\ell^1(\mathbb{Z})$ *is not surjective.*

Given $\alpha \in \mathbf{T}$, define

$$\overset{\vee}{\alpha} : \mathbb{Z} \longrightarrow \mathbf{T}, \quad n \longmapsto \overline{\alpha}^n .$$

Then $\overset{\vee}{\alpha} \in \sigma(\mathbb{Z})$ whenever $\alpha \in \mathbf{T}$ and the map

$$\mathbf{T} \longrightarrow \sigma(\mathbb{Z}), \quad \alpha \longmapsto \overset{\vee}{\alpha}$$

is bijective. Since it is continuous and the spaces involved are compact, it is a homeomorphism. By Theorem 2.4.4.4, the map

$$\mathbf{T} \longrightarrow \sigma(\ell^1(\mathbb{Z})), \quad \alpha \longmapsto \widetilde{\alpha}$$

is a homeomorphism. Take $x \in \ell^1(\mathbb{Z})$ and $n \in \mathbb{Z}$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \widehat{x}(\theta) d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-in\theta} \left(\sum_{p \in \mathbb{Z}} x_p e^{ip\theta} \right) d\theta =$$

$$= \sum_{p \in \mathbb{Z}} x_p \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(p-n)\theta} d\theta = x_n .$$

Take $x \in \ell^1(\mathbb{Z})$. Given $p \in \mathbb{N}$, put

$$f_p : \mathbf{T} \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \sum_{n=-p}^{p} x_n \alpha^n .$$

$(f_p)_{p \in \mathbb{N}}$ converges uniformly to $\widehat{x}$. By Example 1.4.1.10, the Gelfand transform is not surjective. ■

Corollary 2.4.5.7 (Wiener, 1932) *Take* $(x_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{Z})$. *If the function*

$$\mathbf{T} \longrightarrow \mathbb{C}, \quad \alpha \longmapsto \sum_{n \in \mathbb{Z}} x_n \alpha^n$$

does not vanish, then there is a $(y_n)_{n \in \mathbb{N}} \in \ell^1(\mathbb{Z})$ *such that*

$$\sum_{n \in \mathbb{Z}} y_n \alpha^n = \left(\sum_{n \in \mathbb{Z}} x_n \alpha^n \right)^{-1}$$

for every $\alpha \in \mathbf{T}$.

By Corollary 2.4.1.6 and Example 2.4.5.6, $(x_n)_{n\in\mathbb{Z}}$ is invertible in $\ell^1(\mathbb{Z})$. Let $(y_n)_{n\in\mathbb{Z}}$ be its inverse. Then

$$\left(\sum_{n\in\mathbb{Z}} x_n\alpha^n\right)\left(\sum_{n\in\mathbb{Z}} y_n\alpha^n\right) = 1$$

for every $\alpha \in T$ (Proposition 2.4.1.2). ■

Remark. The above proof was given by Gelfand in 1941.

Example 2.4.5.8 *Take $n \in \mathbb{N}$. Given $\alpha \in \mathbf{T}^n$, define*

$$\widetilde{\alpha} : \ell^1(\mathbb{Z}^n) \longrightarrow \mathbb{C}, \quad x \longmapsto \sum_{j\in\mathbb{Z}^n} x_j \prod_{k=1}^{n} \alpha_k^{j(k)}.$$

Then $\widetilde{\alpha} \in \sigma(\ell^1(\mathbb{Z}^n))$ for $\alpha \in \mathbf{T}^n$ and the map

$$\mathbf{T}^n \longrightarrow \sigma(\ell^1(\mathbb{Z}^n)), \quad \alpha \longmapsto \widetilde{\alpha}$$

is a homeomorphism. Identifying $\sigma(\ell^1(\mathbb{Z}^n))$ with $\mathbf{T}^n$ via this homeomorphism,

$$x_j = \int \prod_{k=1}^{n} \alpha_k^{-j(k)} \widehat{x}(\alpha) d\lambda(\alpha)$$

for every $x \in \ell^1(\mathbb{Z}^n)$ and $j \in \mathbb{Z}^n$, where λ is the Haar measure of $\mathbf{T}^n$ with $\lambda(\mathbf{T}^n) = 1$. The Gelfand transform of $\ell^1(\mathbf{T}^n)$ is not surjective.

The assertion follows from Example 2.4.5.6 and Proposition 2.4.4.9. ■

Example 2.4.5.9 *We consider the additive group $\mathbb{Q}$ with the discrete topology and define*

$$m \preceq n :\Longleftrightarrow \frac{n}{m} \in \mathbb{N},$$

for $m, n \in \mathbb{N}$ and

$$\varphi_{m,n} : \mathbf{T} \longrightarrow \mathbf{T}, \quad \alpha \longmapsto \alpha^{\frac{n}{m}}$$

for $m, n \in \mathbb{N}$ with $m \preceq n$. $\preceq$ is an upward directed order relation on $\mathbb{N}$ and $(\varphi_{m,n})_{m,n\in\mathbb{N}}$ is a projective system of compact spaces. Let S denote its projective limit and $\varphi_n : S \to \mathbf{T}$ the canonical maps for $n \in \mathbb{N}$.

a) *Given* $s \in S$, $m, p \in \mathbb{Z}$ *and* $n, q \in \mathbb{N}$

$$\frac{m}{n} = \frac{p}{q} \Longrightarrow \varphi_n(s)^m = \varphi_q(s)^p .$$

Given $s \in S$, *define*

$$\widetilde{s} : \mathbb{Q} \longrightarrow \mathbb{T}, \quad \frac{m}{n} \longmapsto \varphi_n(s)^m$$

$$\overset{\approx}{s} : \ell^1(\mathbb{Q}) \longrightarrow \mathbb{C}, \quad x \longmapsto \sum_{t \in \mathbb{Q}} x(t)\overline{\widetilde{s}(t)} .$$

b) $\widetilde{s} \in \sigma(\mathbb{Q})$ *for every* $s \in S$ *and the map*

$$S \longrightarrow \sigma(\mathbb{Q}), \quad s \longmapsto \widetilde{s}$$

is a homeomorphism.

c) $\overset{\approx}{s} \in \sigma(\ell^1(\mathbb{Q}))$ *for every* $s \in S$ *and the map*

$$S \longrightarrow \sigma(\ell^1(\mathbb{Q})), \quad s \longmapsto \overset{\approx}{s}$$

is a homeomorphism.

a) We may assume that $n \preceq q$. Then

$$\varphi_n(s)^m = \varphi_{n,q}(\varphi_q(s))^m = (\varphi_q(s))^{\frac{q}{n}m} = \varphi_q(s)^p .$$

b) Given $m, p \in \mathbb{Z}$ and $n, q \in \mathbb{N}$,

$$\widetilde{s}(\frac{m}{n})\widetilde{s}(\frac{p}{q}) = \widetilde{s}(\frac{mq}{nq})\widetilde{s}(\frac{np}{nq}) = \varphi_{n,q}(s)^{mq}\varphi_{n,q}(s)^{np} =$$

$$= \varphi_{n,q}(s)^{mq+np} = \widetilde{s}(\frac{mq+np}{nq}) = \widetilde{s}(\frac{m}{n} + \frac{p}{q}) .$$

Hence, $\widetilde{s} \in \sigma(\mathbb{Q})$.

Take $u \in \sigma(\mathbb{Q})$. Given $n \in \mathbb{N}$, put

$$s_n := u(\frac{1}{n}) .$$

Then

$$s_m = u(\frac{1}{m}) = u(\frac{n}{m}\frac{1}{n}) = u(\frac{1}{n})^{\frac{n}{m}} = s_n^{\frac{n}{m}} = \varphi_{m,n}(s_n)$$

for $m, n \in \mathbb{N}$ with $m \preceq n$. Hence there is an $s \in S$ such that

$$s_n = \varphi_n(s)$$

for every $n \in \mathbb{N}$. Then

$$\widetilde{s}(\frac{m}{n}) = \varphi_n(s)^m = s_n^m = u(\frac{1}{n})^m = u(\frac{m}{n})$$

for all $(m,n) \in \mathbb{Z} \times \mathbb{N}$. Thus $\widetilde{s} = u$ and the map

$$S \longrightarrow \sigma(\mathbb{Q}), \quad s \longmapsto \widetilde{s}$$

is surjective. This map is also a homeomorphism, since it is obviously injective and continuous and the spaces involved are compact.

c) follows from b) and Theorem 2.4.4.4. ■

Example 2.4.5.10 (4) *Let λ be the Haar measure of $\mathbf{T}$. Define*

$$\widetilde{n} : \mathbf{T} \longrightarrow \mathbf{T}, \quad \alpha \longmapsto \overline{\alpha}^n,$$

$$\widetilde{\widetilde{n}} : L^1(\mathbf{T}) \longrightarrow \mathbb{C}, \quad x \longmapsto \int x(\alpha)\alpha^n d\lambda(\alpha)$$

for $n \in \mathbb{Z}$. Then $\widetilde{n} \in \sigma(\mathbf{T})$, $\widetilde{\widetilde{n}} \in \sigma(L^1(\mathbf{T}))$ for every $n \in \mathbb{Z}$ and the maps

$$\mathbb{Z} \longrightarrow \sigma(\mathbf{T}), \quad n \longmapsto \widetilde{n},$$

$$\mathbb{Z} \longrightarrow \sigma(L^1(\mathbf{T})), \quad n \longmapsto \widetilde{\widetilde{n}}$$

are homeomorphisms. Identifying $\mathbb{Z}$ with $\sigma(L^1(\mathbf{T}))$ via the above homeomorphisms, then for every $x \in L^1(\mathbf{T})$,

$$\widehat{x} : \mathbb{Z} \longrightarrow \mathbb{C}, \quad n \longmapsto \int x(\alpha)\alpha^n d\lambda(\alpha).$$

It is obvious that $\widetilde{n} \in \sigma(\mathbf{T})$ and that the map

$$\mathbb{Z} \longrightarrow \sigma(\mathbf{T}), \quad n \longmapsto \widetilde{n}$$

is injective. Take $u \in \sigma(\mathbf{T})$, $u \neq \widetilde{0}$. There is an $\alpha > 0$ such that

$$\operatorname{re} u(e^{i\beta}) > 0$$

for every $\beta \in [-\alpha, \alpha]$. Take $\gamma \in]-\frac{\pi}{2}, \frac{\pi}{2}[$ with

$$e^{i\gamma} = u(e^{i\alpha}).$$

Take $n \in \mathbb{N}$. Then

$$\operatorname{re} u\left(e^{i\frac{\alpha}{n}}\right)^k = \operatorname{re} u\left(e^{i\frac{\alpha k}{n}}\right) > 0$$

for every $k \in \mathbb{N}_n$ and so

$$u\left(e^{i\frac{\alpha}{n}}\right) = e^{i\frac{\gamma}{n}}.$$

Hence

$$u\left(e^{i\alpha\frac{p}{n}}\right) = u\left(e^{i\frac{\alpha}{n}}\right)^p = \left(e^{i\frac{\gamma}{n}}\right)^p = e^{i\gamma\frac{p}{n}}$$

for every $p \in \mathbb{Z}$. By continuity,

$$u(e^{i\alpha\beta}) = e^{i\gamma\beta}$$

for every $\beta \in \mathbb{R}$. In particular,

$$e^{i\gamma\frac{2\pi}{\alpha}} = u\left(e^{i\alpha\frac{2\pi}{\alpha}}\right) = u(e^{2\pi i}) = u(1) = 1,$$

$$n := -\frac{\gamma}{\alpha} \in \mathbb{Z},$$

and

$$\widetilde{n}(e^{i\beta}) = e^{-i\beta n} = e^{i\beta\frac{\gamma}{\alpha}} = u\left(e^{i\alpha\frac{\beta}{\alpha}}\right) = u(e^{i\beta})$$

for every $\beta \in \mathbb{R}$. Hence

$$\widetilde{n} = u$$

and the map

$$\mathbb{Z} \longrightarrow \sigma(\mathbf{T}), \quad n \longmapsto \widetilde{n}$$

is surjective. By the above considerations and by Theorem 2.4.4.4, the maps

$$\mathbb{Z} \longrightarrow \sigma(\mathbf{T}), \quad n \longmapsto \widetilde{n},$$

$$\mathbb{Z} \longrightarrow \sigma(L^1(\mathbf{T})), \quad n \longmapsto \widetilde{\widetilde{n}},$$

are bijective. They obviously are continuous. Moreover, for every sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{Z}$ with

$$\lim_{n\to\infty} |k_n| = \infty$$

the sequences $(\widetilde{k_n})_{n\in\mathbb{N}}$, $(\widetilde{\widetilde{k_n}})_{n\in\mathbb{N}}$ have no points of adherence and so the above maps are homeomorphism.

Turning to the last assertion, let $x \in L^1(\mathbf{T})$. Then

$$\widehat{x}(n) = \widehat{x}(\widetilde{n}) = \int x(\alpha)\alpha^n d\lambda(\alpha)$$

for every $n \in \mathbb{Z}$. ■

Example 2.4.5.11 *Take $n \in \mathbb{N}$ and let λ be the Haar measure of $\mathbf{T}^n$. Define*

$$\widetilde{k} : \mathbf{T}^n \longrightarrow \mathbf{T}, \quad (\alpha_j)_{j\in\mathbb{N}_n} \longmapsto \prod_{j=1}^{n} \overline{\alpha}_j^{k_j}$$

$$\widetilde{\widetilde{k}} : L^1(\mathbf{T}^n) \longrightarrow \mathbb{C}, \quad x \longmapsto \int \widetilde{k} x d\lambda$$

for $k = (k_j)_{j\in\mathbb{N}_n} \in \mathbb{Z}^n$. Then $\widetilde{k} \in \sigma(\mathbf{T}^n)$, $\widetilde{\widetilde{k}} \in \sigma(L^1(\mathbf{T}^n))$ for every $k \in \mathbb{Z}^n$ and the maps

$$\mathbb{Z}^n \longrightarrow \sigma(\mathbf{T}^n), \quad k \longmapsto \widetilde{k},$$

$$\mathbb{Z}^n \longrightarrow \sigma(L^1(\mathbf{T}^n)), \quad k \longmapsto \widetilde{\widetilde{k}}$$

are homeomorphisms.

The assertions follow from Example 2.4.5.10 and Proposition 2.4.4.9. ■

2.4.6 The Fourier Transform

In this subsection n is a fixed natural number, λ always denotes the Haar measure on $\mathbb{R}^n$, normalized by $\lambda([0,1]^n) = (2\pi)^{-\frac{n}{2}}$, and $\langle\cdot|\cdot\rangle$ denotes the inner product

$$\langle\cdot|\cdot\rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}, \quad (s,t) \longmapsto \langle s|t\rangle := \sum_{k=1}^{n} s_k t_k .$$

Moreover, we take $\mathbb{K} = \mathbb{C}$.

Proposition 2.4.6.1 (4) *Take $n = 1$. Define*

$$\widetilde{s} : \mathbb{R} \longrightarrow \mathbf{T}, \quad t \longmapsto e^{ist},$$

$$\widetilde{\widetilde{s}} : L^1(\mathbb{R}) \longrightarrow \mathbb{C}, \quad x \longmapsto \int \widetilde{s} x d\lambda$$

for every $s \in \mathbb{R}$. Then $\widetilde{s} \in \sigma(\mathbb{R})$, $\widetilde{\widetilde{s}} \in \sigma(L^1(\mathbb{R}))$ for every $s \in \mathbb{R}$ and the maps

$$\mathbb{R} \longrightarrow \sigma(\mathbb{R}), \quad s \longmapsto \widetilde{s},$$

$$\mathbb{R} \longrightarrow \sigma(L^1(\mathbb{R})), \quad s \longmapsto \widetilde{\widetilde{s}},$$

are homeomorphisms.

It is obvious that $\widetilde{s} \in \sigma(\mathbb{R})$ for every $s \in \mathbb{R}$ and that the map

$$\mathbb{R} \longrightarrow \sigma(\mathbb{R}), \quad s \longmapsto \widetilde{s}$$

is injective and continuous. Take $u \in \sigma(\mathbb{R})$. Then there is an $\varepsilon > 0$ such that

$$\mathrm{re}\, u([-\varepsilon, \varepsilon]) \subset \mathbb{R}_+ .$$

Take $s' \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ with

$$u(\varepsilon) = e^{is'} .$$

Define

$$s := \frac{1}{\varepsilon} s' .$$

Then for $p \in \mathbb{N}$ and $k \in \mathbb{N}_p$,

$$\operatorname{re} u(\frac{\varepsilon}{p})^k = \operatorname{re} u(\frac{\varepsilon k}{p}) \geq 0 ,$$

so that

$$u(\frac{\varepsilon}{p}) = e^{i\frac{s'}{p}} = e^{i\frac{\varepsilon}{p}s} = \widetilde{s}(\frac{\varepsilon}{p}) .$$

Hence

$$u(\varepsilon\frac{m}{p}) = u(\frac{\varepsilon}{p})^m = \widetilde{s}(\frac{\varepsilon}{p})^m = \widetilde{s}(\varepsilon\frac{m}{p})$$

for every $(m,p) \in \mathbb{Z} \times \mathbb{N}$. Thus, by continuity,

$$u = \widetilde{s} ,$$

so that the map

$$\mathbb{R} \longrightarrow \sigma(\mathbb{R}) , \quad s \longmapsto \widetilde{s}$$

is bijective.

Let $(s_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with

$$\lim_{k\to\infty} |s_k| = \infty .$$

Then $(\widetilde{s}_k)_{k\in\mathbb{N}}$ has no points of adherence. It follows that the map

$$\mathbb{R} \longrightarrow \sigma(\mathbb{R}) , \quad s \longmapsto \widetilde{s}$$

is a homeomorphism. By Theorem 2.4.4.4, the map

$$\mathbb{R} \longrightarrow \sigma(L^1(\mathbb{R})) , \quad s \longmapsto \widetilde{\widetilde{s}}$$

is also a homeomorphism. ■

Theorem 2.4.6.2 (4) *Given* $s \in \mathbb{R}^n$, *define*

$$\widetilde{s} : \mathbb{R}^n \longrightarrow \mathbf{T} , \quad t \longmapsto e^{i\langle s|t\rangle} ,$$

$$\widetilde{\widetilde{s}} : L^1(\mathbb{R}^n) \longrightarrow \mathbb{C} , \quad x \longmapsto \int \widetilde{s} x d\lambda .$$

Given $x \in L^1(\lambda)$, *define*

$$\widehat{x} : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad s \longmapsto \widetilde{\widetilde{s}}(x) = \int e^{-\imath\langle s|t\rangle} x(t) d\lambda(t).$$

Then $\widetilde{s} \in \sigma(\mathbb{R}^n)$, $\widetilde{\widetilde{s}} \in \sigma(L^1(\mathbb{R}^n))$ *for every* $s \in \mathbb{R}^n$ *and the maps*

$$\mathbb{R}^n \longrightarrow \sigma(\mathbb{R}^n), \quad s \longmapsto \widetilde{s},$$

$$\mathbb{R}^n \longrightarrow \sigma(L^1(\mathbb{R}^n)), \quad s \longmapsto \widetilde{\widetilde{s}}$$

are homeomorphisms. We identify $\sigma(L^1(\mathbb{R}^n))$ *with* $\mathbb{R}^n$ *via the last homeomorphism. Then, for* $x \in L^1(\mathbb{R}^n)$, $\widehat{x}$ *is the Gelfand transform of* x. *The integral*

$$\int e^{-\imath\langle s|t\rangle} x(t) d\lambda(t)$$

is called the ***Fourier integral of*** x *(Fourier, 1822) and the map*

$$L^1(\mathbb{R}^n) \longrightarrow C_0(\mathbb{R}^n), \quad x \longmapsto \widehat{x}$$

is called the ***Fourier transform.*** *It is an injective, involutive, continuous algebra homomorphism of norm* 1 *and its image is dense in* $C_0(\mathbb{R}^n)$.

The first assertion follows from Proposition 2.4.6.1 and Proposition 2.4.4.9. By Proposition 2.4.4.8, the Fourier transform is an injective, involutive, continuous algebra homomorphism of norm at most 1 and its image is dense in $C_0(\mathbb{R}^n)$. Now

$$\|\widehat{x}\| \geq |\widehat{x}(0)| = \left|\int x d\lambda\right| = \|x\|_1$$

for every positive real function $x \in L^1(\lambda)$, and so the norm of the Fourier transform is 1. ■

Proposition 2.4.6.3 (4) *Take* $x \in L^1(\mathbb{R}^n)$, $a \in \mathbb{R}^n$, *and* $\alpha > 0$. *Define*

$$y : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad t \longmapsto x(t + a),$$

$$z : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad t \longmapsto x(\frac{1}{\alpha} t).$$

Then

$$\widehat{y}(s) = e^{\imath\langle s|a\rangle} \widehat{x}(s),$$

$$\widehat{z}(s) = \alpha^n \widehat{x}(\alpha s)$$

for every $s \in \mathbb{R}^n$.

By Theorem 2.4.6.2,

$$\widehat{y}(s) = \int e^{-i\langle s|t\rangle} x(t+a)d\lambda(t) = \int e^{-i\langle s|t-a\rangle} x(t)d\lambda(t) = e^{i\langle s|a\rangle}\widehat{x}(s),$$

$$\widehat{z}(s) = \int e^{-i\langle s|t\rangle} x(\frac{1}{\alpha}t)d\lambda(t) = \int e^{-i\langle s|\alpha t\rangle} x(t)\alpha^n d\lambda(t) = \alpha^n \widehat{x}(\alpha s). \quad \blacksquare$$

Proposition 2.4.6.4 (4) *If* $x, y \in L^1(\lambda)$, *then*

$$\int \widehat{x} y d\lambda = \int x \widehat{y} d\lambda.$$

By Fubini's Theorem (and Theorem 2.4.6.2),

$$\int \widehat{x} y d\lambda = \int \left(\int e^{-i\langle s|t\rangle} x(t)d\lambda(t) \right) y(s)d\lambda(s) =$$

$$= \int \left(\int e^{-i\langle s|t\rangle} y(s)d\lambda(s) \right) x(t)d\lambda(t) = \int x \widehat{y} d\lambda. \quad \blacksquare$$

Definition 2.4.6.5 (4) (L. Schwartz, 1954) *Let* $\mathcal{S}(\mathbb{R}^n)$ *denote the vector space of infinitely differentiable functions* x *on* $\mathbb{R}^n$ *with the property that the product of any polynomial by any derivative of* x *converges to* 0 *at the infinity.* $\mathcal{S}(\mathbb{R}^n)$ *is called the* ***Schwartz space of rapidly decreasing*** C^∞***-functions.***

Proposition 2.4.6.6 (4) *If*

$$x : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad t \longmapsto e^{-\frac{1}{2}\|t\|^2},$$

then $x \in \mathcal{S}(\mathbb{R}^n)$ *and* $\widehat{x} = x$.

It is clear that $x \in \mathcal{S}(\mathbb{R}^n)$. In order to prove the other assertion we may assume that $n = 1$. Then

$$\frac{dx}{ds}(s) = -sx(s),$$

$$\frac{d\widehat{x}}{ds}(s) = \frac{d}{ds}\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} e^{-\frac{t^2}{2}} dt = -\frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ist} t e^{-\frac{t^2}{2}} dt =$$

$$= \frac{i^2 s}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{-ist} dt = -s\widehat{x}(s),$$

for every $s \in \mathbb{R}$, by partial integration. Hence x and $\widehat{x}$ are proportional. Since

$$\widehat{x}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = 1 = x(0),$$

it follows that

$$\widehat{x} = x. \quad \blacksquare$$

Proposition 2.4.6.7 (4) *Take $x \in L^1(\lambda)$ which is continuously differentiable in the first variable such that the partial derivative $\frac{\partial x}{\partial s_1}$ belongs to $L^1(\lambda)$. Then*

$$\widehat{\frac{\partial x}{\partial s_1}}(s) = is_1\hat{x}(s)$$

for every $s \in \mathbb{R}^n$.

By the method of partial integration and Fubini's Theorem,

$$\widehat{\frac{\partial x}{\partial s_1}}(s) = \int e^{-i\langle s|t\rangle}\frac{\partial x}{\partial s_1}(t)d\lambda(t) = is_1\int e^{-i\langle s|t\rangle}x(t)d\lambda(t) = is_1\hat{x}(s). \quad \blacksquare$$

Theorem 2.4.6.8 (4) (L. Schwartz)

a) *$\hat{x} \in \mathcal{S}(\mathbb{R}^n)$ for every $x \in \mathcal{S}(\mathbb{R}^n)$.*

b) *The map*

$$\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n), \quad x \longmapsto \hat{x}$$

is linear and bijective and

$$x(s) = \hat{\hat{x}}(-s)$$

for every $x \in \mathcal{S}(\mathbb{R}^n)$ and $s \in \mathbb{R}^n$.

c) *If P is a polynomial in n variables with coefficients in $\mathbb{K}$ and D is the associated partial differential operator, then*

$$\widehat{Dx}(s) = P(is)\hat{x}(s)$$

for every $x \in \mathcal{S}(\mathbb{R}^n)$ and $s \in \mathbb{R}^n$.

a) and c) follow from Proposition 2.4.6.7 using complete induction.

b) Take $\alpha > 0$ and $x, y \in \mathcal{S}(\mathbb{R}^n)$. Define

$$x_\alpha : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad t \longmapsto x(\frac{1}{\alpha}t),$$

$$y_\alpha : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad t \longmapsto y(\frac{1}{\alpha}t).$$

Then

$$\int \hat{x} y d\lambda = \int x \hat{y} d\lambda$$

(Proposition 2.4.6.4), so that

$$\int x_\alpha \hat{y} d\lambda = \int \hat{x}_\alpha y d\lambda = \int \alpha^n \hat{x}(\alpha t) y(t) d\lambda(t) =$$

$$= \int \hat{x}(s) y(\frac{1}{\alpha} s) d\lambda(s) = \int \hat{x} y_\alpha d\lambda$$

by Proposition 2.4.6.3. By Lebesgue's dominated Convergence Theorem we see that for $\alpha \to \infty$,

$$x(0) \int \hat{y} d\lambda = y(0) \int \hat{x} d\lambda \,.$$

If we replace y by the function introduced in Proposition 2.4.6.6, then we obtain

$$x(0) = \int \hat{x} d\lambda \,.$$

Define

$$z : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad t \longmapsto x(s+t) \,.$$

By Proposition 2.4.6.3,

$$x(s) = z(0) = \int \hat{z} d\lambda = \int e^{i\langle s|t\rangle} \hat{x}(t) dt = \check{\hat{x}}(-s)$$

for every $s \in \mathbb{R}^n$. It follows immediately that the map

$$\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n), \quad x \longmapsto \hat{x}$$

is bijective. ■

Corollary 2.4.6.9 (4) *Let $\mathcal{F}$ denote the involutive subalgebra $\mathcal{S}(\mathbb{R}^n)$ of $C_0(\mathbb{R}^n)$ with the induced structure. $\mathcal{S}(\mathbb{R}^n)$ is an involutive subalgebra of $L^1(\mathbb{R}^n)$ and the map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{F}$ defined by the Fourier transform is an isomorphism of involutive algebras.*

Take $x, y \in \mathcal{S}(\mathbb{R}^n)$. Then

$$z := \widehat{x * y} = \hat{x}\hat{y} \in \mathcal{S}(\mathbb{R}^n)$$

(Proposition 2.4.1.2, Theorem 2.4.6.2, Theorem 2.4.6.8 a)) and

$$(x * y)(s) = \widehat{\widehat{x * y}}(-s) = \hat{z}(-s)$$

for every $s \in \mathbb{R}^n$ (Theorem 2.4.6.8 b)). Thus $x * y \in \mathcal{S}(\mathbb{R}^n)$ by Theorem 2.4.6.8 a). Hence $\mathcal{S}(\mathbb{R}^n)$ is a subalgebra of $L^1(\mathbb{R}^n)$ and it is obvious that it is an involutive subalgebra of $L^1(\mathbb{R}^n)$. By Theorem 2.4.6.8 a) and Corollary 2.4.4.8, the map $\mathcal{S}(\mathbb{R}^n) \to \mathcal{F}$ defined by the Fourier transform is an isomorphism of involutive algebras. ■

Exercises

E 2.4.1 Let E be the unital Gelfand algebra of E 2.2.6. Given $t \in [a,b]$, define

$$\tilde{t} : E \longrightarrow \mathbb{C}, \quad x \longmapsto x(t) .$$

Prove:

a) $\tilde{t} \in \sigma(E)$ for every $t \in [a,b]$ and the map

$$f : [a,b] \longrightarrow \sigma(E), \quad t \longmapsto \tilde{t}$$

is a homeomorphism.

b) $\hat{x} \circ f = x$ for every $x \in E$.

c) The Gelfand transform is injective but not surjective.

d) $\{\hat{x} \mid x \in E\}$ is dense in $C(\sigma(E))$.

E 2.4.2 Let T be a compact space, S a closed set of T, $S \neq T$, $S \neq \emptyset$, and put

$$\mathcal{F} := \{x \in C(T) \mid x = 0 \text{ on } S\}$$

Show:

a) $\mathcal{F}$ is an involutive closed ideal of $C(T)$.

b) The factorization of the map

$$C(T) \longrightarrow C(S), \quad x \longmapsto x|S$$

through $C(T)/\mathcal{F}$ is an isomorphism of involutive unital Banach algebras.

c) S is homeomorphic to $\sigma(C(T)/\mathcal{F})$.

E 2.4.3 Let E, F be unital Gelfand algebras, such that the Gelfand transform of E is injective but its image is not closed and the Gelfand transform of F is injective but its image is not dense in $C(\sigma(F))$ (Example 2.4.3.7 for $K = \mathbb{K}^{\#}$). Show that the Gelfand transform of $E \times F$ is injective and its image is neither closed nor dense.

E 2.4.4 Let E be a unital Gelfand algebra. If E contains a finite subset A such that E is the closed subalgebra of E generated by A, then $\sigma(E)$ is homeomorphic to a subset of $\mathbb{C}^n$, where $n = \operatorname{Card} A$.

E 2.4.5 Assume $\mathbb{K} = \mathbb{C}$. Let T be a locally compact additive group and λ its Haar measure. Take $p \in [1, \infty[$, $x \in L^p(\lambda)$, and define

$$x_t : T \longrightarrow \mathbb{C}, \quad s \longmapsto x(s-t)$$

for $t \in T$. Show that the map

$$T \longrightarrow L^p(\lambda), \quad t \longmapsto x_t$$

is continuous.

E 2.4.6 Assume $\mathbb{K} = \mathbb{C}$. Take $(x_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$ and define

$$y : \mathbb{Z} \longrightarrow \mathbb{C}, \quad n \longmapsto nx_n .$$

Show that if $y \in \ell^1(\mathbb{Z})$, then $\widehat{x}$ is continuously differentiable and its derivative is $-i\widehat{y}$.

E 2.4.7 Assume $\mathbb{K} = \mathbb{C}$. Let x be a continuously differentiable function on $\mathbf{T}$. Show that

$$\widehat{x'}(n) = -n\widehat{x}(n-1)$$

for every $n \in \mathbb{Z}$.

E 2.4.8 Let E be the underlying real algebra of $\mathbb{C}$. Show that

$$x' : E \longrightarrow E, \quad x \longmapsto x ,$$

$$y' : E \longrightarrow E, \quad x \longmapsto \overline{x}$$

are algebra homomorphisms such that

$$\operatorname{Ker} x' = \operatorname{Ker} y' = \{0\}, \quad x' \neq y' .$$

E 2.4.9 *Let E be the involutive, commutative, complex, unital Banach algebra obtained by endowing the complex Banach algebra $\ell^1(\mathbb{Z})$ (Theorem 2.2.2.7 a),c)) with the involution*

$$\ell^1(\mathbb{Z}) \longrightarrow \ell^1(\mathbb{Z}), \quad x \longmapsto \overline{x}.$$

Take $n \in \mathbb{N} \cup \{0\}$ and put

$$F := \{x \in E | m \in \mathbb{Z}, m < n \Longrightarrow x_m = 0\}$$

and

$$\widetilde{\alpha} : F \longrightarrow \mathbb{C}, \quad x \longmapsto \sum_{m=n}^{\infty} x_m \alpha^m$$

for every $\alpha \in \mathbb{C}^{\#}$. Show:

a) *F is an involutive Banach subalgebra of E. It is a unital subalgebra iff $n = 0$.*

b) *If $n = 0$ (if $n \neq 0$) then $\alpha \in \sigma(F)$ for every $\alpha \in \mathbb{C}^{\#}$ (for every $\alpha \in \mathbb{C}^{\#} \backslash \{0\}$) and the map*

$$\mathbb{C}^{\#} \ (\mathbb{C}^{\#} \backslash \{0\}) \longrightarrow \sigma(F), \quad \alpha \longmapsto \overline{\alpha}$$

is a homomorphism.

c) *F is semi-simple but it is not symmetric.*

d) *If we restrict the scalars of F to $\mathbb{R}$ then $\mathrm{Re}\, F$ is a strongly symmetric involutive unital Banach subalgebra of F.*

e) *If $n = 0$ and if we take $K = \mathbb{C}^{\#}$ in Example 2.4.3.7, then $\widehat{x} \in \mathcal{A}$ for every $x \in F$ and the map*

$$F \longrightarrow \mathcal{A}, \quad x \longmapsto \widehat{x}$$

is an injective continuous involutive algebra homomorphism.

3. Compact Operators

There are two main reasons for developing a mathematical theory of compact operators: they have strong properties which lead to numerous theorems and they have proven to be useful in many branches of mathematics. The first section of this chapter presents the theory and the second provides an example of its usefulness in the study of linear differential equations.

3.1 The General Theory

3.1.1 General Results

Definition 3.1.1.1 (0) (F. Riesz, 1918) *Let E, F be normed spaces. A linear map $u : E \to F$ is called* **compact** *if $u(A)$ is relatively compact whenever A is bounded (or, equivalently, if $u(E^{\#})$ is relatively compact). We write $\mathcal{K}(E, F)$ for the set of compact linear maps of E into F and define*

$$\mathcal{K}(E) := \mathcal{K}(E, E) .$$

Proposition 3.1.1.2 (0) *Let A, B be relatively compact sets of the normed space E. Take $\alpha, \beta \in \mathbb{K}$. Then $\alpha A + \beta B$ is relatively compact.*

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\alpha A + \beta B$. Given $n \in \mathbb{N}$, there are $x_n \in A$, $y_n \in B$ such that

$$z_n = \alpha x_n + \beta y_n .$$

By Lemma 1.1.2.11 b $\Rightarrow$ c, there is a strictly increasing sequence $(k_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ such that $(x_{k_n})_{n\in\mathbb{N}}$ and $(y_{k_n})_{n\in\mathbb{N}}$ converge in E. It follows that $(z_{k_n})_{n\in\mathbb{N}}$ converges. By Lemma 1.1.2.11 c $\Rightarrow$ b, $\alpha A + \beta B$ is relatively compact. ■

Theorem 3.1.1.3 (0) *Let E, F be normed spaces. Then $\mathcal{K}(E, F)$ is a vector subspace of $\mathcal{L}(E, F)$. If F is complete, then $\mathcal{K}(E, F)$ is closed in $\mathcal{L}(E, F)$.*

Take $u \in \mathcal{K}(E,F)$. Then $u(E^\#)$ is relatively compact and therefore bounded. Hence u is continuous and

$$\mathcal{K}(E,F) \subset \mathcal{L}(E,F).$$

Take $u,v \in \mathcal{K}(E,F)$ and $\alpha,\beta \in \mathbb{K}$. Let A be a bounded set of E. Then $u(A)$, $v(A)$ are relatively compact sets of F. By Proposition 3.1.1.2,

$$\alpha u(A) + \beta v(A)$$

is a relatively compact set of F. Since

$$(\alpha u + \beta v)(A) \subset \alpha u(A) + \beta v(A),$$

$(\alpha u + \beta v)(A)$ is relatively compact. Hence $\alpha u + \beta v \in \mathcal{K}(E,F)$ and $\mathcal{K}(E,F)$ is a vector subspace of $\mathcal{L}(E,F)$.

Suppose F is complete and $u \in \overline{\mathcal{K}(E,F)}$. Let $\varepsilon > 0$. There is a $v \in \mathcal{K}(E,F)$ such that

$$\|u - v\| < \frac{\varepsilon}{2}.$$

Since $v(E^\#)$ is relatively compact, it is precompact. Therefore there is a finite subset B of F with

$$v(E^\#) \subset \bigcup_{y \in B} U_{\frac{\varepsilon}{2}}(y).$$

Take $x \in E^\#$. There is a $y \in B$ with $vx \in U_{\frac{\varepsilon}{2}}(y)$. Hence

$$\|ux - y\| \leq \|ux - vx\| + \|vx - y\| \leq$$

$$\leq \|u - v\| \, \|x\| + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

i.e. $ux \in U_\varepsilon(y)$. Hence

$$u(E^\#) \subset \bigcup_{y \in B} U_\varepsilon(y).$$

Thus $u(E^\#)$ is precompact. By Lemma 1.1.2.12, $u(E^\#)$ is relatively compact. Thus $u \in \mathcal{K}(E,F)$ and so $\mathcal{K}(E,F)$ is a closed set of $\mathcal{L}(E,F)$. ■

Proposition 3.1.1.4 (0) *If E,F are normed spaces, then $\mathcal{L}_f(E,F)$ is a vector subspace of $\mathcal{K}(E,F)$. If F is complete then*

$$\overline{\mathcal{L}_f(E,F)} \subset \mathcal{K}(E,F).$$

Take $u \in \mathcal{L}_f(E,F)$ and let A be a bounded set of E. Then $u(A)$ is a bounded set of the finite-dimensional space $\operatorname{Im} u$ and therefore relatively compact (Minkowski's Theorem). Hence u is compact and

$$\mathcal{L}_f(E,F) \subset \mathcal{K}(E,F).$$

It is easy to see that $\mathcal{L}_f(E,F)$ is a vector subspace of $\mathcal{K}(E,F)$.

The last assertion follows from the fact that $\mathcal{K}(E,F)$ is closed (Theorem 3.1.1.3). ■

Corollary 3.1.1.5 (3) *If E is a normed space and F is a Banach space, then*

$$\mathcal{L}^1(E,F) \subset \mathcal{K}(E,F).$$

By Theorem 1.6.1.3 a),

$$\mathcal{L}^1(E,F) \subset \overline{\mathcal{L}_f(E,F)}$$

and by Proposition 3.1.1.4

$$\overline{\mathcal{L}_f(E,F)} \subset \mathcal{K}(E,F). \quad ■$$

Proposition 3.1.1.6 *If E is a normed space and T is a compact space, then*

$$\overline{\mathcal{L}_f(E,\mathcal{C}(T))} = \mathcal{K}(E,\mathcal{C}(T)).$$

Take $u \in \mathcal{K}(E,\mathcal{C}(T))$ and $\varepsilon > 0$. By Ascoli's Theorem (Theorem 1.1.2.16), $u(E^\#)$ is equicontinuous. For $t \in T$, let U_t be an open neighbourhood of t such that

$$|y(s) - y(t)| < \varepsilon$$

for $y \in u(E^\#)$ and $s \in U_t$. $(U_t)_{t\in T}$ is an open covering of T. Let A be a finite subset of T such that

$$T = \bigcup_{t\in A} U_t.$$

Let $(y_t)_{t\in A}$ be a partition of the unit on T subordinate to the covering $(U_t)_{t\in A}$, that is, $(y_t)_{t\in A}$ is a family in $\mathcal{C}(T)$ with

$$\sum_{t\in A} y_t = 1$$

such that each y_t $(t \in A)$ is positive and vanishes on $T \backslash U_t$. Define

$$v : E \longrightarrow C(T), \quad x \longmapsto \sum_{t \in A} (ux)(t) y_t .$$

Then $v \in \mathcal{L}_f(E, C(T))$ and

$$|(ux)(s) - (vx)(s)| = \left| \sum_{t \in A} (ux)(s) y_t(s) - \sum_{t \in A} (ux)(t) y_t(s) \right| \leq$$

$$\leq \sum_{t \in A} |(ux)(s) - (ux)(t)| \, y_t(s) < \varepsilon$$

for $x \in E^{\#}$ and $s \in T$. Hence

$$\|ux - vx\| < \varepsilon$$

for $x \in E^{\#}$ and

$$\|u - v\| \leq \varepsilon .$$

Thus

$$\mathcal{K}(E, C(T)) \subset \overline{\mathcal{L}_f(E, C(T))} .$$

The reverse inclusion follows from Proposition 3.1.1.4. ■

Proposition 3.1.1.7 *Let E, F be normed spaces, $\mathfrak{G}$ an upward directed set of finite-dimensional vector subspaces of F, and for each $G \in \mathfrak{G}$ let p_G be a projection of F onto G such that*

$$\inf_{g \in \mathfrak{G}} \|y - p_G y\| = 0$$

for every $y \in F$ and

$$\|y - p_H y\| \leq \|y - p_G y\|$$

for $y \in F$ and $G, H \in \mathfrak{G}$ with $G \subset H$. Then

$$\mathcal{K}(E, F) \subset \overline{\mathcal{L}_f(E, F)} ,$$

and

$$\mathcal{K}(E, F) = \overline{\mathcal{L}_f(E, F)} ,$$

whenever F is complete.

Take $u \in \mathcal{K}(E,F)$ and $\varepsilon > 0$. Put

$$K := \overline{u(E^{\#})}$$

and

$$f_G : K \longrightarrow \mathbb{R}, \quad y \longmapsto \|y - p_G y\|$$

for $G \in \mathfrak{G}$. By hypothesis, $(f_G)_{G\in\mathfrak{G}}$ is a downward directed family of continuous functions with infimum 0. Since K is compact, there is a $G \in \mathfrak{G}$ such that

$$\|y - p_G y\| < \varepsilon$$

whenever $y \in K$ (Theorem of Dini). Hence

$$\|u - p_G \circ u\| = \sup_{x\in E^{\#}} \|ux - p_G ux\| = \sup_{y\in K} \|y - p_G y\| < \varepsilon .$$

Since $p_G \circ u \in \mathcal{L}_f(E,F)$ and ε is arbitrary,

$$u \in \overline{\mathcal{L}_f(E,F)}, \quad \mathcal{K}(E,F) \subset \overline{\mathcal{L}_f(E,F)} .$$

By Proposition 3.1.1.4,

$$\mathcal{K}(E,F) = \overline{\mathcal{L}_f(E,F)}$$

whenever F is complete. ■

Remark. It is not always true that

$$\mathcal{K}(E,F) = \overline{\mathcal{L}_f(E,F)}$$

(see Per Enflo, A counterexample to the approximation problem in Banach spaces, Acta Math. 130 (1973) 309–317).

Proposition 3.1.1.8 (3) *Let E, F be normed spaces. Take $\alpha \in \mathbb{R}_+$. Let $\mathfrak{G}$ be an upward directed set of finite-dimensional vector subspaces of F such that $\bigcup_{G\in\mathfrak{G}} G$ is dense in F, and for each $G \in \mathfrak{G}$ let p_G be a projection of F onto G with*

$$\|x - p_G x\| \leq \alpha d_G(x)$$

for every $x \in F$. Then

$$\mathcal{K}(E,F) \subset \overline{\mathcal{L}_f(E,F)}$$

and

$$\mathcal{K}(E,F) = \overline{\mathcal{L}_f(E,F)}$$

whenever F is complete.

Take $u \in \mathcal{K}(E,F)$ and $\varepsilon > 0$. Put

$$K = \overline{u(E^{\#})}.$$

Then $(d_G)_{G\in\mathfrak{G}}$ is a downward directed family of continuous real functions on F (Proposition 1.1.4.5) with infimum 0. Since K is compact, there is a $G \in \mathfrak{G}$ such that

$$d_G(y) < \frac{\varepsilon}{1+\alpha}$$

for $y \in K$ (Theorem of Dini). Then

$$\|u - p_G \circ u\| = \sup_{x\in E^{\#}} \|uz - p_G ux\| = \sup_{y\in K} \|y - p_G y\| \leq \alpha \sup_{y\in K} d_G(y) < \alpha\frac{\varepsilon}{1+\alpha} < \varepsilon.$$

Since $p_G \circ u \in \mathcal{L}_f(E,F)$ and since ε is arbitrary,

$$u \in \overline{\mathcal{L}_f(E,F)}, \quad \mathcal{K}(E,F) \subset \overline{\mathcal{L}_f(E,F)}.$$

By Proposition 3.1.1.4,

$$\mathcal{K}(E,F) = \overline{\mathcal{L}_f(E,F)}$$

whenever F is complete. ■

Corollary 3.1.1.9 (3) *Let E be a normed space and T a set. Take $p \in [1,\infty] \cup \{0\}$. Then*

$$\mathcal{K}(E,\ell^p(T)) = \overline{\mathcal{L}_f(E,\ell^p(T))}.$$

Case 1 $p \neq \infty$

Given $A \in \mathfrak{P}_f(T)$, define

$$G_A := \{x \in \ell^p(T) \mid x = 0 \text{ on } T\backslash A\}$$

and

$$p_A : \ell^p(T) \longrightarrow \ell^p(T), \quad x \longmapsto xe_A^T.$$

Then $(G_A)_{A\in\mathfrak{P}_f(T)}$ is an upward directed family of finite-dimensional vector subspaces of $\ell^p(T)$ such that $\bigcup_{A\in\mathfrak{P}_f(T)} G_A$ is dense in $\ell^p(T)$ and that

$$\|x - p_A x\|_p \leq d_{G_A}(x)$$

for $x \in \ell^p(T)$ and $A \in \mathfrak{P}_f(T)$. By Proposition 3.1.1.8,

$$\mathcal{K}(E, \ell^p(T)) = \overline{\mathcal{L}_f(E, \ell^p(T))}.$$

Case 2 $p = \infty$

For every finite partition $A := (A_\iota)_{\iota \in I}$ of T, let G_A be the vector subspace of $\ell^\infty(T)$ generated by $(e_{A_\iota})_{\iota \in I}$, $(t_{A,\iota})_{\iota \in I}$ a fixed element of $\prod_{\iota \in I} A_\iota$, and

$$p_A : \ell^\infty(T) \longrightarrow \ell^\infty(T), \quad x \longmapsto \sum_{\iota \in I} x(t_{A,\iota}) e_{A_\iota}.$$

Then p_A is a projection of $\ell^\infty(T)$ on G_A such that

$$\|x - p_A x\|_\infty \leq 2 d_{G_A}(x)$$

for every $x \in \ell^\infty(T)$. By Proposition 3.1.1.8 (and Proposition 1.1.2.6 d)),

$$\mathcal{K}(E, \ell^\infty(T)) = \overline{\mathcal{L}_f(E, \ell^\infty(T))}. \quad \blacksquare$$

Proposition 3.1.1.10 (0) *Let E be a normed space. The identity operator of E is compact iff E is finite-dimensional. In particular, if a projection u in E is compact then* $\operatorname{Im} u$ *is finite-dimensional.*

If E is finite-dimensional, then the identity operator is compact by Proposition 3.1.1.4. Now suppose that the identity operator of E is compact. Since $E^\#$ is closed and bounded, it is compact. By Theorem 1.1.4.4, E is finite-dimensional.

The final assertion follows from the fact that the identity operator of $\operatorname{Im} u$ is the restriction of u to $\operatorname{Im} u$ and it is therefore compact. $\blacksquare$

Proposition 3.1.1.11 (0) *Let E, F, G, H be normed spaces. Take $u \in \mathcal{L}(E, F)$, $v \in \mathcal{K}(F, G)$, and $w \in \mathcal{L}(G, H)$. Then $w \circ v \circ u \in \mathcal{K}(E, H)$.*

Let A be a bounded set of E. Then $u(A)$ is a bounded set of F and so $v(u(A))$ is a relatively compact set of G and $w(v(u(A)))$ is a relatively compact set of H. Hence $w \circ v \circ u$ is compact. $\blacksquare$

Corollary 3.1.1.12 (2) *Let Ω be a class of Banach spaces. The map*

$$(E, F) \longmapsto \mathcal{K}(E, F)$$

defined on Ω^2 is an $\mathcal{L}_\Omega$-subcategory $\mathcal{K}_\Omega$. The quotient unital $\mathcal{L}_\Omega$-category $\mathcal{L}_\Omega / \mathcal{K}_\Omega$ is called ***the Calkin category of*** *Ω.* $\blacksquare$

Corollary 3.1.1.13 (0) *If E is a Banach space, then $\mathcal{K}(E)$ is a closed ideal of $\mathcal{L}(E)$. It is a proper ideal of $\mathcal{L}(E)$ iff E is inifinite-dimensional. In this case the unital Banach algebra $\mathcal{L}(E)/\mathcal{K}(E)$ is called the Calkin algebra of E.*

By Theorem 3.1.1.3 and Proposition 3.1.1.11, $\mathcal{K}(E)$ is a closed ideal of $\mathcal{L}(E)$. By Proposition 3.1.1.10, $\mathcal{K}(E)$ is a proper ideal of $\mathcal{L}(E)$ iff E is infinite-dimensional. ■

Corollary 3.1.1.14 *Let E, F, G, H be Banach spaces and take $u \in \mathcal{K}(E,F)$, $w \in \mathcal{K}(G,H)$. Then the map*

$$\mathcal{L}(F,G) \longrightarrow \mathcal{K}(E,H), \quad v \longmapsto w \circ v \circ u$$

(Proposition 3.1.1.11) is compact.

Let $\mathcal{F}$ be a bounded set of $\mathcal{L}(F,G)$ and $\mathfrak{F}$ an ultrafilter on $\mathcal{F}$. Put

$$\alpha := \sup_{v \in \mathcal{F}} \|v\| < \infty .$$

Then

$$\sup_{v \in \mathcal{F}} \|vy\| \leq \alpha \|y\|$$

and so $\{wvy \mid v \in \mathcal{F}\}$ is a relatively compact set of H for every $y \in F$. Define

$$v_0 : F \longrightarrow H, \quad y \longmapsto \lim_{v,\mathfrak{F}} wvy .$$

Then v_0 is obviously linear and

$$\|v_0 y\| \leq \alpha \|w\| \, \|y\|$$

for every $y \in F$, so that $v_0 \in \mathcal{L}(F,H)$. By the definition of v_0,

$$\lim_{v,\mathfrak{F}} w \circ v = v_0$$

in the topologly of pointwise convergence. Since $\{w \circ v \mid v \in \mathcal{F}\}$ is equicontinuous and $u(E^{\#})$ is relatively compact, the above convergence holds uniformly on $u(E^{\#})$. Hence

$$\lim_{v,\mathfrak{F}} w \circ v \circ u = v_0 \circ u$$

in $\mathcal{L}(E,H)$. Since $v_0 \circ u \in \mathcal{K}(E,H)$ (Proposition 3.1.1.11), $\{w \circ v \circ u \mid v \in \mathcal{F}\}$ is a relatively compact set of $\mathcal{K}(E,H)$ and our map is compact. ■

Corollary 3.1.1.15 (4) *Let E be a Banach algebra. Given $x \in E$, define*

$$\widetilde{x} : E \longrightarrow E, \quad y \longmapsto xy .$$

If there is an approximate unit $\mathfrak{F}$ of E such that

$$\{x \in E \mid \widetilde{x} \in \mathcal{K}(E)\} \in \mathfrak{F} ,$$

then $\widetilde{x} \in \mathcal{K}(E)$ for every $x \in E$.

Take $x \in E$. Then

$$\|\widetilde{x} - \widetilde{x}\widetilde{y}\| = \|\widetilde{x - xy}\| = \|x - xy\|$$

for $y \in E$ (Proposition 2.2.1.17), so that

$$\lim_{y,\mathfrak{F}} \|\widetilde{x} - \widetilde{x}\widetilde{y}\| = \lim_{y,\mathfrak{F}} \|x - xy\| = 0 .$$

Since $\widetilde{x}\widetilde{y} \in \mathcal{K}(E)$ for $y \in E$ with $\widetilde{y} \in \mathcal{K}(E)$ (Proposition 3.1.1.11), $\widetilde{x} \in \mathcal{K}(E)$ (Theorem 3.1.1.3). ■

Proposition 3.1.1.16 (0) *Let E, F be Banach spaces and take $u \in \mathcal{K}(E,F)$. Then every closed vector subspace of F contained in $\operatorname{Im} u$ is finite-dimensional.*

Let G be a closed vector subspace of F contained in $\operatorname{Im} u$. Then $u^{-1}(G)$ is a closed vector subspace of E and so a Banach space. Define

$$v : u^{-1}(G) \longrightarrow G, \quad x \longmapsto ux ,$$

$$A := \{x \in u^{-1}(G) \mid \|x\| < 1\} .$$

v is surjective and compact, and hence $\overline{v(A)}$ is compact (Lemma 1.1.2.11 b $\Rightarrow$ a). By the Open Mapping Principle, $v(A)$ is an open set of G, i.e. G is locally compact, and therefore finite-dimensional (Theorem 1.1.4.4). ■

Lemma 3.1.1.17 (0) *Every precompact set of a metric space is separable.*

Let A be a precompact set of the metric space T. For each $n \in \mathbb{N}$, there is a finite subset A_n of A with

$$A \subset \bigcup_{s \in A_n} U_{\frac{1}{n}}(s)$$

(Lemma 1.1.2.10 a $\Rightarrow$ c). Put

$$B := \bigcup_{n \in \mathbb{N}} A_n .$$

B is countable. Take $t \in A$ and let $\varepsilon > 0$. Take $n \in \mathbb{N}$, $n > \frac{1}{\varepsilon}$. There is an $s \in A_n$ such that $t \in U_{\frac{1}{n}}(s) \subset U_\varepsilon(s)$. Since $s \in B$ and ε is arbitrary, $x \in \overline{B}$. Hence B is dense in A and A is thus separable. ■

Proposition 3.1.1.18 *Let E, F be normed spaces. Then $\operatorname{Im} u$ is separable whenever $u \in \mathcal{K}(E, F)$.*

Take $n \in \mathbb{N}$. Then $u(nE^\#)$ is relatively compact and hence separable (Lemma 1.1.2.12, Lemma 3.1.1.17). Hence there is a countable set A_n such that

$$A_n \subset u(nE^\#) \subset \overline{A}_n .$$

Put

$$A := \bigcup_{n \in \mathbb{N}} A_n .$$

Then A is countable and

$$A \subset \operatorname{Im} u = \bigcup_{n \in \mathbb{N}} u(nE^\#) \subset \bigcup_{n \in \mathbb{N}} \overline{A}_n \subset \overline{A}.$$

Thus, $\operatorname{Im} u$ is separable. ■

Proposition 3.1.1.19 (0) *Let E, F be normed spaces and take $u \in \mathcal{L}(E, F)$. Then the following are equivalent:*

a) *u is compact.*

b) *Every bounded sequence in E has a subsequence $(x_n)_{n \in \mathbb{N}}$ for which $(ux_n)_{n \in \mathbb{N}}$ converges.*

a $\Rightarrow$ b. Let $(x_n)_{n \in \mathbb{N}}$ be a bounded sequence in E. Then $\{ux_n \mid n \in \mathbb{N}\}$ is a relatively compact set of F. By Lemma 1.1.2.11 b $\Rightarrow$ c, there is a subsequence $(y_n)_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ such that $(uy_n)_{n \in \mathbb{N}}$ converges.

b $\Rightarrow$ a. Let A be a bounded set of E and $(y_n)_{n \in \mathbb{N}}$ a sequence in $u(A)$. Given $n \in \mathbb{N}$, take $x_n \in A$ with $ux_n = y_n$. By b), there is a subsequence $(x_{k_n})_{n \in \mathbb{N}}$ of $(x_n)_{n \in \mathbb{N}}$ for which $(ux_{k_n})_{n \in \mathbb{N}}$ converges. Hence $(y_n)_{n \in \mathbb{N}}$ has a

convergent subsequence. By Lemma 1.1.2.11 c $\Rightarrow$ b, $u(A)$ is relatively compact. Hence, u is compact. ■

Remark. It follows from b $\Rightarrow$ a that u is compact if its restrictions to all separable closed vector subspaces of E are compact.

Proposition 3.1.1.20 (0) *Let E, F be Banach spaces and take $u \in \mathcal{L}(E, F)$. If E is reflexive, then the following are equivalent:*

a) *u is compact.*

b) *If $\mathfrak{F}$ is a weakly convergent filter on E containing a bounded set of E, then $u(\mathfrak{F})$ converges.*

c) *If A is a bounded set of E, then the map*

$$A_{E'} \longrightarrow F, \quad x \longmapsto ux$$

is continuous.

a $\Rightarrow$ b. Let A be a bounded set of E belonging to $\mathfrak{F}$, x the weak limit of $\mathfrak{F}$, and y a point of adherence of $u(\mathfrak{F})$. Since $u(\mathfrak{F})$ converges weakly to ux (Corollary 1.4.1.6 a $\Rightarrow$ c), we have that $y = ux$. Hence ux is the only possible point of adherence of $u(\mathfrak{F})$. Since $u(A)$ is relatively compact, $u(\mathfrak{F})$ converges.

b $\Rightarrow$ c. Take $x \in A$ and let $\mathfrak{F}$ be a filter on A converging weakly to x. By Corollary 1.4.1.6 a $\Rightarrow$ c, $u(\mathfrak{F})$ converges weakly to ux. By b), $u(\mathfrak{F})$ converges. Hence it converges to ux, so that the map

$$A_{E'} \longrightarrow F, \quad x \longmapsto ux$$

is continuous.

c $\Rightarrow$ a. Let A be a bounded set of E. Since E is reflexive, A is weakly relatively compact (Proposition 1.3.8.3). By c), $u(A)$ is relatively compact and so u is compact. ■

Remark. The reflexivity of E was used only in proving that c implies a.

Proposition 3.1.1.20' *The above equivalent statements are also equivalent to:*

d) *If $(x_n)_{n\in\mathbb{N}}$ is a weakly convergent sequence in E, then $(ux_n)_{n\in\mathbb{N}}$ converges.*

c $\Rightarrow$ d is trivial.

d $\Rightarrow$ a. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in E. Since E is reflexive, by Proposition 1.3.8.13, there is a weakly convergent subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$. By d), $(uy_n)_{n\in\mathbb{N}}$ converges and so, by Proposition 3.1.1.19 b $\Rightarrow$ a, u is compact. ■

Remark. Since the weakly convergent sequences in ℓ^1 are precisely the convergent ones (Theorem 1.3.6.11), we may not drop the reflexivity of E for the implication d $\Rightarrow$ a.

Proposition 3.1.1.21 *Let E, F be Banach spaces and take $u \in \mathcal{L}(E,F)$. If F is separable then the following are equivalent:*

a) *u is compact.*

b) *If $(y'_n)_{n\in\mathbb{N}}$ is a sequence in F' converging to 0 in the topology of pointwise convergence, then $(u'y'_n)_{n\in\mathbb{N}}$ converges to 0 in norm.*

a $\Rightarrow$ b. Since $u(E^\#)$ is relatively compact, y'_n converges to 0 uniformly on $u(E^\#)$ (Proposition 1.4.1.9 a $\Rightarrow$ b). Hence $(u'y'_n)_{n\in\mathbb{N}}$ converges to 0 uniformly on $E^\#$ i.e. in the norm-topology.

b $\Rightarrow$ a. Let $(y'_n)_{n\in\mathbb{N}}$ be a sequence in F' converging to 0 in the topology of pointwise convergence. By b), $(u'y'_n)_{n\in\mathbb{N}}$ converges to 0 in norm. Hence $(y'_n)_{n\in\mathbb{N}}$ converges to 0 uniformly on $u(E^\#)$. By Proposition 1.4.1.9 b $\Rightarrow$ a, $u(E^\#)$ is relatively compact and hence u is compact. ■

Theorem 3.1.1.22 (0) (Schauder, 1930) *Let E, F be Banach spaces and take $u \in \mathcal{L}(E,F)$. Then the following are equivalent:*

a) *u is compact.*

b) *u' is compact.*

a $\Rightarrow$ b. Let $(y'_n)_{n\in\mathbb{N}}$ be a bounded sequence in F'. Put

$$\alpha := \sup_{n\in\mathbb{N}} \|y'_n\| .$$

Since $u(E^\#)$ is relatively compact, it is precompact (Lemma 1.1.2.12) and hence separable (Lemma 3.1.1.17). Let B be a countable dense set of $u(E^\#)$. By the diagonal procedure, we can find a subsequence $(z'_n)_{n\in\mathbb{N}}$ of $(y'_n)_{n\in\mathbb{N}}$ such that

$(z'_n(y))_{n\in\mathbb{N}}$ converges for every $y \in B$. Take $\varepsilon > 0$. There is a finite subset C of B such that

$$u(E^{\#}) \subset \bigcup_{y\in C} U_{\frac{\varepsilon}{3(\alpha+1)}}(y)$$

(Lemma 1.1.2.10 a $\Rightarrow$ c). Take $m \in \mathbb{N}$ with

$$|z'_m(y) - z'_n(y)| < \frac{\varepsilon}{3}$$

whenever $y \in C$ and $n \in \mathbb{N}, n \geq m$. Take $x \in E^{\#}$ and $n \in \mathbb{N}, n \geq m$. There is a $y \in C$ such that

$$\|ux - y\| < \frac{\varepsilon}{3(\alpha+1)}.$$

Hence

$$|\langle x, u'z'_m - u'z'_n\rangle| = |\langle x, u'(z'_m - z'_n)\rangle| = |\langle ux, z'_m - z'_n\rangle| \leq$$

$$\leq |\langle ux - y, z'_m\rangle| + |\langle y, z'_m - z'_n\rangle| + |\langle y - ux, z'_n\rangle| \leq$$

$$\leq \|ux - y\|\,\|z'_m\| + \frac{\varepsilon}{3} + \|y - ux\|\,\|z'_n\| \leq$$

$$\leq \frac{\varepsilon}{3(\alpha+1)}\alpha + \frac{\varepsilon}{3} + \frac{\varepsilon}{3(\alpha+1)}\alpha < \varepsilon.$$

Since x is arbitrary,

$$\|u'z'_m - u'z'_n\| \leq \varepsilon.$$

Hence $(u'z'_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and therefore convergent. By Proposition 3.1.1.19 b $\Rightarrow$ a, u' is compact.

b $\Rightarrow$ a. Let $j_E : E \to E''$, $j_F : F \to F''$ be the evaluation maps. By the above, u'' is compact. Let A be a bounded set of E. Then $j_E(A)$ is bounded in E'' and so $u''(j_E(A))$ is a relative compact set of F''. Then

$$j_F(u(A)) = u''(j_E(A))$$

(Proposition 1.3.6.16). Since F is complete, $j_F(F)$ is closed. $j_F(u(A))$ is thus a relatively compact set of $j_F(F)$ and $u(A)$ is a relatively compact set of F. Hence u is compact. ■

Proposition 3.1.1.23 (3) *Let T be a set and E a normed space. For every $u \in \mathcal{L}^1(c_0(T), E'')$ there is a unique $\widetilde{u} \in \mathcal{K}(E, c_0(T))'$ such that*

$$\widetilde{u}(\langle \cdot, x' \rangle y) = \langle uy, x' \rangle$$

for every $(x', y) \in E' \times c_0(T)$ and the map

$$\mathcal{L}^1(c_0(T), E'') \longrightarrow \mathcal{K}(E, c_0(T))', \quad u \longmapsto \widetilde{u}$$

is an isometry.

Take $u \in \mathcal{L}^1(c_0(T), E'')$. By Proposition 1.6.1.14, there is a unique $\overline{u} \in \mathcal{L}_f(E, c_0(T))'$ such that

$$\overline{u}(\langle \cdot, x' \rangle y) = \langle uy, x' \rangle$$

for every $(x', y) \in E' \times c_0(T)$. Moreover, $\|\overline{u}\| \leq \|u\|_1$. By Corollary 3.1.1.9,

$$\mathcal{K}(E, c_0(T)) = \overline{\mathcal{L}_f(E, c_0(T))}.$$

Hence, there is a unique $\widetilde{u} \in \mathcal{K}(E, c_0(T))'$ whose restriction to $\mathcal{L}_f(E, c_0(T))$ is $\overline{u}$ and

$$\|\widetilde{u}\| = \|\overline{u}\| \leq \|u\|_1.$$

Take $\theta \in \mathcal{K}(E, c_0(T))'$. Given $t \in T$, define

$$x_t'' : E' \longrightarrow \mathbb{K}, \quad x' \longmapsto \theta\Big(\langle \cdot, x' \rangle e_t^T\Big).$$

Then $x_t'' \in E''$ for every $t \in T$. Take $y \in \mathbb{K}^{(T)}$. Let $\alpha \in]0, 1[$. For each $t \in T$, there is an $x_t' \in E'^{\#}$ with

$$\langle x_t'', x_t' \rangle = \alpha \|x_t''\|.$$

Then

$$\left\| \sum_{t \in T} y(t) \langle x, x_t' \rangle e_t^T \right\|_\infty = \sup_{t \in T} |y(t)| \, |\langle x, x_t' \rangle| \leq \sup_{t \in T} |y(t)| = \|y\|_\infty$$

for every $x \in E^{\#}$, so that

$$\left\| \sum_{t \in T} y(t) \langle \cdot, x_t' \rangle e_t^T \right\| \leq \|y\|_\infty.$$

Hence

$$\left|\sum_{t\in T} y(t)\|x_t''\|\right| = \left|\sum_{t\in T} y(t)\frac{1}{\alpha}\langle x_t'', x_t'\rangle\right| =$$

$$= \left|\frac{1}{\alpha}\sum_{t\in T} y(t)\theta\Big(\langle\cdot, x_t'\rangle e_t^T\Big)\right| = \frac{1}{\alpha}\left|\theta\left(\sum_{t\in T} y(t)\langle\cdot, x_t'\rangle e_t^T\right)\right| \leq$$

$$\leq \frac{1}{\alpha}\|\theta\|\left\|\sum_{t\in T} y(t)\langle\cdot, x_t'\rangle e_t^T\right\| \leq \frac{1}{\alpha}\|\theta\|\,\|y\|_\infty$$

and

$$\left|\sum_{t\in T} y(t)\|x''\|\right| \leq \|\theta\|\,\|y\|_\infty\,,$$

since α is arbitrary. Since y is also arbitrary,

$$(\|x_t''\|)_{t\in T} \in \ell^1(T)\,, \quad \sum_{t\in T}\|x_t''\| \leq \|\theta\|$$

(Proposition 1.2.2.2). Define

$$u : c_0(T) \longrightarrow E''\,, \quad z \longmapsto \sum_{t\in T} z(t)x_t''\,.$$

Then $u \in \mathcal{L}^1(c_0(T), E'')$ and

$$\|u\|_1 \leq \sum_{t\in T}\|x_t''\| \leq \|\theta\|\,.$$

Take $z \in c_0(T)$. Then

$$\langle\cdot, x'\rangle z = \sum_{t\in T} z(t)\langle\cdot, x'\rangle e_t^T$$

for every $x' \in E'$. Hence

$$\widetilde{u}(\langle\cdot, x'\rangle z) = \langle uz, x'\rangle = \langle\sum_{t\in T} z(t)x_t'', x'\rangle = \sum_{t\in T} z(t)\langle x_t'', x'\rangle =$$

$$= \sum_{t\in T} z(t)\theta\Big(\langle\cdot, x'\rangle e_t^T\Big) = \theta\left(\sum_{t\in T} z(t)\langle\cdot, x'\rangle e_t^T\right) = \theta(\langle\cdot, x'\rangle z)$$

for every $x' \in E'$. Thus

$$\widetilde{u} = \theta$$

(Corollary 1.3.3.4, Corollary 3.1.1.9) and

$$\|u\|_1 \leq \|\theta\| = \|\widetilde{u}\| \leq \|u\|_1\,,$$

$$\|\widetilde{u}\| = \|u\|_1\,.$$

■

Lemma 3.1.1.24 (3) *Let S,T be Hausdorff spaces and X a subspace of $S \times T$ such that for each $(s_0,t_0) \in S \times T$, the sets*

$$\{t \in T \mid (s_0,t) \in X\}, \quad \{s \in S \mid (s,t_0) \in X\}$$

contains at most one element. Let $(\mu_\iota)_{\iota \in I}$, $(\nu_\iota)_{\iota \in I}$ be families of Radon measures on S and T, respectively, such that $((\mu_\iota \otimes \nu_\iota)(K))_{\iota \in I}$ is summable whenever K is a compact set of $S \times T$. Then the restriction of the Radon measure

$$\sum_{\iota \in I} \mu_\iota \otimes \nu_\iota$$

to X is atomic.

For $\iota \in I$, let μ_ι' and μ_ι'' (resp. ν_ι' and ν_ι'') be the atomic and atomless parts of μ_ι (resp. ν_ι), respectively. Let K be a compact set of X. Take $\iota \in I$. Then

$$(\mu_\iota'' \otimes \nu_\iota)(K) = \int \left(\int e_K(s,t) d\mu_\iota''(s) \right) d\nu_\iota(t) = 0$$

and

$$(\mu_\iota' \otimes \nu_\iota'')(K) = \int \left(\int e_K(s,t) d\nu_\iota''(t) \right) d\mu_\iota'(s) = 0 .$$

Hence

$$(\mu_\iota \otimes \nu_\iota)(K) = (\mu_\iota' \otimes \nu_\iota')(K) .$$

Thus

$$\sum_{\iota \in I} (\mu_\iota \otimes \nu_\iota) \quad \text{and} \quad \sum_{\iota \in I} (\mu_\iota' \otimes \nu_\iota')$$

have the same restrictions to X. Since the latter measure is atomic, the restriction of

$$\sum_{\iota \in I} \mu_\iota \otimes \nu_\iota$$

to X is atomic. ■

Definition 3.1.1.25 (0) *Let S,T be sets. Given $x \in \mathbb{K}^S$ and $y \in \mathbb{K}^T$, define*

$$x \otimes y : S \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto x(s)y(t) .$$

Example 3.1.1.26 *For $u \in \mathcal{L}^1(\ell^\infty, (\ell^1)'')$ let $\widetilde{u}$ be the unique element of $\mathcal{K}(\ell^1, \ell^\infty)'$ satisfying*

$$\widetilde{u}(\langle \cdot, x' \rangle y) = \langle uy, x' \rangle$$

for every $(x', y) \in (\ell^1)' \times \ell^\infty$ (Proposition 1.6.1.14, Corollary 3.1.1.9). The map

$$\mathcal{L}^1(\ell^\infty, (\ell^1)'') \longrightarrow \mathcal{K}(\ell^1, \ell^\infty)', \quad u \longmapsto \widetilde{u}$$

is not surjective.

Let $\mathfrak{F}$ be a free ultrafilter on $\mathbb{N}$. Define

$$\theta : \mathcal{L}_f(\ell^1, \ell^\infty) \longrightarrow \mathbb{K}, \quad v \longmapsto \lim_{n,\mathfrak{F}} \frac{1}{n} \sum_{k=1}^{n} \langle ve_k, e_k \rangle .$$

Then $\theta \in \mathcal{L}_f(\ell^1, \ell^\infty)'$. By Corollary 3.1.1.9, we may assume that $\theta \in \mathcal{K}(\ell^1, \ell^\infty)'$. Let $\beta\mathbb{N}$ be the Stone–Čech compactification of $\mathbb{N}$. For $x \in \ell^\infty$, let $\widetilde{x}$ be the continuous extension of x to $\beta\mathbb{N}$ and for $A \subset \mathbb{N}$, let $\overline{A}$ be the closure of A in $\beta\mathbb{N}$. Furthermore, let λ be the Radon measure on $\beta\mathbb{N} \times \beta\mathbb{N}$ such that

$$\int (\widetilde{x}' \otimes \widetilde{y}) d\lambda = \theta(\langle \cdot, x' \rangle y)$$

whenever $(x', y) \in (\ell^1)' \times \ell^\infty$, where we have identified $(\ell^1)'$ canonically with ℓ^∞ (Weierstrass–Stone Theorem). Then

$$\lambda(\overline{A} \times \overline{B}) = \int (\widetilde{e}_A \otimes \widetilde{e}_B) d\lambda = \theta(\langle \cdot, e_A \rangle e_B) =$$

$$= \lim_{n,\mathfrak{F}} \frac{1}{n} \sum_{k=1}^{n} \langle e_k, e_A \rangle \langle e_B, e_k \rangle = \lim_{n,\mathfrak{F}} \frac{1}{n} \operatorname{Card}(A \cap B \cap \mathbb{N}_n)$$

whenever A, B are subsets of $\mathbb{N}$. In particular,

$$\lambda(\overline{A} \times \overline{B}) = 0$$

for disjoint subsets A, B of $\mathbb{N}$. Hence the support of λ is contained in the diagonal of $\beta\mathbb{N} \times \beta\mathbb{N}$.

Take $t \in \beta\mathbb{N}$ and let $\mathfrak{G}$ be the ultrafilter on $\mathbb{N}$ converging to t. Put

$$\alpha := \lambda(\{(t, t)\}) \in \mathbb{R}_+ .$$

Assume that $\alpha > 0$. Then $t \notin \mathbb{N}$. Take $\varepsilon \in]0, \frac{\alpha}{5}[$. There is an $A \in \mathfrak{G}$ with

$$\lambda(\overline{B} \times \overline{B}) \in]\alpha - \varepsilon, \alpha + \varepsilon[$$

for every $\mathcal{B} \in \mathfrak{G}$, $B \subset A$. Take $C \in \mathfrak{F}$ such that

$$\frac{1}{n} \operatorname{Card}(A \cap \mathbb{N}_n) \in]\alpha - \varepsilon, \alpha + \varepsilon[$$

for every $n \in C$. Put $k_0 := 0$ and let $(k_n)_{n \in \mathbb{N}}$ be a sequence in C satisfying

$$\frac{k_{n-1}}{k_n} \frac{\alpha + 3\varepsilon}{2} < \varepsilon$$

for every $n \in \mathbb{N}$. We construct inductively a sequence $(j_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, j_n is the smallest natural number satisfying

$$\frac{1}{k_n} \operatorname{Card}\left(\bigcup_{m=1}^{n} (A \cap]k_{m-1}, k_{m-1} + j_m])\right) > \frac{\alpha - \varepsilon}{2}.$$

Since

$$\frac{1}{k_1} \operatorname{Card}(A \cap \mathbb{N}_{k_1}) > \alpha - \varepsilon,$$

the existence of j_{n_1} is clear. Take $n \in \mathbb{N}$, $n > 1$, and assume the sequence was constructed up to $n - 1$. Then

$$\frac{1}{k_{n-1}} \operatorname{Card}\left(\bigcup_{m=1}^{n-1} (A \cap]k_{m-1} + j_m, k_m])\right) =$$

$$= \frac{1}{k_{n-1}} \operatorname{Card}(A \cap \mathbb{N}_{k_{n-1}}) - \frac{1}{k_{n-1}} \operatorname{Card}\left(\bigcup_{m=1}^{n-1} (A \cap]k_{m-1}, k_{m-1} + j_m])\right) <$$

$$< \alpha + \varepsilon - \frac{\alpha - \varepsilon}{2} = \frac{\alpha + 3\varepsilon}{2}.$$

Hence

$$\frac{1}{k_n} \operatorname{Card}\left((A \cap]k_{n-1}, k_n]) \cup \left(\bigcup_{m=1}^{n-1} (A \cap]k_{m-1}, k_{m-1} + j_m])\right)\right) =$$

$$= \frac{1}{k_n} \operatorname{Card}(A \cap \mathbb{N}_{k_n}) - \frac{1}{k_n} \operatorname{Card}\left(\bigcup_{m=1}^{n-1} (A \cap]k_{m-1} + j_m, k_m])\right) >$$

$$> \alpha - \varepsilon - \frac{k_{n-1}}{k_n} \frac{\alpha + 3\varepsilon}{2} > \alpha - 2\varepsilon > \frac{\alpha - \varepsilon}{2}.$$

This establishes the existence of j_n.

Put

$$B := A \cap \left(\bigcup_{m \in \mathbb{N}}]k_{m-1}, k_{m-1} + j_m] \right) .$$

Then

$$\lambda(\overline{B} \times \overline{B}) = \lim_{n,\mathfrak{F}} \frac{1}{n} \operatorname{Card}(B \cap \mathbb{N}_n) =$$

$$= \lim_{n \to \infty} \frac{1}{k_n} \operatorname{Card} \left(\bigcup_{m=1}^{n} (B \cap]k_{m-1}, k_{m-1} + j_m]) \right) =$$

$$= \frac{\alpha - \varepsilon}{2} < \alpha - \varepsilon$$

and so $B \notin \mathfrak{G}$. Hence $A \backslash B \in \mathfrak{G}$ and

$$\alpha - \varepsilon < \lambda(\overline{A \backslash B} \times \overline{A \backslash B}) = \lambda(\overline{A} \times \overline{A}) - \lambda(\overline{B} \times \overline{B}) < \alpha + \varepsilon - \frac{\alpha - \varepsilon}{2} =$$

$$= \frac{\alpha + 3\varepsilon}{2} < \alpha - \varepsilon ,$$

which is a contradiction. Hence $\alpha = 0$ and (t, t) is not an atom of λ. Therefore, λ is atomless.

Assume that there is a $u \in \mathcal{L}^1(\ell^\infty, (\ell^1)'')$ with $\widetilde{u} = \theta$. Let $((y_\iota', x_\iota''))_{\iota \in I}$ be a family in $(\ell^\infty)' \times (\ell^1)''$ such that

$$\sum_{\iota \in I} \|y_\iota'\| \, \|x_\iota''\| < \infty$$

and

$$uy = \sum_{\iota \in I} \langle y, y_\iota' \rangle x_\iota''$$

for every $y \in \ell^\infty$. For $\iota \in I$, let μ_ι and ν_ι denote the Radon measure on $\beta\mathbb{N}$ satisfying

$$\int \widetilde{x} d\mu_\iota = \langle x, x_\iota'' \rangle , \quad \int \widetilde{x} d\nu_\iota = \langle x, y_\iota' \rangle$$

for every $x \in \ell^\infty$. Then

$$\int (\widetilde{x'} \otimes \widetilde{y}) d\lambda = \theta(\langle \cdot, x' \rangle y) = \widetilde{u}(\langle \cdot, x' \rangle y) = \langle uy, x' \rangle =$$

$$= \sum_{\iota \in I} \langle y, y_\iota' \rangle \langle x_\iota'', x' \rangle = \sum_{\iota \in I} \left(\int \tilde{y} d\nu_\iota \right) \left(\int \tilde{x}' d\mu_\iota \right) =$$

$$= \sum_{\iota \in I} \int (\tilde{x} \otimes \tilde{y}) d(\mu_\iota \otimes \nu_\iota) = \int (\tilde{x} \otimes \tilde{y}) d \left(\sum_{\iota \in I} \mu_\iota \otimes \nu_\iota \right)$$

for every $(x', y) \in (\ell^1)' \times \ell^\infty$ and so

$$\lambda = \sum_{\iota \in I} \mu_\iota \otimes \nu_\iota$$

(Weierstrass–Stone Theorem). By Lemma 3.1.1.24, λ is atomic, which is a contradiction. Hence for no $u \in L^1(\ell^\infty, (\ell^1)'')$ is $\tilde{u} = \theta$ and thus the map

$$\mathcal{L}^1(\ell^\infty, (\ell^1)'') \longrightarrow \mathcal{K}(\ell^1, \ell^\infty)', \quad u \longmapsto \tilde{u}$$

is not surjective. ■

3.1.2 Examples

Proposition 3.1.2.1 *Let S, T be sets. Take $p \in [1, \infty[$ and $k \in \ell^{p,\infty}(S,T)$. Define*

$$\overset{\cap}{k} : \ell^1(T) \longrightarrow \ell^p(S), \quad x \longmapsto \overset{\cap}{k}x$$

(Proposition 1.2.3.4 a)). Then the following are equivalent:

a) $\overset{\cap}{k} \in \mathcal{K}(\ell^1(T), \ell^p(S))$.

b) *For each $\varepsilon > 0$, there is a finite subset S' of S such that*

$$\sum_{s \in S \setminus S''} |k(s,t)|^p < \varepsilon$$

for every $t \in T$.

a $\Rightarrow$ b. Now $e_t^T \in \ell^1(T)^\#$ and

$$k(\cdot, t) = \overset{\cap}{k} e_t^T \in \overset{\cap}{k}(\ell^1(T)^\#)$$

for every $t \in T$. Hence b) follows from Proposition 1.1.2.13 a $\Rightarrow$ b.

b $\Rightarrow$ a. Let A be a bounded set of $\ell^1(T)$ and

$$\alpha := \sup_{x \in A} \|x\|_1 .$$

Take $\varepsilon > 0$. By b), there is a finite subset S' of S such that

$$\left(\sum_{s \in S \setminus S'} |k(s,t)|^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{3(\alpha + 1)}$$

for every $t \in T$. Let B be a finite subset of A such that for each $x \in A$, there is a $y_x \in B$ with

$$\left(\sum_{s \in S'} |(\overset{\cap}{k}x)(s) - (\overset{\cap}{k}y_x)(s)|^p \right)^{\frac{1}{p}} < \frac{\varepsilon}{3} .$$

Then

$$\|\overset{\cap}{k}x - \overset{\cap}{k}y_x\|_p \leq \left(\sum_{s \in S'} |(\overset{\cap}{k}x)(s) - (\overset{\cap}{k}y_x)(s)|^p \right)^{\frac{1}{p}} +$$

$$+\left(\sum_{s\in S\setminus S'}|(\overset{\cap}{k}x)(s)|^p\right)^{\frac{1}{p}}+\left(\sum_{s\in S\setminus S'}|(\overset{\cap}{k}y_x)(s)|^p\right)^{\frac{1}{p}}<$$

$$<\frac{\varepsilon}{3}+\frac{\varepsilon}{3(\alpha+1)}\|x\|_1+\frac{\varepsilon}{3(\alpha+1)}\|y_x\|_1<\varepsilon$$

for every $x\in A$ (Proposition 1.2.3.4 a)). Hence

$$\overset{\cap}{k}(A)\subset\bigcup_{y\in B}U_\varepsilon(y)$$

and $\overset{\cap}{k}(A)$ is therefore a precompact set of $\ell^p(S)$. By Lemma 1.1.2.12 and Example 1.1.2.5, $\overset{\cap}{k}(A)$ is a relatively compact set of $\ell^p(S)$. Thus $\overset{\cap}{k}$ is compact. ■

Proposition 3.1.2.2 (1) *Let S,T be sets. Take $p\in[1,\infty[\cup\{0\}$ and let q be the conjugate exponent of p. Take $k\in\ell^{\infty,q}(S,T)$ such that $k(\cdot,t)\in c_0(S)$ (resp. $c(S)$) for every $t\in T$. Define*

$$\overset{\cap}{k}:\ell^p(T)\longrightarrow c_0(S)\quad(resp.\ c(S)),\qquad x\longmapsto\overset{\cap}{k}x$$

(Proposition 1.2.3.4 a),d)). Then the following are equivalent:

a) $\overset{\cap}{k}\in\mathcal{K}(\ell^p(T),c_0(S))$ (resp. $\overset{\cap}{k}\in\mathcal{K}(\ell^p(T),c(S))$.

b) *Given $\varepsilon>0$, there is a finite subset S' of S with*

$$\sup_{s\in S\setminus S'}\|k(s,\cdot)\|_q<\varepsilon\quad(\ resp.\quad\sup_{r,s\in S\setminus S'}\|k(r,\cdot)-k(s,\cdot)\|_q<\varepsilon).$$

a $\Rightarrow$ b. $\overset{\cap}{k}(\ell^p(T)^\#)$ is relatively compact. By Proposition 1.1.2.13 a $\Rightarrow$ b, there is a finite subset S' of S with

$$\sup_{r,s\in S\setminus S'}|(\overset{\cap}{k}x)(s)|<\varepsilon\quad(\text{ resp. }\sup_{r,s\in S\setminus S'}|(\overset{\cap}{k}x)(r)-(\overset{\cap}{k}x)(s)|<\varepsilon)$$

for every $x\in\ell^p(R)^\#$. Hence, by Example 1.2.2.3 b),c),

$$\sup_{s\in S\setminus S'}\|k(s,\cdot)\|_q\le\varepsilon\quad(\text{ resp. }\sup_{r,s\in S\setminus S'}\|k(r,\cdot)-k(s,\cdot)\|_q\le\varepsilon).$$

b $\Rightarrow$ a. Let A be a bounded set of $\ell^p(T)$ and put

$$\alpha:=\sup_{x\in A}\|x\|_p.$$

Take $\varepsilon > 0$. By b), there is a finite subset S' of S such that

$$\sup_{s \in S \setminus S'} \|k(s,\cdot)\|_q < \frac{\varepsilon}{\alpha + 1}$$

$$(\text{resp. } \sup_{r,s \in S \setminus S'} \|k(r,\cdot) - k(s,\cdot)\|_q < \frac{\varepsilon}{\alpha + 1}).$$

Then, by Example 1.2.2.3 b),c),

$$\sup_{s \in S \setminus S'} |(\overset{\cap}{k}x)(s)| \leq \frac{\varepsilon\alpha}{\alpha + 1} < \varepsilon$$

$$(\text{resp. } \sup_{r,s \in S \setminus S'} |(\overset{\cap}{k}x)(r) - (\overset{\cap}{k}x)(s)| \leq \frac{\varepsilon\alpha}{\alpha + 1} < \varepsilon).$$

By Proposition 1.1.2.13 b $\Rightarrow$ a, $\overset{\cap}{k}(A)$ is relatively compact. ■

Remark. In the case of c_0, b) asserts precisely that $k \in \ell^{0,q}$.

Proposition 3.1.2.3 (1) *Let S, T be sets. Take conjugate exponents p and q with $p \neq \infty$. Take $r \in \{0\} \cup [1, \infty[$ and $k \in \ell^{r,q}(S,T)$. Then the operator*

$$\ell^p(T) \longrightarrow \ell^r(S), \quad x \longmapsto \overset{\cap}{k}x$$

(Proposition 1.2.3.4 b)) is compact.

Take $\varepsilon > 0$. There is a finite subset S' of S such that

$$\|(\|k(s,\cdot)\|_q)_{s \in S \setminus S'}\|_r < \varepsilon.$$

Then

$$\|(\overset{\cap}{k}x)(s)\|_r \leq \varepsilon$$

for every $x \in \ell^p(T)^\#$ and $s \in S \setminus S'$ (Proposition 1.2.3.4 a)). By Proposition 1.1.2.13 b $\Rightarrow$ a, $\{\overset{\cap}{k}x \mid x \in \ell^p(T)^\#\}$ is relatively compact. ■

Remark. Take $k \in \ell^{2,2}(T,T)$. The map

$$\ell^{2,2}(T,T) \longrightarrow \ell^{2,2}(T,T), \quad h \longmapsto k \circ h$$

is, in general, not compact. Indeed, take $x \in \ell^2(T)\backslash\{0\}$,

$$k : T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto \delta_{s,1}(t),$$

and, for each $n \in \mathbb{N}$,

$$h_n : T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto x(s)\delta_{t,n}.$$

Then $\|h_n\| = \|x\|$ and

$$k \circ h_n(s,t) = \delta_{s,1}\delta_{t,n}\|x\|^2$$

for $n \in \mathbb{N}$ and $(s,t) \in T \times T$. The sequence $(k \circ h_n)_{n\in\mathbb{N}}$ has no convergent subsesquence and so the map

$$\ell^{2,2}(T,T) \longrightarrow \ell^{2,2}(T,T), \quad h \longmapsto k \circ h$$

is not compact (Proposition 3.1.1.19 a $\Rightarrow$ b).

Proposition 3.1.2.4 (1) *Let S,T be sets and k a function $S \times T \longrightarrow \mathbb{K}$ with $k(s,\cdot) \in \ell^1(T)$ whenever $s \in S$. Then the following are equivalent:*

a) $\overset{\cap}{k}x \in c_0(S)$ *for every* $x \in \ell^\infty(T)$ *and the map*

$$\ell^\infty(T) \longrightarrow c_0(S), \quad x \longmapsto \overset{\cap}{k}x$$

is compact.

b) $\overset{\cap}{k}x \in c_0(S)$ *for every* $x \in c(T)$ *and the map*

$$c(T) \longrightarrow c_0(S), \quad x \longmapsto \overset{\cap}{k}x$$

is compact.

c) $\overset{\cap}{k}x \in c_0(S)$ *for every* $x \in c_0(T)$ *and the map*

$$c_0(T) \longrightarrow c_0(S), \quad x \longmapsto \overset{\cap}{k}x$$

is compact.

d) $k \in \ell^{0,1}(S,T)$.

e) $\overset{\cap}{k}e_B \in c_0(S)$ *for every countable subset B of T.*

a $\Rightarrow$ b $\Rightarrow$ c and a $\Rightarrow$ e are trivial.

c $\Rightarrow$ d $\Rightarrow$ a follows from Proposition 3.1.2.2.

e $\Rightarrow$ d. We may assume that S is infinite. Put

$$\mathfrak{F} := \{S \backslash A \mid A \in \mathfrak{P}_f(S)\}.$$

Then

$$\lim_{s,\mathfrak{F}} \sum_{t \in B} k(s,t) = \lim_{s,\mathfrak{F}} (\overset{\cap}{k} e_B)(s) = 0$$

for every countable subset B of T. By Corollary 1.2.3.13,

$$\lim_{s,\mathfrak{F}} \|k(s,\cdot)\|_1 = 0\,,$$

and so $k \in \ell^{0,1}(S,T)$. ■

Corollary 3.1.2.5 (1) *Let S,T be sets. Given $k \in \ell^{0,1}(S,T)$ define*

$$\overset{\cap}{k} : c_0(T) \longrightarrow c_0(S)\,, \quad x \longmapsto \overset{\cap}{k} x$$

(Proposition 3.1.2.4 d $\Rightarrow$ c). Then the map

$$\ell^{0,1}(S,T) \longrightarrow \mathcal{K}(c_0(T), c_0(S))\,, \quad k \longmapsto \overset{\cap}{k}$$

(Proposition 3.1.2.4 d $\Rightarrow$ c) is an isometry.

Take $u \in \mathcal{K}(c_0(T), c_0(S))$. By Proposition 1.2.3.4 e), there is a $k \in \ell_0^{\infty,1}(S,T)$ such that

$$\overset{\cap}{k} = u\,, \quad \|k\| = \|u\|\,.$$

Hence, by Proposition 3.1.2.4 c $\Rightarrow$ d, $k \in \ell^{0,1}(S,T)$. ■

Example 3.1.2.6 *Let T be a locally compact group and take $\mu \in \mathcal{M}_b(T) \backslash \{0\}$. If T is not compact then the operator*

$$\mathcal{M}_b(T) \longrightarrow \mathcal{M}_b(T)\,, \quad \nu \longmapsto \mu * \nu$$

is not compact.

By Corollary 2.2.2.21, there is a sequence $(t_n)_{n \in \mathbb{N}}$ in T such that

$$\|\mu * \delta_{t_m} - \mu * \delta_{t_n}\| > \|\mu\|$$

for distinct $m,n \in \mathbb{N}$. Hence the sequence $(\mu * \delta_{t_n})_{n \in \mathbb{N}}$ has no Cauchy subsequence. The conclusion now follows from Proposition 3.1.1.19 a $\Rightarrow$ b. ■

Example 3.1.2.7 *Let T be a unimodular locally compact group. Take $p \in [1, \infty[$ and $\mu \in \mathcal{M}_b(T) \backslash \{0\}$. If T is not compact, then the operator*

$$L^p(T) \longrightarrow L^p(T), \quad x \longmapsto \mu * x$$

(Proposition 2.2.2.14 a)) is not compact.

There is an $x \in L^p(T)$ with

$$\mu * x \neq 0.$$

By Corollary 2.2.2.21, there is a sequence $(x_n)_{n \in \mathbb{N}}$ in $L^p(T)^\#$ with

$$\|\mu * x_m - \mu * x_n\|_p > \|\mu * x\|_p$$

for distinct $m, n \in \mathbb{N}$. Hence the sequence $(\mu * x_n)_{n \in \mathbb{N}}$ has no Cauchy subsequence. The conclusion now follows from Proposition 3.1.1.19 a $\Rightarrow$ b. ■

Proposition 3.1.2.8 *Let T be a topological space and $\mathcal{F}$ a vector subspace of $\mathbb{K}^T$. Given a norm on $\mathcal{F}$, the following are equivalent:*

a) *Given $t \in T$, the map*

$$\mathcal{F} \longrightarrow \mathbb{K}, \quad x \longmapsto x(t)$$

is continuous and $\{x|K \mid x \in \mathcal{F}^\#\}$ is equicontinuous whenever K is a compact set of T.

b) *The map*

$$\mathcal{F} \longrightarrow \mathcal{C}(K), \quad x \longmapsto x|K$$

is a compact operator whenever K is a compact set of T.

a $\Rightarrow$ b. Since $\{x|K \mid x \in \mathcal{F}^\#\}$ is equicontinous, every $t \in K$ is contained in an open neighbourhood U_t with

$$|x(s) - x(t)| < 1$$

for every $x \in \mathcal{F}^\#$ and every $s \in K \cap U_t$. Since K is compact, there is a finite set A of K such that

$$K \subset \bigcup_{t \in A} U_t.$$

Put

$$\alpha := \sup\{|x(t)| \mid x \in \mathcal{F}^{\#}, t \in A\}.$$

Since, for every $t \in A$, the map

$$\mathcal{F} \longrightarrow \mathbb{K}, \quad x \longmapsto x(t)$$

is continuous, it follows that α is finite. Take $x \in \mathcal{F}^{\#}$ and $s \in K$. Then there is a $t \in A$ with $s \in U_t$. We get

$$|x(s)| \leq |x(s) - x(t)| + |x(t)| \leq 1 + \alpha$$

and so

$$\|x|K\|_\infty \leq 1 + \alpha.$$

Hence $\{x|K \mid x \in \mathcal{F}^{\#}\}$ is bounded. Being equicontinuous als well, it is, by Ascoli's Theorem (Theorem 1.1.2.16), a relatively compact set of $\mathcal{C}(K)$. Hence

$$\mathcal{F} \longrightarrow \mathcal{C}(K), \quad x \longmapsto x|K$$

defines a compact operator.

b $\Rightarrow$ a follows from Ascoli's Theorem (Theorem 1.1.2.16). ■

Example 3.1.2.9 *Let T be an open set of an Euclidean space and λ the Lebesgue measure on T. Take $p \in [1, \infty]$, and put*

$$\mathcal{F} := \{x \in L^p(\lambda) \mid x \text{ is harmonic}\}.$$

Then $\mathcal{F}$ is a closed vector subspace of $L^p(\lambda)$, $\mathcal{F}^{\#}$ is equicontinuous, and for each compact set K of T,

$$\mathcal{F} \longrightarrow \mathcal{C}(K), \quad x \longmapsto x|K$$

defines a compact operator.

$\mathcal{F}$ is obviously a vector subspace of $L^p(\lambda)$. Let q be the conjugate exponent of p. Take $t \in T$ and $r > 0$ with

$$U_{3r}(t) \subset T.$$

Take further $x \in \mathcal{F}^{\#}$ and $s \in U_{2r}(t)$. By Gauss's Mean Value Theorem and Hölder's inequality,

$$|x(s)| = \frac{1}{\lambda(U_r(0))}\left|\int\limits_{U_r(s)} x d\lambda\right| \leq \frac{1}{\lambda(U_r(0))}\int\limits_{U_r(s)} |x| d\lambda \leq$$

$$\leq \frac{1}{\lambda(U_r(0))}\left(\int\limits_{U_r(s)} |x|^p d\lambda\right)^{\frac{1}{p}}\left(\int\limits_{U_r(s)} d\lambda\right)^{\frac{1}{q}} \leq \frac{\lambda(U_r(s))^{\frac{1}{q}}}{\lambda(U_r(s))} = \frac{1}{\lambda(U_r(0))^{\frac{1}{p}}}$$

if $p \in]1, \infty[$. It is easy to see that the same inequality holds if $p = 1$ or $p = \infty$.

Now take $x \in \mathcal{F}^{\#}$ and $s \in U_r(t)$. Put

$$A := (U_r(s) - U_r(t)) \cup (U_r(t) - U_r(s)).$$

By Gauss's Mean Value Theorem and the above result,

$$|x(s) - x(t)| = \frac{1}{\lambda(U_r(0))}\left|\int\limits_{U_r(s)} x d\lambda - \int\limits_{U_r(t)} x d\lambda\right| \leq$$

$$\leq \frac{1}{\lambda(U_r(0))}\int\limits_{A} |x| d\lambda \leq \frac{\lambda(A)}{\lambda(U_r(0))^{1+\frac{1}{p}}}.$$

Hence $\mathcal{F}^{\#}$ is equicontinuous. By Proposition 3.1.2.8 a $\Rightarrow$ b, given a compact set K of T, the map

$$\mathcal{F} \longrightarrow C(K), \quad x \longmapsto x|K$$

is a compact operator.

Take $x \in \overline{\mathcal{F}}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{F}$ converging to x in $L^p(\lambda)$. By the above proof, $(x_n)_{n\in\mathbb{N}}$ converges to x uniformly on the compact sets of T. Hence x is harmonic and $\mathcal{F}$ is a closed vector subspace of $L^p(\lambda)$. ■

Example 3.1.2.10 *Take $\mathbb{K} = \mathbb{C}$. Let T be an open set of $\mathbb{C}$ and let λ denote Lebesgue measure on T. Take $p \in [1, \infty]$, and put*

$$\mathcal{A} := \{x \in L^p(\lambda) \mid x \text{ is analytic}\}.$$

Then $\mathcal{A}$ is a closed vector subspace of $L^p(\lambda)$, for each compact set K of T the map

$$\mathcal{A} \longrightarrow C(K), \quad x \longmapsto x|K$$

is a compact operator, and $\mathcal{A}^{\#}$ is equicontinuous.

The assertion follows from Example 3.1.2.9. ■

Proposition 3.1.2.11 (0) *Let T be a set and take $p \in [1,\infty[\cup\{0\}$. Given $x \in \ell^\infty(T)$, define*

$$\widetilde{x} : \ell^p(T) \longrightarrow \ell^p(T), \quad x \longmapsto xy,$$

and given $u \in \mathcal{L}(\ell^p(T))$, define

$$\dot{u} : T \longrightarrow \mathbb{K}, \quad t \longmapsto (ue_t)(t).$$

a) *$\widetilde{x} \in \mathcal{L}(\ell^p(T))$ and $\|\widetilde{x}\| = \|x\|_\infty$ whenever $x \in \ell^\infty(T)$.*

b) *Given $u \in \mathcal{K}(\ell^p(T))$, $\dot{u} \in c_0(T)$ and $\|\dot{u}\|_\infty \leq \|u\|$.*

c) *Given $x \in \ell^\infty(T)$, $\dot{\widetilde{x}} = x$.*

d) *If $x \in \ell^\infty(T)$, then*

$$x \in c_0(T) \Longleftrightarrow \widetilde{x} \in \mathcal{K}(\ell^p(T)).$$

Define

$$\psi : c_0(T) \longrightarrow \mathcal{K}(\ell^p(T)), \quad x \longmapsto \widetilde{x}.$$

e) *ψ is linear and the map*

$$c_0(T) \longrightarrow \operatorname{Im}\psi, \quad x \longmapsto \psi x$$

is an isometry.

f) *The map*

$$\mathcal{K}(\ell^p(T)) \longrightarrow \mathcal{K}(\ell^p(T)), \quad u \longmapsto \widetilde{\dot{u}}$$

is a projection of $\mathcal{K}(\ell^p(T))$ onto $\operatorname{Im}\psi$ of norm 1.

g) *$\operatorname{Im}\psi$ is a complemented subspace of $\mathcal{K}(\ell^p(T))$.*

a) is easy to see.

b) Take $\varepsilon > 0$. Since $u(\ell^p(T)^\#)$ is relatively compact, there is a finite subset S of T such that

$$\|e_{T\setminus S}ux\|_p < \varepsilon$$

whenever $x \in \ell^p(T)^\#$ (Proposition 1.1.2.13 a $\Rightarrow$ b). Thus

$$|\dot{u}(t)| = |(ue_t)(t)| \leq \|e_{T\setminus S} u e_t\|_p < \varepsilon$$

for every $t \in T\setminus S$. Hence $\dot{u} \in c_0(T)$. It is easy to see that $\|\dot{u}\|_\infty \leq \|u\|$.

c) is easy to see.

d) Assume $x \in c_0(T)$. Take $\varepsilon > 0$. There is a finite subset S of T such that $|x| < \varepsilon$ on $T\setminus S$. Thus

$$\|e_{T\setminus S}\tilde{x}y\|_p = \|e_{T\setminus S}xy\|_p \leq \varepsilon\|y\|_p \leq \varepsilon$$

for all $y \in \ell^p(T)^\#$. By Proposition 1.1.2.13 b $\Rightarrow$ a, $\tilde{x}(\ell^p(T))^\#$ is relatively compact, so $\tilde{x}$ is compact.

If $\tilde{x}$ is compact, then, by c) and b),

$$x = \dot{\tilde{x}} \in c_0(T)\,.$$

e) follows from d) and a).

f) By a), b), and d), the map is a well-defined operator of norm at most 1. By c), it is a projection of $\mathcal{K}(\ell^p(T))$ onto $\operatorname{Im}\psi$ and so it has norm 1.

g) follows from f) and Murray's Theorem. ■

Corollary 3.1.2.12 (0) *If S,T are infinite sets and if $p \in [1,\infty[\,\cup\{0\}$, then $\mathcal{K}(\ell^p(S),\ell^p(T))$ is not a complemented subspace of $\mathcal{L}(\ell^p(S),\ell^p(T))$.*

First consider $S = T$. By Proposition 1.2.5.16 b) and Proposition 3.1.2.11 d), f), there is an operator

$$u : \ell^\infty(T) \longrightarrow \mathcal{L}(\ell^p(T))$$

such that the map

$$\ell^\infty(T) \longrightarrow \operatorname{Im} u\,, \quad x \longmapsto ux$$

is an isometry,

$$u(c_0(T)) \subset \mathcal{K}(\ell^p(T))\,,$$

and there is a projection of $\mathcal{K}(\ell^p(T))$ on $u(c_0(T))$. Since $c_0(T)$ is not a complemented subspace of $\ell^\infty(T)$ (Corollary 1.2.5.14), it follows from Corollary 1.2.5.10 that $\mathcal{K}(\ell^p(T))$ is not a complemented subspace of $\mathcal{L}(\ell^p(T))$.

Now let S and T be arbitrary. We may assume that

$$R := S \cap T$$

is infinite. Given $x \in \ell^p(R)$, define

$$\tilde{x} : T \longrightarrow \mathbb{R}, \quad t \longmapsto \begin{cases} x(t) & \text{if} \quad t \in R \\ 0 & \text{if} \quad t \in T \backslash R. \end{cases}$$

Given $u \in \mathcal{L}(\ell^p(R))$, define

$$\tilde{u} : \ell^p(S) \longrightarrow \ell^p(T), \quad x \longmapsto \widetilde{u(x|R)}.$$

Given $v \in \mathcal{K}(\ell^p(S), \ell^p(T))$, define

$$\varphi : \mathcal{L}(\ell^p(R)) \longrightarrow \mathcal{L}(\ell^p(S), \ell^p(T)), \quad u \longmapsto \tilde{u},$$

$$\dot{v} : \ell^p(S) \longrightarrow \ell^p(T), \quad x \longmapsto e_R v(e_R x)$$

and

$$\psi : \mathcal{K}(\ell^p(S), \ell^p(T)) \longrightarrow \mathcal{K}(\ell^p(S), \ell^p(T)), \quad v \longmapsto v.$$

Then

$$\varphi(\mathcal{K}(\ell^p(R))) \subset \mathcal{K}(\ell^p(S), \ell^p(T)),$$

the map

$$\mathcal{L}(\ell^p(R)) \longrightarrow \operatorname{Im} \varphi, \quad u \longmapsto \varphi u$$

is an isometry, and ψ is a projection of $\mathcal{K}(\ell^p(S), \ell^p(T))$ onto $\varphi(\mathcal{K}(\ell^p(R)))$. By the above considerations and by Corollary 1.2.5.10, $\mathcal{K}(\ell^p(S), \ell^p(T))$ is not a complemented subspace of $\mathcal{L}(\ell^p(S), \ell^p(T))$. ■

Example 3.1.2.13 *Let $(a_\iota)_{\iota \in I}$ be a summable family in the Banach space E, and define*

$$u : \ell^\infty(I) \longrightarrow E, \quad x \longmapsto \sum_{\iota \in I} x(\iota) a_\iota$$

(Corollary 1.1.6.8). Then u is a compact operator.

Take $\varepsilon > 0$. There is a $J \in \mathfrak{P}_f(I)$ such that

$$\left\| \sum_{\iota \in K} a_\iota \right\| < \frac{\varepsilon}{5}$$

for every $K \in \mathfrak{P}_f(I\backslash J)$ (Proposition 1.1.6.6). Thus

$$\left\| \sum_{\iota \in K} x(\iota) a_\iota \right\| \leq \frac{4\varepsilon}{5}$$

whenever $K \in \mathfrak{P}_f(I\backslash J)$ and $x \in \ell^\infty(I)^\#$ (Proposition 1.1.1.5). Hence

$$\left\| \sum_{\iota \in I\backslash J} x(\iota) a_\iota \right\| \leq \frac{4\varepsilon}{5} < \varepsilon$$

for every $x \in \ell^\infty(I)^\#$. If we now put

$$v : \ell^\infty(I) \longrightarrow E, \quad x \longmapsto \sum_{\iota \in J} x(\iota) a_\iota ,$$

then $v \in \mathcal{L}_f(\ell^\infty(I), E)$ and, by the above considerations,

$$\|u - v\| < \varepsilon .$$

Hence

$$u \in \overline{\mathcal{L}_f(E,F)} \subset \mathcal{K}(E,F)$$

(Proposition 3.1.1.4). ■

Proposition 3.1.2.14 *Take $u \in \mathcal{L}(c_0, \ell^\infty)$ and let $\mathfrak{F}$ be a free ultrafilter on $\mathbb{N}$. Then*

$$\operatorname{Card}\{n \in \mathbb{N} \mid |\lim_{k,\mathfrak{F}} (ue_n)_k| \geq 1\} \leq \|u\| .$$

Define

$$f : \mathbb{N} \longrightarrow \mathbb{K}, \quad n \longmapsto \lim_{k,\mathfrak{F}} (ue_n)_k ,$$

and put

$$A := \{n \in \mathbb{N} \mid |f(n)| \geq 1\} .$$

Take $B \in \mathfrak{P}_f(A)$ and put

$$x := \sum_{n \in B} \frac{1}{f(n)} e_n .$$

Then

$$x \in c_0, \quad \|x\| \leq 1$$

so that

$$\|u\| \geq \|ux\| = \left\| \sum_{n \in B} \frac{1}{f(n)} ue_n \right\| \geq \lim_{k,\mathfrak{F}} \sum_{n \in B} \frac{1}{f(n)} (ue_n)_k =$$

$$= \sum_{n \in B} \frac{1}{f(n)} \lim_{k,\mathfrak{F}} (ue_n)_k = \operatorname{Card} B .$$

Since B is arbitrary,

$$\operatorname{Card} A \leq \|u\| . \qquad \blacksquare$$

Proposition 3.1.2.15 *Take $u \in \mathcal{K}(c_0, \ell^\infty)$ and let $\mathfrak{F}$ be a free ultrafilter on $\mathbb{N}$. Then for every $\varepsilon > 0$, there is an $A \in \mathfrak{F}$ and an $n_0 \in \mathbb{N}$ such that*

$$|(ue_n)_k| < \varepsilon$$

whenever $k \in A$ and $n \geq n_0$ $(n \in \mathbb{N})$.

Since $u(c_0^{\#})$ is relatively compact, there is a finite subset $\mathcal{F}$ of ℓ^∞ such that

$$u(c_0^{\#}) \subset \bigcup_{x \in \mathcal{F}} U_{\frac{\varepsilon}{3}}^{\ell^\infty}(x) .$$

Define

$$f : \mathbb{N} \longrightarrow \mathbb{K}, \quad n \longmapsto \lim_{k,\mathfrak{F}} (ue_n)_k ,$$

$$\mathcal{G} := \{x \in \mathcal{F} \mid |\lim_{k,\mathfrak{F}} x_k| < \frac{2\varepsilon}{3}\} .$$

Then there is an $A \in \mathfrak{F}$ such that

$$|x_k| < \frac{2\varepsilon}{3}$$

for every $x \in \mathcal{G}$ and $k \in A$. By Proposition 3.1.2.14, there is an n_0 such that

$$|f(n)| < \frac{\varepsilon}{3}$$

whenever $n \geq n_0$ $(n \in \mathbb{N}_0)$.

Take $n \in \mathbb{N}$, $n \geq n_0$. There is an $x \in \mathcal{F}$ such that

$$ue_n \in U_{\frac{\varepsilon}{3}}^{\ell^\infty}(x)\,.$$

Thus

$$|x_k| \leq |x_k - (ue_n)_k| + |(ue_n)_k| < \frac{\varepsilon}{3} + |(ue_n)_k|$$

for every $k \in \mathbb{N}$ and so

$$\lim_{k,\mathfrak{F}} |x_k| \leq \frac{\varepsilon}{3} + f(n) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}\,,$$

$$x \in \mathcal{G}\,.$$

It follows that

$$|(ue_n)_k| \leq |(ue_n)_k - x_k| + |x_k| < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon$$

for every $k \in A$. ■

Example 3.1.2.16 *Take $p \in \{0\} \cup [1, \infty]$. Let q be the quotient map*

$$\mathcal{L}(\ell^p) \longrightarrow \mathcal{L}(\ell^p)/\mathcal{K}(\ell^p)\,,$$

and let u_r, u_ℓ be the right and left shift operators of ℓ^p, respectively. Then

$$\|qu_r\| = \|qu_\ell\| = 1\,, \quad (qu_r)^{-1} = qu_\ell$$

and this still holds when ℓ^p is replaced by c.

That

$$(qu_r)^{-1} = qu_\ell\,,$$

follows immediately from

$$u_\ell u_r = 1\,, \quad u_r u_\ell - 1 \in \mathcal{K}(\ell^p)\,.$$

Case 1 $p \in \{0\} \cup [1, \infty[$

Take $v \in \mathcal{K}(\ell^p)$ and $\varepsilon > 0$. Since $v((\ell^p)^\#)$ is relatively compact, there is a finite subset $\mathcal{F}$ of ℓ^p such that

$$v((\ell^p)^\#) \subset \bigcup_{x \in \mathcal{F}} U_{\frac{\varepsilon}{2}}^{\ell^p}(x)\,.$$

There is an $n_0 \in \mathbb{N}$ such that

$$|x_n| < \frac{\varepsilon}{2},$$

whenever $x \in \mathcal{F}$ and every $n \in \mathbb{N}$, $n \geq n_0$. Take $y \in (\ell^p)^{\#}$. There is an $x \in \mathcal{F}$ such that

$$vy \in U_{\frac{\varepsilon}{2}}^{\ell^p}(x).$$

Thus, given $n \in \mathbb{N}$ with $n \geq n_0$,

$$|(vy)_n| \leq |(vy)_n - x_n| + |x_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

In particular, if $n := n_0 + 1$, then

$$\|u_r - v\| \geq \|(u_r - v)e_n\| = \|e_{n+1} - ve_n\| \geq$$

$$\geq |1 - (ve_n)_{n+1}| \geq 1 - |(ve_n)_{n+1}| > 1 - \varepsilon,$$

$$\|u_\ell - v\| \geq \|(u_r - v)e_n\| \geq \|e_{n-1} - ve_n\| \geq$$

$$\geq |1 - (ve_n)_{n-1}| \geq 1 - |(ve_n)_{n-1}| > 1 - \varepsilon.$$

Since v is arbitrary, it follows that

$$\|qu_r\| \geq 1 - \varepsilon, \quad \|qu_\ell\| \geq 1 - \varepsilon.$$

Since ε is also arbitrary, we deduce that

$$\|qu_r\| = \|qu_\ell\| = 1.$$

Case 2 ℓ^∞ and c

Take $v \in \mathcal{K}(\ell^\infty)$ (resp. $v \in \mathcal{K}(c)$). Let $\mathfrak{F}$ be a free ultrafilter on $\mathbb{N}$ and take $\varepsilon > 0$. By Proposition 3.1.2.15, there is an $A \in \mathfrak{F}$ and an $n_0 \in \mathbb{N}$ such that

$$|(ve_n)_k| < \varepsilon$$

whenever $k \in A$ and $n \in \mathbb{N}$, $n \geq n_0$. Take $k \in A$ with $k > n_0$. Then

$$\|u_r - v\| \geq \|(u_r - v)e_{k-1}\| = \|e_k - ve_{k-1}\| \geq$$

$$\geq |1 - (ve_{k-1})_k| \geq 1 - |(ve_{k-1})_k| > 1 - \varepsilon,$$

$$\|u_\ell - v\| \geq \|(u_\ell - v)e_{k+1}\| = \|e_k - ve_{k+1}\| \geq$$

$$\geq |1 - (ve_{k+1})_k| \geq 1 - |(ve_{k+1})_k| > 1 - \varepsilon .$$

Since v is arbitrary, it follows that

$$\|qu_r\| \geq 1 - \varepsilon , \quad \|qu_\ell\| \geq 1 - \varepsilon .$$

Since ε is also arbitrary, we deduce that

$$\|qu_r\| = \|qu_\ell\| = 1 . \qquad \blacksquare$$

Proposition 3.1.2.17 *Let $p \in \{0\} \cup [1, \infty]$. Let $(E_t)_{t\in T}$, $(F_t)_{t\in T}$ be families of Banach spaces, and E (resp. F) the Banach space defined in Proposition 1.1.2.7 b) for p and $(E_t)_{t\in T}$ (resp. $(F_t)_{t\in T}$). If*

$$(u_t)_{t\in T} \in \prod_{t\in T} \mathcal{K}(E_t, F_t)$$

such that

$$(\|u_t\|)_{t\in T} \in c_0(T) ,$$

then the map

$$u : E \longrightarrow F , \quad x \longmapsto (u_t x_t)_{t\in T}$$

is a compact operator.

It is easy to see that u is an operator. Let $(x^{(n)})_{n\in\mathbb{N}}$ be a sequence in $E^\#$. Since $\{t \in T \mid u_t \neq 0\}$ is countable, the diagonal procedure provides a subsequence $(x^{(k_n)})_{n\in\mathbb{N}}$ of $(x^{(n)})_{n\in\mathbb{N}}$ such that $(u_t x_t^{(k_n)})_{n\in\mathbb{N}}$ converges whenever $t \in T$ (Proposition 3.1.1.19 a $\Rightarrow$ b). Given $t \in T$, put

$$y_t := \lim_{n\to\infty} u_t x_t^{(k_n)} .$$

Then $y := (y_t)_{t\in T} \in F$ and $(\|y_t\|)_{t\in T} \in c_0(T)$.

Take $\varepsilon > 0$ and $p \in [1, \infty[$. There is a finite subset K of T such that

$$\sup_{t\in F\setminus K} \|u_t\| < \varepsilon , \quad \sum_{t\in T\setminus K} \|y_t\|^p < \varepsilon .$$

Furthermore, there is an $n_0 \in \mathbb{N}$ such that for every $n \in \mathbb{N}$ with $n \geq n_0$,

$$\sum_{t\in K} \|u_t x_t^{(k_n)} - y_t\|^p < \varepsilon .$$

Thus, given $n \in \mathbb{N}$ with $n \geq n_0$,

$$\begin{aligned}
\|ux^{k_n} - y\|^p &= \sum_{t\in T} \|u_t x_t^{(k_n)} - y_t\|^p = \\
&= \sum_{t\in K} \|u_t x_t^{(k_n)} - y_t\|^p + \sum_{t\in T\setminus K} \|u_t x_t^{(k_n)} - y_t\|^p \leq \\
&\leq \varepsilon + \sum_{t\in T\setminus K} \left(\|u_t\| \, \|x_t^{(k_n)}\| + \|y_t\| \right)^p \leq \\
&\leq \varepsilon + \left(\left(\sum_{t\in T\setminus K} \|u_t\|^p \, \|x_t^{(k_n)}\|^p \right)^{\frac{1}{p}} + \left(\sum_{t\in T\setminus K} \|y_t\|^p \right)^{\frac{1}{p}} \right)^p \leq \\
&\leq \varepsilon + \left(\left(\sup_{t\in T\setminus K} \|u_t\|^p \sum_{t\in T\setminus K} \|x_t^{(k_n)}\|^p \right)^{\frac{1}{p}} + \varepsilon^{\frac{1}{p}} \right)^p \leq \\
&\leq \varepsilon + \left(\varepsilon + \varepsilon^{\frac{1}{p}} \right)^p .
\end{aligned}$$

ε being arbitrary, it follows that $(ux^{k_n})_{n\in\mathbb{N}}$ converges to y. By Proposition 3.1.1.19 b $\Rightarrow$ a u is compact.

The case $p \in \{0, \infty\}$ is treated similarly. ■

Proposition 3.1.2.18 (4) (7) *Let T be an infinite group,*

$$f : T \times T \longrightarrow \{\alpha \in \mathbb{K} \mid |\alpha| = 1\},$$

$p \in \{0\} \cup [1, \infty[$, and q the conjugate exponent of p. We set

$$x * y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \sum_{s\in T} f(ts^{-1}, s) x(ts^{-1}) y(s)$$

for every $(x, y) \in \ell^p(T) \times \ell^q(T)$. We have:

a) *If $x \in c_0(T)\setminus\{0\}$, then $x * y \in \ell^\infty(T)$ for every $y \in \ell^1(T)$ and the map*

$$\ell^1(T) \longrightarrow \ell^\infty(T), \quad y \longmapsto x * y$$

is continuous but not compact.

b) *If $x \in \ell^2(T)\setminus\{0\}$ such that $x * y \in \ell^2(T)$ for every $y \in \ell^2(T)$, then the map*

$$\ell^2(T) \longrightarrow \ell^2(T), \quad y \longmapsto x * y$$

is not compact.

a) Take $y \in \ell^1(T)$. Then

$$|(x * y)(t)| \leq \|x\|_\infty \|y\|_1$$

for every $t \in T$. Hence $x * y \in \ell^\infty(T)$ and the map is continuous.

Take $t \in T$. Then

$$(x * e_t)(s) = \sum_{r \in T} f(sr^{-1}, r) x(sr^{-1}) e_t(r) = f(st^{-1}, t) x(st^{-1})$$

for every $s \in T$.

Take $t_0 \in T$ such that

$$|x(t_0)| = \|x\|_\infty .$$

We construct a sequence $(t_n)_{n \in \mathbb{N}}$ in T inductively such that

$$\|x * e_{t_m} - x * e_{t_n}\| \geq \frac{1}{2} \|x\|_\infty$$

for all $m, n \in \mathbb{N}$, $m < n$. Take $n \in \mathbb{N}$ and assume the sequence to be constructed up to n. Then for every $t \in T$ and $m \in \mathbb{N}_n$

$$\|x * e_{t_m} - x * e_t\|_\infty \geq |(x * e_{t_m})(t_0 t_m) - (x * e_t)(t_0 t_m)| \geq$$

$$\geq |f(t_0, t_m) x(t_0) - f(t_0 t_m t^{-1}, t) x(t_0 t_m t^{-1})| \geq$$

$$\geq \|x\|_\infty - |x(t_0 t_m t^{-1})| .$$

Since there is an $S \in \mathfrak{P}_f(T)$ such that

$$|x(t_0 t_m t^{-1})| \leq \frac{1}{2} \|x\|_\infty$$

for every $t \in T \backslash S$ and $m \in \mathbb{N}_n$ we can take t_{n+1} arbitrarily in $T \backslash S$. By Proposition 3.1.1.19 a $\Rightarrow$ b, the map is not compact.

b) Assume the map is compact. By Proposition 3.1.1.11 (and Proposition 1.1.2.6 b),e)), the map

$$\ell^1(T) \longrightarrow \ell^\infty(T), \quad y \longmapsto x * y$$

is compact which contradicts a). ■

3.1.3 Fredholm Operators

Definition 3.1.3.1 (0) *Let E, F be Banach spaces. The operator $u : E \to F$ is called a* ***Fredholm operator*** *if* $\operatorname{Ker} u$ *and* $\operatorname{Coker} u$ *are finite-dimensional. The number*

$$\operatorname{Ind} u := \operatorname{Dim} \operatorname{Ker} u - \operatorname{Dim} \operatorname{Coker} u$$

is called ***the index of*** *u. We write by $\mathcal{F}(E, F)$ for the set of Fredholm operators $E \to F$ and define*

$$\mathcal{F}(E) := \mathcal{F}(E, E) .$$

The right and left shift operators on ℓ^p ($p \in [1, \infty] \cup \{0\}$) are examples of Fredholm operators (Example 1.2.2.9).

The index was introduced by F. Noether (1921) in the case of singular integral equations. For selfadjoint Fredholm operators on a Hilbert space it was defined by Carleman (1923) under some special suppositions and by J. von Neumann (1929) in the general case.

Proposition 3.1.3.2 (0) *Let E, F be Banach spaces and $u \in \mathcal{F}(E, F)$. Then* $\operatorname{Im} u$ *is a complemented subspace (and therefore a closed subspace) of F and there is a projection of E onto* $\operatorname{Ker} u$ *and a projection of F onto* $\operatorname{Im} u$.

The existence of a projection of E onto $\operatorname{Ker} u$ follows from Corollary 1.3.3.2 b). By Corollary 1.4.2.7, $\operatorname{Im} u$ is a closed subspace and a complemented subspace of F. By Murray's Theorem (Theorem 1.2.5.8 a $\Rightarrow$ b), there is a projection of F onto $\operatorname{Im} u$. ■

Proposition 3.1.3.3 (0) *Let E, F, G, H be Banach spaces and take $u \in \mathcal{L}(E, F)$, $v \in \mathcal{L}(F, G)$, $w \in \mathcal{L}(G, H)$. If* $\operatorname{Ker}(w \circ v)$ *and* $G/\operatorname{Im}(v \circ u)$ *are finite-dimensional then $v \in \mathcal{F}(F, G)$.*

The assertion follows from

$$\operatorname{Ker} v \subset \operatorname{Ker}(w \circ v) , \quad \operatorname{Im} v \supset \operatorname{Im}(v \circ u) . \qquad ■$$

Proposition 3.1.3.4 (0) *Let E, F be Banach spaces and take $u \in \mathcal{F}(E, F)$. Then*

$$\operatorname{Dim} \operatorname{Ker} u' = \operatorname{Dim} \operatorname{Coker} u ,$$

$$\operatorname{Dim}\operatorname{Coker} u' = \operatorname{Dim}\operatorname{Ker} u\,,$$

$$u' \in \mathcal{F}(F', E')\,, \quad \operatorname{Ind} u' = -\operatorname{Ind} u\,,$$

$$\operatorname{Im} u = {}^{\circ}(\operatorname{Ker} u')\,.$$

If $\operatorname{Ind} u = 0$, *then*

$$\operatorname{Dim}\operatorname{Ker} u = \operatorname{Dim}\operatorname{Ker} u' = \operatorname{Dim}\operatorname{Coker} u = \operatorname{Dim}\operatorname{Coker} u' < \infty\,.$$

Now

$$\operatorname{Im} u = {}^{\circ}(\operatorname{Ker} u')$$

follows from Proposition 1.3.5.8 and Proposition 3.1.3.2. We have

$$\operatorname{Dim}\operatorname{Coker} u = \operatorname{Dim}(\operatorname{Coker} u)'\,,$$

$$\operatorname{Dim}\operatorname{Ker} u = \operatorname{Dim}(\operatorname{Ker} u)'$$

(Corollary 1.2.4.10),

$$\operatorname{Dim}(\operatorname{Coker} u)' = \operatorname{Dim}\operatorname{Ker} u'$$

(Corollary 1.3.5.10), and

$$\operatorname{Dim}\operatorname{Coker} u' = \operatorname{Dim}(\operatorname{Ker} u)'$$

(Proposition 1.4.2.9 c)) which proves the assertion. ■

Corollary 3.1.3.5 (0) *Let* E, F *be Banach spaces and take* $u \in \mathcal{L}(E, F)$ *such that* $\operatorname{Im} u$ *is closed or* E *is reflexive. Then* u *is Fredholm iff* u' *is Fredholm.*

Assume that u' is Fredholm. By Proposition 3.1.3.4, $u'' \in \mathcal{F}(E'', F'')$. Now

$$j_F \circ u = u'' \circ j_E$$

(Proposition 1.3.6.16), so that

$$\operatorname{Ker} u \subset \overset{-1}{j_E}(\operatorname{Ker} u'')$$

and $\operatorname{Ker} u$ is finite-dimensional. Moreover, if E is reflexive then

$$\operatorname{Im} u = \overset{-1}{j_F}(\operatorname{Im} u'')$$

so that $\operatorname{Im} u$ is closed. By Corollary 1.3.5.10, $(F/\operatorname{Im} u)'$ is isomorphic to $\operatorname{Ker} u'$, which is finite-dimensional. Hence $F/\operatorname{Im} u$ is finite-dimensional (Corollary 1.3.6.4) and u is thus Fredholm.

The converse was proved in Proposition 3.1.3.4. ■

Lemma 3.1.3.6 (0) *Let E, F, G be vector spaces. Take $u : E \to F$ linear and injective and $v : F \to G$ linear and surjective such that*

$$\operatorname{Im} u = \operatorname{Ker} v .$$

Then F is finite-dimensional iff E and G are finite-dimensional and in this case

$$\operatorname{Dim} F = \operatorname{Dim} E + \operatorname{Dim} G .$$

Let $(x_\iota)_{\iota \in I}$ be an algebraic basis for E. There is a family $(y_\iota)_{\iota \in J}$ in F such that $(y_\iota)_{\iota \in J}$ together with $(ux_\iota)_{\iota \in I}$ forms an algebraic basis for F. We show that $(vy_\iota)_{\iota \in J}$ is an algebraic basis for G.

Let $(\alpha_\iota)_{\iota \in J}$ be a family in $\mathbb{K}$ such that

$$\{\iota \in J \mid \alpha_\iota \neq 0\}$$

is finite and

$$\sum_{\iota \in J} \alpha_\iota v y_\iota = 0 .$$

Then

$$v\left(\sum_{\iota \in J} \alpha_\iota y_\iota\right) = 0 ,$$

so that

$$\sum_{\iota \in J} \alpha_\iota y_\iota \in \operatorname{Ker} v = \operatorname{Im} u .$$

Hence $\alpha_\iota = 0$ for all $\iota \in J$, i.e. $(vy_\iota)_{\iota \in J}$ is linearly independent.

Take $z \in G$. There is a $y \in F$ such that $vy = z$. There are families $(\beta_\iota)_{\iota \in I}$, $(\gamma_\iota)_{\iota \in J}$ in $\mathbb{K}$ such that

$$\{\iota \in I \mid \beta_\iota \neq 0\}, \quad \{\iota \in J \mid \gamma_\iota \neq 0\}$$

are finite and

$$y = \sum_{\iota \in I} \beta_\iota u x_\iota + \sum_{\iota \in J} \gamma_\iota y_\iota .$$

Thus

$$z = vy = \sum_{\iota \in J} \gamma_\iota v y_\iota .$$

Hence $(vy_\iota)_{\iota\in J}$ is an algebraic basis for G.

Thus

$$\operatorname{Dim} E = \operatorname{Card} I\,, \quad \operatorname{Dim} G = \operatorname{Card} J\,,$$

$$\operatorname{Dim} F = \operatorname{Card} I + \operatorname{Card} J\,,$$

proving the assertion. ■

Proposition 3.1.3.7 (0) (Atkinson, 1951) *Let E, F, G be Banach spaces and take $u \in \mathcal{F}(E,F)$, $v \in \mathcal{F}(F,G)$. Then $v \circ u \in \mathcal{F}(E,G)$ and*

$$\operatorname{Ind}(v \circ u) = \operatorname{Ind} u + \operatorname{Ind} v\,.$$

First consider the maps

$$\operatorname{Ker} u \longrightarrow \operatorname{Ker}(v \circ u) \xrightarrow{\overline{u}} (\operatorname{Im} u) \cap (\operatorname{Ker} v)\,,$$

$$(\operatorname{Im} u) \cap (\operatorname{Ker} v) \longrightarrow \operatorname{Ker} v \longrightarrow (\operatorname{Ker} v)/((\operatorname{Im} u) \cap (\operatorname{Ker} v))\,,$$

where $\overline{u}$ is the map defined by u and the others are inclusion maps or quotient maps. By Lemma 3.1.3.6, all vector spaces appearing above are finite-dimenisonal and

$$\operatorname{Dim} \operatorname{Ker}(v \circ u) = \operatorname{Dim} \operatorname{Ker} u + \operatorname{Dim}((\operatorname{Im} u) \cap (\operatorname{Ker} v))\,,$$

$$\operatorname{Dim} \operatorname{Ker} v = \operatorname{Dim}((\operatorname{Im} u) \cap (\operatorname{Ker} v)) + \operatorname{Dim}(\operatorname{Ker} v)/((\operatorname{Im} u) \cap (\operatorname{Ker} v))\,.$$

Now consider the maps

$$\operatorname{Ker} v \longrightarrow F \xrightarrow{\overline{v}} \operatorname{Im} v\,,$$

$$\operatorname{Im} v \longrightarrow G \longrightarrow G/\operatorname{Im} v\,,$$

where $\overline{v}$ is the map defined by v and the others are inclusion maps or quotient maps. By factorization, we get the maps

$$(\operatorname{Ker} v)/((\operatorname{Im} u) \cap (\operatorname{Ker} v)) \longrightarrow F/\operatorname{Im} u \longrightarrow \operatorname{Im} v/\operatorname{Im}(v \circ u)\,,$$

$$\operatorname{Im} v/\operatorname{Im}(v \circ u) \longrightarrow G/\operatorname{Im}(v \circ u) \longrightarrow G/\operatorname{Im} v\,.$$

By Lemma 3.1.3.6, all vector spaces appearing above are finite–dimensional and

$$\operatorname{Dim} F/\operatorname{Im} u = \operatorname{Dim}(\operatorname{Ker} v)/((\operatorname{Im} u) \cap (\operatorname{Ker} v)) + \operatorname{Dim} \operatorname{Im} v/\operatorname{Im}(v \circ u) ,$$

$$\operatorname{Dim} G/\operatorname{Im}(v \circ u) = \operatorname{Dim} \operatorname{Im} v/\operatorname{Im}(v \circ u) + \operatorname{Dim} G/\operatorname{Im} v .$$

Hence $v \circ u$ is Fredholm and

$$\operatorname{Ind}(v \circ u) = \operatorname{Dim} \operatorname{Ker}(v \circ u) - \operatorname{Dim} G/\operatorname{Im}(v \circ u) =$$

$$= \operatorname{Dim} \operatorname{Ker} u + \operatorname{Dim}((\operatorname{Im} u) \cap (\operatorname{Ker} v)) - \operatorname{Dim} \operatorname{Im} v/\operatorname{Im}(v \circ u) - \operatorname{Dim} G/\operatorname{Im} v =$$

$$= \operatorname{Dim} \operatorname{Ker} u + \operatorname{Dim} \operatorname{Ker} v - \operatorname{Dim} F/\operatorname{Im} u - \operatorname{Dim} G/\operatorname{Im} v =$$

$$= \operatorname{Ind} u + \operatorname{Ind} v . \qquad \blacksquare$$

Theorem 3.1.3.8 (0) (F. Riesz, 1918) *If E is a Banach space, then $1-u$ is a Fredholm operator whenever $u \in \mathcal{K}(E)$.*

Define

$$v := 1 - u , \quad F := \operatorname{Ker} v , \quad G := \operatorname{Im} v .$$

Then

$$x - ux = vx = 0 ,$$

$$x = ux \in u(F^{\#}) \subset \overline{u(F^{\#})}$$

for every $x \in F^{\#}$, such that

$$F^{\#} \subset \overline{u(F^{\#})} .$$

Hence $F^{\#}$ is compact and F is finite–dimensional (Theorem 1.1.4.4). By Corollary 1.3.3.2 b), there is a projection w of E onto F. Define

$$H := \operatorname{Ker} w ,$$

$$v_0 : H \longrightarrow G , \quad x \longmapsto vx .$$

Then H is a Banach space and v_0 is linear and continuous.

Step 1 v_0 is injective

Take $x \in H$ with

$$v_0 x = 0 .$$

Then

$$vx = 0 ,$$

$$x \in F ,$$

$$x = wx = 0 .$$

Hence v_0 is injective.

Step 2 v_0 is surjective

Take $x \in G$. There is a $y \in E$ with

$$x = vy .$$

Put

$$z := y - wy .$$

Then

$$wz = wy - w(wy) = 0 ,$$

$$z \in H .$$

Since $wy \in F$,

$$v_0 z = vz = vy - vwy = vy = x .$$

Hence v_0 is surjective.

Step 3 v_0^{-1} is continuous

Let $(y_n)_{n \in \mathbb{N}}$ be a sequence in G converging to 0. For $n \in \mathbb{N}$, define

$$x_n := v_0^{-1} y_n .$$

Assume that $(x_n)_{n\in\mathbb{N}}$ does not converge to 0. Replacing $(y_n)_{n\in\mathbb{N}}$ with a subsequence if necessary, we may assume that there is an $\varepsilon > 0$ such that

$$\|x_n\| \geq \varepsilon$$

for every $n \in \mathbb{N}$. Since $\left(\frac{1}{\|x_n\|}x_n\right)_{n\in\mathbb{N}}$ is a bounded sequence, we may assume (by replacing $(y_n)_{n\in\mathbb{N}}$ with a subsequence if necessary), that $\left(u\left(\frac{1}{\|x_n\|}x_n\right)\right)_{n\in\mathbb{N}}$ converges (Proposition 3.1.1.19 a $\Rightarrow$ b). Put

$$x := \lim_{n\to\infty} u\left(\frac{1}{\|x_n\|}x_n\right).$$

Then

$$\left\|\frac{1}{\|x_n\|}x_n - u\left(\frac{1}{\|x_n\|}x_n\right)\right\| = \left\|v\left(\frac{1}{\|x_n\|}x_n\right)\right\| =$$

$$= \frac{1}{\|x_n\|}\|v_0(x_n)\| = \frac{1}{\|x_n\|}\|y_n\| \leq \frac{\|y_n\|}{\varepsilon}$$

for every $n \in \mathbb{N}$ and therefore

$$\lim_{n\to\infty}\left(\frac{1}{\|x_n\|}x_n - u\left(\frac{1}{\|x_n\|}x_n\right)\right) = 0,$$

$$x = \lim_{n\to\infty}\frac{1}{\|x_n\|}x_n \in H,$$

$$\|x\| = 1,$$

$$v_0 x = \lim_{n\to\infty} v_0\left(\frac{1}{\|x_n\|}x_n\right) = \lim_{n\to\infty}\frac{1}{\|x_n\|}\|y_n\| = 0.$$

Thus, by Step 1, $x = 0$, which is a contradiction. Hence $(x_n)_{n\in\mathbb{N}}$ converges to 0 and v_0^{-1} is continuous.

Step 4 G is a closed set of E

By Steps 1 to 3, v_0 is an isomorphism. Since H is complete, G is also complete. Hence G is a closed set of E.

Step 5 $E/\operatorname{Im} v$ is finite-dimensional

Assume the contrary. By Corollary 1.1.4.3, there is a sequence $(X_n)_{n\in\mathbb{N}}$ in $E/\operatorname{Im} v$ such that

$$\|X_n\| = 1\,, \quad \|X_m - X_n\| \geq \frac{1}{2}$$

whenever $m, n \in \mathbb{N}$ are distinct. For each $n \in \mathbb{N}$, choose $x_n \in X_n$ such that

$$\|x_n\| < 2\,.$$

Replacing $(X_n)_{n\in\mathbb{N}}$ with a subsequence if nesessary, we may assume that $(ux_n)_{n\in\mathbb{N}}$ converges (Proposition 3.1.1.19 a $\Rightarrow$ b). Then

$$x_n - ux_n = vx_n \in \operatorname{Im} v\,,$$

$$ux_n \in X_n$$

for every $n \in \mathbb{N}$. It follows that $(X_n)_{n\in\mathbb{N}}$ converges, which is a contradiction. Hence $E/\operatorname{Im} v$ is finite-dimensional. ■

Theorem 3.1.3.9 (0) (Dieudonné, 1943) *Let E, F be Banach spaces. Then $\mathcal{F}(E,F)$ is an open set of $\mathcal{L}(E,F)$, the connected components of $\mathcal{F}(E,F)$ are open, and the map*

$$\mathcal{F}(E,F) \longrightarrow \mathbb{Z}\,, \quad u \longmapsto \operatorname{Ind} u$$

is continuous (i.e. it is constant on each connected component of $\mathcal{F}(E,F)$).

Take $u \in \mathcal{F}(E,F)$. $\operatorname{Ker} u$ being finite-dimensional, it is a complemented subspace of E (Corollary 1.3.3.2 c)). Let G be a complement of $\operatorname{Ker} u$ in E. Since $\operatorname{Im} u$ has finite codimension, there is a finite-dimensional subspace H of F such that

$$F = H \oplus \operatorname{Im} u$$

(Corollary 1.2.5.6).

Given $v \in \mathcal{L}(E,F)$, define

$$\overline{v} : G \times H \longrightarrow F\,, \quad (x,y) \longmapsto vx + y\,.$$

$\overline{v}$ is linear and

$$\|\overline{v}(x,y)\| \leq \|vx\| + \|y\| \leq \|v\|\,\|x\| + \|y\| \leq$$

$$\leq (\|v\|^2 + 1)^{\frac{1}{2}}((\|x\|^2 + \|y\|^2)^{\frac{1}{2}} = (\|v\|^2 + 1)^{\frac{1}{2}}\|(x, y)\|_2$$

for every $(x, y) \in G \times H$. Hence $\overline{v}$ is continuous.

Define

$$\mathcal{U} := \{v \in \mathcal{L}(E, F) \mid \overline{v} \text{ is an isomorphism}\},$$

$$\mathcal{V} := \{w \in \mathcal{L}(G \times H, F) \mid w \text{ is an isomorphism}\},$$

$$f : \mathcal{L}(E, F) \longrightarrow \mathcal{L}(G \times H, F), \quad v \longmapsto \overline{v}.$$

Take $v, w \in \mathcal{L}(E, F)$. Then

$$\|(\overline{v} - \overline{w})(x, y)\| = \|vx - wx\| \leq \|v - w\|\,\|x\| \leq \|v - w\|\,\|(x, y)\|_2$$

for every $(x, y) \in G \times H$, so that

$$\|\overline{v} - \overline{w}\| \leq \|v - w\|.$$

Hence f is uniformly continuous. Since $\mathcal{V}$ is open (Corollary 2.2.4.15) and

$$\mathcal{U} = \overset{-1}{f}(\mathcal{V}),$$

$\mathcal{U}$ is open.

Take $z \in F$. Then

$$z = z_0 + y$$

for some $(z_0, y) \in (\operatorname{Im} u) \times H$. Take $x_1 \in E$ with $ux_1 = z_0$. There is an $(x_2, x) \in (\operatorname{Ker} u) \times G$ such that

$$x_1 = x_2 + x.$$

Then $(x, y) \in G \times H$ and

$$\overline{u}(x, y) = ux + y = ux_1 + y = z_0 + y = z.$$

Hence $\overline{u}$ is surjective.

Suppose that

$$(x, y) \in \operatorname{Ker} \overline{u}.$$

Then $(ux, y) \in (\operatorname{Im} u) \times H$ and

$$ux + y = 0.$$

It follows in succession that

$$ux = y = 0,$$

$$x \in \operatorname{Ker} u,$$

$$x = 0.$$

Hence $\overline{u}$ is injective. By the Principle of Inverse Operators, $\overline{u}$ is an isomorphism, i.e. $u \in \mathcal{U}$.

Take $v \in \mathcal{U}$. Then

$$\overline{v}(x, 0) = vx + 0 = 0$$

and so

$$x = 0$$

for every $x \in G \cap \operatorname{Ker} v$. Hence $\operatorname{Ker} v$ is finite-dimensional and the map

$$G \times \operatorname{Ker} v \longrightarrow E, \quad (x_1, x_2) \longmapsto x_1 + x_2$$

is injective. Let Z be a vector subspace of E, such that the map

$$Z \times G \times \operatorname{Ker} v \longrightarrow E, \quad (z, x_1, x_2) \longmapsto z + x_1 + x_2$$

is bijective. Then Z is finite-dimensional,

$$\operatorname{Dim} Z + \operatorname{Dim} \operatorname{Ker} v = \operatorname{Dim} \operatorname{Ker} u,$$

and the restriction of v on $Z \times G$ is injective. Also

$$F = \overline{v}(G \times H) = v(G) \oplus H.$$

Hence $F/v(G)$ is finite-dimensional. Since

$$v(G) \subset \operatorname{Im} v,$$

$F/\operatorname{Im} v$ is finite-dimensional. Thus $v \in \mathcal{F}(E, F)$, $\mathcal{U} \subset \mathcal{F}(E, F)$, $\mathcal{F}(E, F)$ is a neighbourhood of u, and $\mathcal{F}(E, F)$ is open. Since $\mathcal{L}(E, F)$ is locally connected, the connected componentes of $\mathcal{F}(E, F)$ are also open. The map

$$Z \times G \longrightarrow \operatorname{Im} v\,, \quad (x_1, x_2) \longmapsto v(x_1 + x_2)$$

is bijective, so that

$$\operatorname{Dim} v(Z) = \operatorname{Dim} Z$$

and

$$\operatorname{Dim} F/\operatorname{Im} v = \operatorname{Dim} H - \operatorname{Dim} Z = \operatorname{Dim} F/\operatorname{Im} u - \operatorname{Dim} Z\,,$$

$$\operatorname{Ind} v = \operatorname{Dim}\operatorname{Ker} v - \operatorname{Dim}\operatorname{Coker} v =$$

$$= \operatorname{Dim}\operatorname{Ker} u - \operatorname{Dim}\operatorname{Coker} u = \operatorname{Ind} u\,.$$

Hence the map

$$\mathcal{F}(E,F) \longrightarrow \mathbb{Z}\,, \quad u \longmapsto \operatorname{Ind} u$$

is continuous. ■

Remark. Observe that we have also proved the map

$$\mathcal{F}(E,F) \longrightarrow \mathbb{N}\,, \quad a \longmapsto \operatorname{Dim}\operatorname{Ker} u \quad (\text{resp. } \operatorname{Dim}\operatorname{Coker} u)$$

to be upper semicontinuous.

Theorem 3.1.3.10 (0) *Let E, F be Banach spaces, $u \in \mathcal{F}(E,F)$, p a projection of E onto $\operatorname{Ker} u$, and q a projection of F onto $\operatorname{Im} u$ (p and q exist by Proposition 3.1.3.2). Then there is a $v \in \mathcal{F}(F,E)$ such that*

$$\operatorname{Ind} v = -\operatorname{Ind} u\,,$$

$$1_E - v \circ u = p\,, \quad u \circ v = q\,,$$

$$1_E - v \circ u \in \mathcal{L}_f(E) \subset \mathcal{K}(E)\,, \quad 1_F - u \circ v \in \mathcal{L}_f(F) \subset \mathcal{K}(F)\,,$$

$$\operatorname{Ind} u \geq 0 \Longrightarrow v \text{ injective and } \operatorname{Ind} u \leq 0 \Longrightarrow v \text{ surjective.}$$

In particular, if $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in E such that

$$\|x_m - x_n\| \geq 1$$

for all distinct $m, n \in \mathbb{N}$, then $(ux_n)_{n\in\mathbb{N}}$ does not converge.

By Proposition 1.4.2.13, there is a $w \in \mathcal{L}(F, E)$ with

$$1_E - w \circ u = p, \quad u \circ w = q, \quad w \circ q = w .$$

Now

$$\operatorname{Dim} \operatorname{Ker} q = \operatorname{Dim} F/q(F) = \operatorname{Dim} \operatorname{Coker} u < \infty$$

(Theorem 1.2.5.8 b $\Rightarrow$ a, Proposition 1.2.5.2 a $\Rightarrow$ b), so

$$\operatorname{Dim} \operatorname{Ker} u - \operatorname{Dim} \operatorname{Ker} q = \operatorname{Ind} u .$$

Let

$$w_0 : \operatorname{Ker} q \longrightarrow \operatorname{Ker} u$$

be a linear map. We assume that w_0 is injective (surjective) whenever $\operatorname{Ind} u \geq 0$ ($\operatorname{Ind} u \leq 0$). Then

$$\operatorname{Dim} \operatorname{Ker} q = \operatorname{Dim} \operatorname{Ker} w_0 + \operatorname{Dim} w_0(\operatorname{Ker} q) ,$$

such that

$$\operatorname{Ind} u = \operatorname{Dim} \operatorname{Ker} u - \operatorname{Dim} \operatorname{Ker} w_0 - \operatorname{Dim} w_0(\operatorname{Ker} q) .$$

Define

$$v : F \longrightarrow E , \quad y \longmapsto wqy + w_0(y - qy) .$$

Then v is linear and continuous. We have

$$v \circ u = w \circ u , \quad u \circ v = u \circ w = q ,$$

hence

$$1_E - v \circ u = 1_E - w \circ u = p \in \mathcal{L}_f(E) \subset \mathcal{K}(E) ,$$

$$1_F - u \circ v = 1_F - q \in \mathcal{L}_f(F) \subset \mathcal{K}(F) ,$$

(Proposition 3.1.1.4). We have

$$E = (\operatorname{Im} p) \oplus (\operatorname{Ker} p) = (\operatorname{Ker} u) \oplus (\operatorname{Ker} p) ,$$

$$F = (\operatorname{Ker} q) \oplus (\operatorname{Im} q) = (\operatorname{Ker} q) \oplus (\operatorname{Im} u)$$

(Theorem 1.2.5.8 b ⇒ a), and the map

$$\operatorname{Im} u \longrightarrow \operatorname{Ker} p\,, \quad y \longmapsto wy$$

is bijective. It follows that

$$\operatorname{Ker} v = \operatorname{Ker} w_0\,,$$

$$\operatorname{Dim} \operatorname{Ker} v = \operatorname{Dim} \operatorname{Ker} w_0 < \infty\,,$$

$$\operatorname{Im} v = (\operatorname{Ker} p) \oplus (w_0(\operatorname{Ker} q))\,,$$

$$\operatorname{Dim} \operatorname{Coker} v = \operatorname{Dim} E/\operatorname{Ker} p - \operatorname{Dim} w_0(\operatorname{Ker} q) =$$

$$= \operatorname{Dim} \operatorname{Ker} u - \operatorname{Dim} w_0(\operatorname{Ker} q) < \infty\,,$$

i.e. v is a Fredholm operator and

$$\operatorname{Ind} v = \operatorname{Dim} \operatorname{Ker} v - \operatorname{Dim} \operatorname{Coker} v =$$

$$= \operatorname{Dim} \operatorname{Ker} w_0 - \operatorname{Dim} \operatorname{Ker} u + \operatorname{Dim} w_0(\operatorname{Ker} q) = -\operatorname{Ind} u\,.$$

Moreover v is injective (surjective) whenever $\operatorname{Ind} u \geq 0$ $(\operatorname{Ind} u \leq 0)$.

Now we prove the last assertion. By Proposition 3.1.1.19 a ⇒ b, there is a subsequence $(y_n)_{n\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ such that $((1_E - v\circ u)y_n)_{n\in\mathbb{N}}$ converges. Since $(y_n)_{n\in\mathbb{N}}$ does not converge, it follows that $((vuy_n)_{n\in\mathbb{N}}$ does not converge. Hence $(uy_n)_{n\in\mathbb{N}}$ and a fortiori $(ux_n)_{n\in\mathbb{N}}$ does not converge. ■

Theorem 3.1.3.11 (0) (F.V. Atkinson, B. Yood, 1951) *Let E, F be Banach spaces. The following are equivalent for $u \in \mathcal{L}(E,F)$:*

a) $u \in \mathcal{F}(E,F))$.

b) *There are $v, w \in \mathcal{L}(F,E)$ such that*

$$1_E - v \circ u \in \mathcal{K}(E)\,.$$

$$1_F - u \circ w \in \mathcal{K}(F)\,.$$

a $\Rightarrow$ b follows from Theorem 3.1.3.10.

b $\Rightarrow$ a. It follows from

$$v \circ u = 1_E - (1_E - v \circ u),$$

$$u \circ w = 1_F - (1_F - u \circ w),$$

and Theorem 3.1.3.8 that $v \circ u$ and $u \circ w$ are Fredhold operators. By Proposition 3.1.3.3, u is a Fredholm operator. ∎

Corollary 3.1.3.12 (0) (Mihlin, 1947, Atkinson, Gohberg, Yood, 1951) *Let E, F be Banach spaces. Take $u \in \mathcal{F}(E,F)$ and $v \in \mathcal{K}(E,F)$. Then $u+v \in \mathcal{F}(E,F)$, u and $u+v$ belong to the same connected component of $\mathcal{F}(E,F)$, and*

$$\operatorname{Ind}(u+v) = \operatorname{Ind} u.$$

By Theorem 3.1.3.10, there is a $w \in \mathcal{L}(E,F)$ such that

$$1_E - w \circ u \in \mathcal{K}(E),$$

$$1_F - u \circ w \in \mathcal{K}(F).$$

We deduce

$$1_E - w \circ (u+v) = 1_E - w \circ u - w \circ v \in \mathcal{K}(E),$$

$$1_F - (u+v) \circ w = 1_F - u \circ w - v \circ w \in \mathcal{K}(F)$$

(Theorem 3.1.1.3, Proposition 3.1.1.11). By Theorem 3.1.3.11 b $\Rightarrow$ a, $u+v \in \mathcal{F}(E,F)$.

The map

$$\mathbb{K} \longrightarrow \mathcal{F}(E,F), \quad \alpha \longmapsto u + \alpha v$$

is continuous and so u and $u+v$ belong to the same connected component of $\mathcal{F}(E,F)$. It follows that the map

$$\mathbb{K} \longrightarrow \mathbb{Z}, \quad \alpha \longmapsto \operatorname{Ind}(u+\alpha v)$$

is continuous (Theorem 3.1.3.9), i.e. it is constant. Hence

$$\operatorname{Ind}(u+v) = \operatorname{Ind} u.$$ ∎

Corollary 3.1.3.13 (0) *If E is a finite-dimensional Banach space then every $u \in \mathcal{L}(E)$ is a Fredholm operator with index 0.*

By Proposition 3.1.1.4, $u - 1$ is compact and by Corollary 3.1.3.12,

$$u = (u - 1) + 1$$

is Fredholm with index 0. ■

Corollary 3.1.3.14 *Let E, F, G be Banach spaces. Take $u \in \mathcal{L}(E, F)$ and $v \in \mathcal{L}(F, G)$. If $v \circ u$ is Fredholm, then the following are equivalent:*

a) $u \in \mathcal{F}(E, F)$.

b) $v \in \mathcal{F}(F, G)$.

c) $F/\operatorname{Im} u$ *is finite-dimensional.*

d) $\operatorname{Ker} v$ *is finite-dimensional.*

a $\Rightarrow$ b. By Theorem 3.1.3.10, there is a $w \in \mathcal{F}(F, E)$ such that

$$1_F - u \circ w \in \mathcal{K}(F).$$

Hence

$$v - (v \circ u) \circ w = v \circ (1_F - u \circ w) \in \mathcal{K}(F, G)$$

(Proposition 3.1.1.11). Since $(v \circ u) \circ w$ is Fredholm (Proposition 3.1.3.7), it follows from

$$v = (v \circ u) \circ w + (v - (v \circ u) \circ w)$$

that v is Fredholm (Corollary 3.1.3.12).

b $\Rightarrow$ a. By Theorem 3.1.3.10, there is a $w \in \mathcal{F}(G, F)$ with

$$1_F - w \circ v \in \mathcal{K}(F).$$

Hence

$$u - w \circ (v \circ u) = (1_F - w \circ v) \circ u \in \mathcal{K}(E, F)$$

(Proposition 3.1.1.11). Since $w \circ (v \circ u)$ is Fredholm (Proposition 3.1.3.7), it follows from

$$u = w \circ (v \circ u) + (u - w \circ (v \circ u))$$

that u is Fredholm (Corollary 3.1.3.12).

a $\Rightarrow$ c and b $\Rightarrow$ d are trivial.

c $\Rightarrow$ a. Since

$$\operatorname{Ker} u \subset \operatorname{Ker}(v \circ u),$$

$\operatorname{Ker} u$ is finite-dimensional.

d $\Rightarrow$ b. Since

$$\operatorname{Im} v \supset \operatorname{Im} v \circ u,$$

$G/\operatorname{Im} v$ is finite-dimensional. ■

Corollary 3.1.3.15 *If E, F are Banach spaces, then any $u \in \mathcal{F}(E,F)$ with $\operatorname{Ind} u = 0$ may be connected by a continuous path in $\mathcal{F}(E,F)$ with an isomorphism $E \to F$.*

By Theorem 3.1.3.10. there is a bijective $v \in \mathcal{L}(F,E)$ such that

$$1_E - v \circ u \in \mathcal{K}(E).$$

By the Principle of Inverse Operators, v is an isomorphism. Now

$$v^{-1} - u = v^{-1} \circ (1_E - v \circ u) \in \mathcal{K}(E)$$

(Proposition 3.1.1.11). Hence

$$[0,1] \longrightarrow \mathcal{F}(E,F), \quad \alpha \longmapsto u + \alpha(v^{-1} - u)$$

(Corollary 3.1.3.12) is a continuous path in $\mathcal{F}(E,F)$ connecting u to v^{-1}. ■

Proposition 3.1.3.16 *Let E be a Banach space and take $u \in \mathcal{K}(E)$. The following are equivalent for $\alpha \in \mathbb{K}\backslash\{0\}$:*

a) $\alpha \in \sigma(u)$.

b) $\alpha \in \sigma(u)$ *and there are projections p, q of E such that*

$$\operatorname{Dim} \operatorname{Coker} u = \operatorname{Dim} \operatorname{Im} q,$$

$$up = \alpha p, \quad qu = \alpha q.$$

c) *There is a* $v \in \mathcal{L}(E)\backslash\{0\}$ *such that*

$$uv = \alpha v .$$

d) *There is a* $v \in \mathcal{L}(E)\backslash\{0\}$ *such that*

$$vu = \alpha v .$$

a $\Rightarrow$ b. Define

$$F := \mathrm{Ker}(\alpha 1 - u) , \quad G = \mathrm{Im}(\alpha 1 - u) .$$

By Corollary 3.1.3.12, $\alpha 1 - u$ is a Fredholm operator with index 0. By Proposition 3.1.3.2, there are projections p, r of E onto F and G, respectively. Define

$$q := 1 - r .$$

Then q is a projection of E and

$$\mathrm{Im}\, q = \mathrm{Ker}\, r$$

is isomorphic to $\mathrm{Coker}\, u$ (Proposition 1.2.5.2 a $\Rightarrow$ c). Hence

$$\mathrm{Dim\,Coker}\, u = \mathrm{Dim\,Im}\, q .$$

We have

$$0 = (\alpha 1 - u)p = \alpha p - up , \quad up = \alpha p ,$$

$$0 = q(\alpha 1 - u) = \alpha q - qu , \quad qu = \alpha q .$$

b $\Rightarrow$ c & d is trivial.
c $\Rightarrow$ a (d $\Rightarrow$ a). Define

$$\widetilde{u} : \mathcal{L}(E) \longrightarrow \mathcal{L}(E) , \quad w \longmapsto uw \quad (\text{resp. } wu) .$$

Then $\alpha \in \sigma(\widetilde{u})$. By Proposition 2.2.1.17, $\alpha \in \sigma(u)$. ■

Theorem 3.1.3.17 (6) (F. Riesz, 1918) *Let* E *be a Banach space and take* $u \in \mathcal{L}(E)$ *with* $1 - u \in \mathcal{K}(E)$. *Given* $n \in \mathbb{N} \cup \{0\}$, *define*

$$F_n := \mathrm{Ker}\, u^n , \quad G_n := \mathrm{Im}\, u^n .$$

There is a $p \in \mathbb{N} \cup \{0\}$ *with:*

a) $F_{n-1} \underset{\neq}{\subset} F_n$ *and* $G_{n-1} \underset{\neq}{\supset} G_n$ *for every* $n \in \mathbb{N}_p$ *with* $n \leq p$.

b) $F_n = F_p$ *and* $G_n = G_p$ *for every* $n \in \mathbb{N}$ *with* $n \geq p$.

c) *The map*

$$G_p \longrightarrow G_p, \quad x \longmapsto ux$$

is an isomorphism.

d) F_p *is finite-dimensional and*

$$E = F_p \oplus G_p.$$

e) *If* v *is the projection of* E *onto* F_p *with* $\operatorname{Ker} v = G_p$ *and*

$$i : G_p \longrightarrow E, \quad x \longmapsto x,$$

$$j : G_p \longrightarrow G_p, \quad x \longmapsto ux,$$

$$(1-v)_0 : E \longrightarrow G_p, \quad x \longmapsto (1-v)x,$$

$$w := i \circ j^{-1} \circ (1-v)_0,$$

then

$$uv = vu, \quad (uv)^p = 0, \quad (uv)^{p-1} \neq 0,$$

$$vw = wv = 0, \quad uw = wu = 1 - v$$

and

$$G_p \longrightarrow G_p, \quad x \longmapsto wx$$

is the inverse map of j.

If $\mathbb{K} = \mathbb{C}$ *and* $p \geq 1$, *then* $E \neq \{0\}$ *and (using the notations from e))*

$$\sum_{n=1}^{p} t^{-n} u^{n-1} v - \sum_{n=0}^{\infty} t^n w^{n+1}$$

is the Laurent series of the resolvent of u *in* 0. *It converges in*

$$\left\{ t \in \mathbb{C} \,\middle|\, 0 < |t| < \frac{1}{\|w\|} \right\}.$$

The resolvent of u *has in* 0 *a pole of order* p *and* v *is its residue.*

u is a Fredholm operator with index 0 (Corollary 3.1.3.12). Hence u^n is a Fredholm operator with index 0 for every $n \in \mathbb{N} \cup \{0\}$ (Proposition 3.1.3.7). So, $(F_n)_{n\in\mathbb{N}\cup\{0\}}$ is an increasing sequence of finite-dimensional subspaces of E, $(G_n)_{n\in\mathbb{N}\cup\{0\}}$ is a drecreasing sequence of closed subspaces of E $(G_n)_{n\in\mathbb{N}\cup\{0\}}$ is a decreasing sequence of closed subspaces of E, and

$$\operatorname{Dim} F_n = \operatorname{Dim} E/G_n$$

for every $n \in \mathbb{N} \cup \{0\}$.

Assume that for every $n \in \mathbb{N}$, $F_n \neq F_{n+1}$. Then there is a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that

$$\|x_n\| = 1\,, \quad x_n \in F_{n+1}\,, \quad d_{F_n}(x_n) \geq \frac{1}{2}$$

for every $n \in \mathbb{N}$ (Proposition 1.1.4.2). Take $m, n \in \mathbb{N}$ with $m < n$. Then

$$ux_n + x_m - ux_m \in F_n\,,$$

so that

$$\|(1-u)x_n - (1-u)x_m\| = \|x_n - (ux_n + x_m - ux_m)\| \geq d_{F_n}(x_n) \geq \frac{1}{2}\,.$$

Hence $((1-u)x_n)_{n\in\mathbb{N}}$ has no Cauchy subsequence and this contradicts Proposition 3.1.1.19 a $\Rightarrow$ b. Hence there is a $p \in \mathbb{N} \cup \{0\}$ fulfilling a) and b). c),d),e), and the last assertion follow from Theorem 2.2.6.6. ■

Remark. Let E be a Banach space and take $u \in \mathcal{F}(E)$ such that (continuing the above notation)

$$F_n = F_{n+1}\,, \quad G_n = G_{n+1}$$

for some $n \in \mathbb{N}$. Then

$$n \operatorname{Ind} u = \operatorname{Ind} u^n = \operatorname{Dim} F_n - \operatorname{Dim} E/G_n =$$

$$= \operatorname{Dim} F_{n+1} - \operatorname{Dim} E/G_{n+1} = \operatorname{Ind} u^{n+1} = (n+1)\operatorname{Ind} u$$

(Proposition 3.1.3.7) and so

$$\operatorname{Ind} u = 0\,.$$

Hence, in order for the above theorem to hold, the index of u must vanish. But this ist not sufficient, as the following example shows. Define

$$v : \ell^\infty(\mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z}), \quad (x_n)_{n\in\mathbb{Z}} \longmapsto (x_{n+1})_{n\in\mathbb{Z}},$$

$$w : \ell^\infty(\mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z}), \quad x \longmapsto x(1 - e_0^{\mathbb{Z}})$$

and

$$u = vw.$$

Then u is a Fredholm operator with index 0, but the above theorem does not hold for u.

Define

$$v_0 : \ell^\infty(\mathbb{Z}) \longrightarrow \ell^\infty(\mathbb{Z}), \quad (x_n)_{n\in\mathbb{Z}} \longmapsto (y_n)_{n\in\mathbb{Z}},$$

where

$$y_{2n} := x_{2n+1}, \quad y_{2n+1} := x_{2n}$$

for $n \in \mathbb{N}$. If w is the above operator and

$$u = v_0 w,$$

then the above theorem holds for u, but $1 - u$ is not compact.

Definition 3.1.3.18 (6) *Let E be a Banach space and take $u \in \mathcal{L}(E)$ with $1 - u \in \mathcal{K}(E)$. The number p in the preceding theorem is called* ***the order of*** *u. Define*

$$E_a(u) := F_p, \quad E_b(u) := G_p.$$

Corollary 3.1.3.19 (6) *Let E be a complex Banach space. Take $u \in \mathcal{K}(E)$,*

$$\alpha \in \sigma(u)\backslash\{0\}$$

and put

$$u_0 := 1 - \frac{1}{\alpha}u.$$

Let p be the order of u_0, and v the projection of E onto $E_a(u)$ with

$$\operatorname{Ker} v = E_b(u).$$

Define

$$i : E_b \longrightarrow E\,, \quad x \longmapsto x\,,$$

$$j : E_b(u) \longrightarrow E_b(u)\,, \quad x \longmapsto ux\,,$$

$$(1-v)_0 : E \longrightarrow E_b(u)\,, \quad x \longmapsto (1-v)x\,,$$

and

$$w := i \circ j^{-1} \circ (1-v)_0\,.$$

Then $p \geq 1$ *and*

$$\sum_{n=1}^{p}(t-\alpha)^{-n}(u-\alpha 1)^{n-1}v + \sum_{n=0}^{\infty}(t-\alpha)^n \frac{(-1)^n}{\alpha^{n+1}} w^{n+1}$$

is the Laurent series of the resolvent of u *at* α. *It converges in*

$$\left\{ t \in \mathbb{C} \;\middle|\; 0 < |t-\alpha| < \frac{\alpha}{\|w\|} \right\}.$$

Thus the resolvent of u *has a pole of order* p *at* α *and* v *is its residue.*

Since u_0 is not invertible, $p \neq 0$ by the Principle of Inverse Operators. By the last part of Theorem 3.1.3.17,

$$\sum_{n=1}^{p} t^{-n} u_0^{n-1} v - \sum_{n=0}^{\infty} t^n w^{n+1}$$

is the Laurent series of the resolvent of u_0 in 0 and it converges in

$$\left\{ t \in \mathbb{C} \;\middle|\; 0 < |t| < \frac{\alpha}{\|w\|} \right\}.$$

Now

$$(\beta 1 - u)^{-1} = ((\beta-\alpha)1 + (\alpha 1 - u))^{-1} = -\frac{1}{\alpha}\left(\frac{\alpha-\beta}{\alpha} 1 - u_0\right)^{-1} =$$

$$= -\frac{1}{\alpha}\sum_{n=1}^{p}\left(\frac{\alpha-\beta}{\alpha}\right)^{-n} u_0^{n-1} v + \frac{1}{\alpha}\sum_{n=0}^{\infty}\left(\frac{\alpha-\beta}{\alpha}\right)^{n} w^{n+1} =$$

$$= \sum_{n=1}^{p}(\beta-\alpha)^{-n}(-1)^{n-1}\alpha^{n-1}u_0^{n-1}v + \sum_{n=0}^{\infty}(\beta-\alpha)^n \frac{(-1)^n}{\alpha^{n+1}} w^{n+1} =$$

$$= \sum_{n+1}^{p} (\beta - \alpha)^{-n} (u - \alpha 1)^{n-1} v + \sum_{n=0}^{\infty} (\beta - \alpha)^n \frac{(-1)^n}{a^{n+1}} w^{n+1}$$

for every $\beta \in \mathbb{C}$, so that

$$0 < |\frac{\alpha - \beta|}{\alpha} < \frac{1}{\|w\|}.$$

The assertions now follow. ■

Proposition 3.1.3.20 (0) *If E is a Banach space and $u \in \mathcal{F}(E)$ satisfies* $\operatorname{Ind} u = 0$, *then the following are equivalent:*

a) $uv = 1$ *for some* $v \in \mathcal{L}(E)$.

b) $vu = 1$ *for some* $v \in \mathcal{L}(E)$.

c) u *is invertible in* $\mathcal{L}(E)$.

d) u *is injective.*

e) u *is surjective.*

c ⇒ a & b, b ⇒ d, and a ⇒ e are trivial.
d ⇔ e follows from

$$\operatorname{Dim} \operatorname{Ker} u = \operatorname{Dim} \operatorname{Coker} u.$$

d & e ⇒ c follows from the Principle of Inverse Operators. ■

Proposition 3.1.3.21 (2) *Let E, F be Banach spaces, $\mathcal{L}/\mathcal{K}$ the Calkin category of (E, F), φ the quotient functor $\mathcal{L} \to \mathcal{L}/\mathcal{K}$, $\mathcal{A}(E, F)$ the set of isomorphisms of $\mathcal{L}/\mathcal{K}$ belonging to $(\mathcal{L}/\mathcal{K})(E, F)$, and $\mathfrak{F}$ (resp. $\mathfrak{A}$) the set of connected components of $\mathcal{F}(E, F)$ (resp. $\mathcal{A}(E, F)$).*

a) (Atkinson, 1951) $\mathcal{A}(E, F)$ *is an open set of* $(\mathcal{L}/\mathcal{K})(E, F)$ *and*

$$\varphi^{-1}(\mathcal{A}(E, F)) = \mathcal{F}(E, F).$$

b) $\varphi(\mathcal{F}) \in \mathfrak{A}$ *for every* $\mathcal{F} \in \mathfrak{F}$ *and the map*

$$\mathfrak{F} \longrightarrow \mathfrak{A}, \quad \mathcal{F} \longmapsto \varphi(\mathcal{F})$$

is bijective.

c) *There is a unique continuous map* $\psi : \mathcal{A}(E,F) \to \mathbb{Z}$ *satisfying*

$$\psi \circ \varphi(u) = \operatorname{Ind} u$$

for every $u \in \mathcal{F}(E,F)$. *We define*

$$\operatorname{Ind} U := \psi(U) \qquad (\textit{index of } U)$$

for every $U \in \mathcal{A}(E,F)$.

d) *The followng are equivalent for* $U \in \mathcal{A}(E,F)$:

d_1) $\operatorname{Ind} U = 0$.

d_2) *There is an isomorphism* $u : E \to F$ *belonging to* U.

a) Take $u \in \mathcal{F}(E,F)$. By Theorem 3.1.3.10, there is a $v \in \mathcal{L}(F,E)$ such that

$$1_E - v \circ u \in \mathcal{K}(E), \quad 1_F - u \circ v \in \mathcal{K}(F).$$

Put

$$V := \varphi(v) \in (\mathcal{L}/\mathcal{K})(F,E).$$

Then

$$1_E \in V\varphi(u), \quad 1_F \in \varphi(u)V,$$

i.e.

$$\varphi(u) \in \mathcal{A}(E,F) \text{ and } \mathcal{F}(E,F) \subset \varphi^{-1}(\mathcal{A}(E,F)).$$

Now take $u \in \varphi^{-1}(\mathcal{A}(E,F))$. There is a $V \in (\mathcal{L}/\mathcal{K})(F,E)$ such that $\varphi(u)V$ and $V\varphi(u)$ are units. By the uniqueness of the units,

$$1_E \in V\varphi(u), \quad 1_F \in \varphi(u)V.$$

Hence

$$1_E - vu \in \mathcal{K}(E), \quad 1_F - uv \in \mathcal{K}(F)$$

for every $v \in V$. By Theorem 3.1.3.11 b $\Rightarrow$ a, $u \in \mathcal{F}(E,F)$ so that $\varphi^{-1}(\mathcal{A}(E,F)) \subset \mathcal{F}(E,F)$.

Since $\mathcal{F}(E,F)$ is an open set of $\mathcal{L}(E,F)$ (Theorem 3.1.3.9), $\mathcal{A}(E,F)$ is an open set of $(\mathcal{L}/\mathcal{K})(E,F)$ (Theorem 1.2.4.2 b)).

b) Let $\mathcal{G}, \mathcal{H}$ be distinct connected components of $\mathcal{F}(E,F)$. Assume that

$$\varphi(\mathcal{G}) \cap \varphi(\mathcal{H}) \neq \emptyset .$$

Then there are $u \in \mathcal{G}$, $v \in \mathcal{H}$ such that

$$\varphi(u) = \varphi(v) .$$

Hence

$$u - v \in \mathcal{K}(E,F) .$$

By Corollary 3.1.3.12, u and v belong to the same connected component of $\mathcal{F}(E,F)$, which is a contradiction. Hence

$$\varphi(\mathcal{G}) \cap \varphi(\mathcal{H}) = \emptyset .$$

Now take $\mathcal{G} \in \mathfrak{F}$. Since $\mathcal{G}$ is open (Theorem 3.1.3.9), $\varphi(\mathcal{A})$ is also open (Theorem 1.2.4.2 b)) and it is obviously connected. If it is not a connected component of $\mathcal{A}(E,F)$, then it has a boundary point U in $\mathcal{A}(E,F)$. Take $u \in U$. By a), $u \in \mathcal{F}(E,F)$. Let $\mathcal{H}$ be the connected component of $\mathcal{F}(E,F)$ containing u. Then $\varphi(\mathcal{H})$ is open, so that

$$\varphi(\mathcal{G}) \cap \varphi(\mathcal{H}) \neq \emptyset .$$

By the above, $\mathcal{G} = \mathcal{H}$ and

$$U \in \varphi(\mathcal{H}) = \varphi(\mathcal{G}) ,$$

which is a contradiction. Hence $\varphi(\mathcal{G}) \in \mathfrak{A}$.

By the above, the map

$$\mathfrak{F} \longrightarrow \mathfrak{A} , \quad \mathcal{G} \longmapsto \varphi(\mathcal{G})$$

is well defined and injective. By a), it is surjective.

c) By Corollary 3.1.3.12, there is a map

$$\psi : \mathcal{A}(E,F) \longrightarrow \mathbb{Z}$$

such that

$$\psi \circ \varphi(u) = \operatorname{Ind} u$$

for every $u \in \mathcal{F}(E,F)$. By a), φ is surjective, so that the above relation uniquely defines ψ. Since $\psi \circ \varphi$ is continuous (Theorem 3.1.3.9), ψ is continuous as well (Theorem 1.2.4.2 d)).

$d_1 \Rightarrow d_2$. Take $u \in U$. Then

$$\operatorname{Ind} u = \psi \circ \varphi(u) = \psi(U) = 0.$$

By Theorem 3.1.3.10, there is a bijective $v \in \mathcal{F}(F, E)$ with

$$1_E - v \circ u \in \mathcal{K}(E), \quad 1_F - u \circ v \in \mathcal{K}(F).$$

By the Principle of Inverse Operators, v is an isomorphism. Since

$$v^{-1} - u = v^{-1}(1_E - v \circ u) \in \mathcal{K}(E, F)$$

(Proposition 3.1.1.11), $v^{-1} \in U$.

$d_2 \Rightarrow d_1$. Let u be an isomorphism $E \to F$ belonging to U. By c),

$$\operatorname{Ind} U = \psi \circ \varphi(u) = \operatorname{Ind} u = 0. \qquad \blacksquare$$

Corollary 3.1.3.22 (2) *Let E, F, G be Banach spaces, $\mathcal{L}/\mathcal{K}$ the Calkin category of (E, F, G), and U, V isomorphisms of $\mathcal{L}/\mathcal{K}$ belonging to*

$$(\mathcal{L}/\mathcal{K})(E, F)$$

and $(\mathcal{L}/\mathcal{K})(F, G)$, respectively. Then $V \circ U$ is an isomorphism of $\mathcal{L}/\mathcal{K}$ and

$$\operatorname{Ind}(V \circ U) = \operatorname{Ind} U + \operatorname{Ind} V.$$

Take $u \in U$, $v \in V$. By Proposition 3.1.3.21 a), $u \in \mathcal{F}(E, F)$, $v \in \mathcal{F}(F, G)$ and by Proposition 3.1.3.7, $v \circ u \in \mathcal{F}(E, G)$ and

$$\operatorname{Ind}(v \circ u) = \operatorname{Ind} u + \operatorname{Ind} v.$$

Since $v \circ u \in V \circ U$,

$$\operatorname{Ind}(V \circ U) = \operatorname{Ind} U + \operatorname{Ind} V. \qquad \blacksquare$$

Proposition 3.1.3.23 (2) *Let Ω be a class of Banach spaces, $\mathcal{L}/\mathcal{K}$ the Calkin category of Ω, $E, F \in \Omega$ and take $u \in \mathcal{L}(E, F)$. Then the following are equivalent:*

a) $u \in \mathcal{F}(E, F)$.

b) *The maps*

$$(\mathcal{L}/\mathcal{K})(G, E) \longrightarrow (\mathcal{L}/\mathcal{K})(G, F), \quad V \longmapsto uV,$$

$$(\mathcal{L}/\mathcal{K})(F, G) \longrightarrow (\mathcal{L}/\mathcal{K})(E, G), \quad V \longmapsto Vu$$

are isomorphisms of Banach spaces for all $G \in \Omega$.

c) *The maps*

$$(\mathcal{L}/\mathcal{K})(F,E) \longrightarrow (\mathcal{L}/\mathcal{K})(F), \quad V \longmapsto uV,$$

$$(\mathcal{L}/\mathcal{K})(F,E) \longrightarrow (\mathcal{L}/\mathcal{K})(E), \quad V \longmapsto Vu$$

are surjective.

a $\Rightarrow$ b. By Theorem 3.1.3.10, there is a $w \in \mathcal{F}(F,E)$ with

$$1_E - w \circ u \in \mathcal{K}(E), \quad 1_F - u \circ w \in \mathcal{K}(F).$$

Since

$$V \in (\mathcal{L}/\mathcal{K})(G,E) \Longrightarrow V = 1_E V = wuV,$$

$$V \in (\mathcal{L}/\mathcal{K})(G,F) \Longrightarrow V = 1_F V = uwV$$

(Proposition 3.1.1.11), the map

$$(\mathcal{L}/\mathcal{K})(G,E) \longrightarrow (\mathcal{L}/\mathcal{K})(G,F), \quad V \longmapsto uV$$

is an isomorphism of Banach spaces. Similarly, the map

$$(\mathcal{L}/\mathcal{K})(F,G) \longrightarrow (\mathcal{L}/\mathcal{K})(E,G), \quad V \longmapsto Vu$$

is an isomorphism of Banach spaces.

b $\Rightarrow$ c is trivial.

c $\Rightarrow$ a. By c), there are $V, W \in (\mathcal{L}/\mathcal{K})(F,E)$ with

$$uV = 1_F, \quad Wu = 1_E.$$

If U is the equivalence class of u in $(\mathcal{L}/\mathcal{K})(E,F)$, then U is invertible. Hence u is a Fredholm operator by Proposition 3.1.3.21 a). ■

Definition 3.1.3.24 (0) *Let E be a Banach space. Take $u \in \mathcal{L}(E)$ and let*

$$q : \mathcal{L}(E) \longrightarrow \mathcal{L}(E)/\mathcal{K}(E)$$

be the quotient map. Then $\sigma(qu)$ is called the ***essential spectrum of*** *u and is denoted by $\sigma^e(u)$.*

Proposition 3.1.3.25 (0) *Let E be a Banach space and $u \in \mathcal{L}(E)$.*

a) $\sigma^e(u) = \{\alpha \in \mathbb{K} \mid \alpha 1 - u \notin \mathcal{F}(E)\}$.

b) $\sigma^e(u) \subset \sigma(u)$.

c) *If E is reflexive, then $\sigma^e(u) = \sigma^e(u')$.*

a) follows from Proposition 3.1.2.11.

b) follows from Corollary 2.1.3.12.

c) Take $\alpha \in \mathbb{K}$. By Corollary 3.1.3.5 (and Corollary 1.3.4.3),

$$\alpha 1 - u \in \mathcal{F}(E) \Longleftrightarrow \alpha 1 - u' \in \mathcal{F}(E').$$

Hence by a),

$$(\alpha \in \sigma^e(u)) \Longleftrightarrow (\alpha 1 - u \notin \mathcal{F}(E)) \Longleftrightarrow (\alpha 1 - u' \notin \mathcal{F}(E')) \Longleftrightarrow (\alpha \in \sigma^e(u')). \blacksquare$$

Lemma 3.1.3.26 *If $p \in]0,1[\cup]1,\infty[$ and $n \in \mathbb{N}$, then*

$$\frac{(n+1)^{1-p} - 1}{1-p} \leq \sum_{k=1}^{n} \frac{1}{k^p} \leq \frac{n^{1-p}}{1-p}.$$

Given $k \in \mathbb{N}$,

$$\frac{1}{(k+1)^p} \leq \int_k^{k+1} \frac{dt}{t^p} \leq \frac{1}{k^p}$$

and so

$$\sum_{k=1}^{n} \frac{1}{k^p} = 1 + \sum_{k=1}^{n-1} \frac{1}{(k+1)^p} \leq 1 + \sum_{k=1}^{n-1} \int_k^{k+1} \frac{dt}{t^p} =$$

$$= 1 + \int_1^n \frac{dt}{t^p} = 1 + \frac{1}{1-p}\left[t^{1-p}\right]_1^n = 1 + \frac{n^{1-p}}{1-p} - \frac{1}{1-p} \leq \frac{n^{1-p}}{1-p}$$

$$\sum_{k=1}^{n} \frac{1}{k^p} \geq \sum_{k=1}^{n} \int_k^{k+1} \frac{dt}{t^p} = \int_1^{n+1} \frac{dt}{t^p} = \frac{1}{1-p}\left[t^{1-p}\right]_1^{n+1} = \frac{(n+1)^{1-p} - 1}{1-p}. \blacksquare$$

Example 3.1.3.27 *Take $p \in \{0\} \cup [1,\infty]$ and let u_r, u_ℓ be the right and left shift operators of ℓ^p, respectively. Then*

$$\sigma^e(u_r) = \sigma^e(u_\ell) = \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}$$

and

$$\mathrm{Ind}(\alpha 1 - u_r) = -1, \quad \mathrm{Ind}(\alpha 1 - u_\ell) = 1$$

for every $\alpha \in \mathbb{K}$ with $|\alpha| < 1$. The above still holds for c in place of ℓ^p.

Step 1 $u_r, u_\ell \in \mathcal{F}(\ell^p)$, $\operatorname{Ind} u_r = -1$, $\operatorname{Ind} u_\ell = 1$.

The assertion is easy to see.

Step 2 $\{\alpha \in \mathbb{K} \mid |\alpha| = 1\} \subset \sigma^e(u_r) \cap \sigma^e(u_\ell)$

Take $\alpha \in \mathbb{K}$ with $|\alpha| = 1$. Define

$$A := \begin{cases}]\frac{1}{p}, \frac{p+1}{p}[& \text{if} \quad p \in [1, \infty[\\ ]0, 1[& \text{if} \quad p \in \{0, \infty\} \end{cases}.$$

Given $q \in A$, define

$$y_q := \left(\frac{\overline{\alpha}^n}{n^q}\right)_{n \in \mathbb{N}} \in \ell^p.$$

Let Q be a finite subset of A and take $(\beta_q)_{q \in Q} \in \mathbb{K}^Q \backslash \{0\}$. Assume that

$$\sum_{q \in Q} \beta_q y_q \in \operatorname{Im}(\alpha 1 - u_r).$$

Then there is an $x \in \ell^p$ such that

$$(\alpha 1 - u_r)x = \sum_{q \in Q} \beta_q y_q.$$

By Example 1.2.2.9 d),

$$x_n = \frac{1}{\alpha^{n+1}} \sum_{q \in Q} \beta_q \left(\sum_{k=1}^{n} \frac{1}{k^q}\right) \tag{*}$$

for every $n \in \mathbb{N}$. Let r be the smallest element of $\{q \in Q \mid \beta_q \neq 0\}$ and put

$$R := Q \backslash \{r\}.$$

Then, by Lemma 3.1.3.26,

$$\gamma_n(q) := \left(\sum_{k=1}^{n} \frac{1}{k^q}\right) : \left(\sum_{k=1}^{n} \frac{1}{k^r}\right) \leq \frac{n^{1-q}}{1-q} \frac{1-r}{(n+1)^{1-r} - 1}$$

for every $n \in \mathbb{N}$ and $q \in R$, so that

$$\lim_{n \to \infty} \frac{|\beta_q|}{|\beta_r|} \gamma_n(q) = 0$$

for every $q \in R$. Hence there is an $n_0 \in \mathbb{N}$ with

$$\sum_{q \in R} \frac{|\beta_q|}{|\beta_r|} \gamma_n(q) < \frac{1}{2}$$

for every $n \in \mathbb{N}$ such that $n \geq n_0$. It now follows from (*) and Lemma 3.1.3.26, that

$$|x_n| \geq |\beta_r| \sum_{k=1}^{n} \frac{1}{k^r} - \sum_{q \in R} |\beta_q| \left(\sum_{k=1}^{n} \frac{1}{k^q} \right) \geq$$

$$\geq |\beta_r| \sum_{k=1}^{n} \frac{1}{k^r} \left(1 - \sum_{q \in R} \frac{|\beta_q|}{|\beta_r|} \gamma_n(q) \right) \geq$$

$$\geq |\beta_r| \sum_{k=1}^{n} \frac{1}{k^r} (1 - \frac{1}{2}) \geq \frac{|\beta_r|}{2(1-r)} \left((n+1)^{1-r} - 1 \right)$$

for every $n \in \mathbb{N}$, $n \geq n_0$. Thus $(x_n)_{n \in \mathbb{N}} \notin \ell^p$ and this is the contradiction sought. Hence

$$\sum_{q \in Q} \beta_q y_q \notin \operatorname{Im}(\alpha 1 - u_r) .$$

Now suppose that $\alpha 1 - u_r \in \mathcal{F}(\ell^p)$. Then there is a finite-dimensional vector subspace F of ℓ^p such that

$$\ell^p = F \oplus \operatorname{Im}(\alpha 1 - u) .$$

Let $v : \ell^p \to \ell^p / \operatorname{Im}(\alpha 1 - u)$ be the quotient map and d the dimension of F. Let Q be a finite subset of A with

$$\operatorname{Card} Q > d .$$

Then $(vy_q)_{q \in Q}$ is linearly dependent. Hence there is a $(\beta_q)_{q \in Q} \in \mathbb{K}^Q \backslash \{0\}$ such that

$$\sum_{q \in Q} \beta_q v y_q = 0 .$$

It follows that

$$\sum_{q \in Q} \beta_q y_q \in \operatorname{Im}(\alpha 1 - u)$$

which contradicts the above result. Hence $\alpha 1 - u_r \notin \mathcal{F}(\ell^p)$.

From

$$\overline{\alpha} 1 - u_r = (\overline{\alpha} u_\ell - 1) u_r = -\overline{\alpha} (\alpha 1 - u_\ell) u_r$$

(Example 1.2.2.9 c)), it follows (Step 1 and Proposition 3.1.3.7) $\alpha 1 - u_\ell \notin \mathcal{F}(\ell^p)$.

By Proposition 3.1.3.25 a),

$$\{\alpha \in \mathbb{K} \mid |\alpha| = 1\} \subset \sigma^e(u_r) \cap \sigma^e(u_\ell)\,.$$

Step 3 $\sigma^e(u_\ell) = \sigma^e(u_r) = \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}$

Let q be the quotient map

$$\mathcal{L}(\ell^p) \longrightarrow \mathcal{L}(\ell^p)/\mathcal{K}(\ell^p)\,.$$

Since

$$u_\ell u_r = 1\,, \quad u_r u_\ell - 1 \in \mathcal{K}(\ell^p)\,,$$

we see that

$$(qu_\ell)(qu_r) = q(u_\ell u_r) = 1\,,$$

$$(qu_r)(qu_\ell) = q(u_r u_\ell) = 1\,.$$

By Example 3.1.2.16,

$$\|(qu_r)^{-1}\| = \|qu_\ell\| = 1\,, \quad \|(qu_\ell)^{-1}\| = \|qu_r\| = 1\,.$$

By Proposition 2.2.4.7,

$$\sigma^e(u_r) = \sigma(qu_r) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,,$$

$$\sigma^e(u_\ell) = \sigma(qu_\ell) \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,.$$

By Step 2,

$$\sigma^e(u_r) = \sigma^e(u_\ell) = \{\alpha \in \mathbb{K} \mid |\alpha| = 1\}\,.$$

Step 4 $\alpha \in \mathbb{K}\,,\ |\alpha| < 1 \Rightarrow \mathrm{Ind}(\alpha 1 - u_r) = -1\,,\ \mathrm{Ind}(\alpha 1 - u_\ell) = 1$

We have

$$u_\ell(\alpha 1 - u_r) = \alpha u_\ell - 1\,,$$

$$(\alpha 1 - u_\ell)u_r = \alpha u_r - 1\,.$$

Since

$$\|u_\ell\| = \|u_r\| = 1\,,$$

$\alpha u_\ell - 1$ and $\alpha u_r - 1$ are invertible (Proposition 2.2.4.1). By Step 1 and Proposition 3.1.3.7,

$$0 = \mathrm{Ind}\,(\alpha u_\ell - 1) = \mathrm{Ind}\,u_\ell + \mathrm{Ind}(\alpha 1 - u_r) = 1 + \mathrm{Ind}(\alpha 1 - u_r)\,,$$

$$0 = \mathrm{Ind}\,(\alpha u_r - 1) = \mathrm{Ind}\,(\alpha 1 - u_\ell) + \mathrm{Ind}\,u_r = \mathrm{Ind}(\alpha 1 - u_\ell) - 1\,,$$

such that

$$\mathrm{Ind}(\alpha 1 - u_r) = -1\,, \quad \mathrm{Ind}(\alpha 1 - u_\ell) = 1\,. \qquad \blacksquare$$

3.1.4 Point Spectrum

Definition 3.1.4.1 (0) *Let E be a normed space and take $u \in \mathcal{L}(E)$. Given $\alpha \in \mathbb{K}$, **the multiplicity of α with respect to u** is the dimension of* $\mathrm{Ker}(\alpha 1 - u)$. *If the multiplicity is non-zero (equivalently: if $\alpha 1 - u$ is not injective), then α is said to be an **eigenvalue of u** and $\mathrm{Ker}(\alpha 1 - u)$ **is the eigenspace of u corresponding to the eigenvalue α**. The elements of $\mathrm{Ker}(\alpha 1 - u)$ are called **eigenvectors of u corresponding to the eigenvalue α**. We use $\sigma_p(u)$ to denote the set of eigenvalues of u and call it **the point spectrum of u**.*

Proposition 3.1.4.2 (0) *Let E be a normed space and $u \in \mathcal{L}(E)$.*

a) $\sigma_p(u) \subset \sigma(u)$.

b) *If α is an eigenvalue of u, then $x \in E$ is an eigenvector of u corresponding to α iff*

$$ux = \alpha x .$$

c) *If E is complete, then*

$$\sigma(u) \backslash \sigma_p(u) \subset \{\alpha \in \mathbb{K} \mid \alpha 1 - u \text{ is not surjective}\} .$$

d) *If E is finite-dimensional, then*

$$\sigma_p(u) = \sigma(u) .$$

a) is trivial.

b)

$$x \in \mathrm{Ker}(\alpha 1 - u) \Longleftrightarrow ux = \alpha x .$$

c) follows from the Principle of Inverse Operators (Corollary 1.4.2.4).

d) follows from the fact that in the finite-dimensional case u is bijective iff it is injective (Corollary 3.1.3.13). ■

Remark. Example 3.1.5.7 shows that a) above is the only restriction on $\sigma_p(u)$.

Proposition 3.1.4.3 (0) *Let E be a normed space and take $u \in \mathcal{L}(E)$. For each $\alpha \in \sigma_p(u)$ let $x_\alpha \neq 0$ be an eigenvector of u corresponding to α. Then the family $(x_\alpha)_{\alpha \in \sigma_p(u)}$ is linearly independent.*

Let $(\beta_\alpha)_{\alpha\in\sigma_p(u)}$ be a family in $\mathbb{K}$ for which

$$A := \{\alpha \in \sigma_p(u) | \beta_\alpha \neq 0\}$$

is finite and

$$\sum_{\alpha\in A} \beta_\alpha x_\alpha = 0\,.$$

We show, by means of complete induction on the cardinality of A, that A must be empty. Now (Proposition 3.1.4.2 b))

$$0 = \sum_{\alpha\in A} \beta_\alpha u x_\alpha = \sum_{\alpha\in A} \beta_\alpha \alpha x_\alpha\,.$$

Take $\gamma \in A$. Then

$$0 = \gamma \sum_{\alpha\in A} \beta_\alpha x_\alpha - \sum_{\alpha\in A} \beta_\alpha \alpha x_\alpha = \sum_{\alpha\in A} \beta_\alpha(\gamma-\alpha)x_\alpha = \sum_{\alpha\in A\setminus\{\gamma\}} \beta_\alpha(\gamma-\alpha)x_\alpha\,.$$

By the inductive hypothesis,

$$\beta_\alpha(\gamma-\alpha) = 0$$

for every $\alpha \in A\setminus\{\gamma\}$. Hence $\beta_\alpha = 0$. Thus

$$\beta_\gamma x_\gamma = 0\,,$$

i.e. $\beta_\gamma = 0$, so that $A = \emptyset$. Hence $(x_\alpha)_{\alpha\in\sigma_p(u)}$ is linearly independent. ■

Definition 3.1.4.4 (0) *Let E be a vector space and $u : E \to E$ a linear map. A vector subspace F of E is called u-**invariant** if $u(F) \subset F$. If $\mathcal{F}$ is a set of linear maps $E \to E$, then a vector subspace of E is called $\mathcal{F}$-**invariant** if it is u-invariant for every $u \in \mathcal{F}$.*

Proposition 3.1.4.5 *Let E be a Banach space and $u \in \mathcal{L}(E)$. If every vector of E belongs to an u-invariant finite-dimensional vector subspace of E, then*

$$\sigma_p(u) = \sigma(u)\,.$$

Take $\alpha \in \mathbb{K}\setminus\sigma_p(u)$ and $x \in E$. Let F be an u-invariant finite-dimensional vector subspace of E containing x. Then

$$(\alpha 1 - u)(F) \subset F$$

and the map

$$v : F \longrightarrow F, \quad y \longmapsto (\alpha 1 - u)y$$

is injective. Since F is finite–dimensional, v is also surjective. Hence

$$x = vy = (\alpha 1 - u)y$$

for some $y \in F$. Thus $\alpha 1 - u$ is surjective and $\alpha \notin \sigma(u)$ (Proposition 3.1.4.2 c)). Hence

$$\sigma(u) \subset \sigma_p(u).$$

The reverse inclusion is trivial. ■

Corollary 3.1.4.6 *Let E be a Banach space and take $u \in \mathcal{L}(E)$. If there is a polynomial P with coefficients in $\mathbb{K}$ such that $P(u) = 0$, then every vector of E belongs to some u–invariant finite–dimensional vector subspace of E and*

$$\sigma_p(u) = \sigma(u).$$

Take $x \in E$. Since

$$P(u)x = 0,$$

there is an $n \in \mathbb{N}$ for which

$$A := \{u^{k-1}x | k \in \mathbb{N}_n\}$$

is linearly dependent. Let F be the vector subspace of E generated by A. Then F is an u–invariant finite–dimensional vector subspace of E containing x. By Proposition 3.1.4.5,

$$\sigma_p(u) = \sigma(u).$$ ■

Proposition 3.1.4.7 *Let E be a Banach space and take $u \in \mathcal{L}(E)$. Let α be a boundary point of $\sigma(u)$. If $\mathrm{Im}(\alpha 1 - u)$ is closed, then $\alpha \in \sigma_p(u)$.*

Assume that

$$\mathrm{Ker}(\alpha 1 - u) = \{0\}.$$

Then the map

$$v : E \longrightarrow \mathrm{Im}(\alpha 1 - u), \quad x \longmapsto (\alpha 1 - u)x$$

is bijective. By Corollary 1.4.2.8 a ⇒ b, there is a $\beta > 0$ such that

$$\|vx\| \geq \beta\|x\|$$

for every $x \in E$. By Proposition 2.2.4.26, there is a sequence $(u_n)_{n\in\mathbb{N}}$ in $\mathcal{L}(E)$ for which

$$\lim_{n\to\infty}(\alpha 1 - u)u_n = 0$$

and

$$\|u_n\| = 1$$

for every $n \in \mathbb{N}$. Given $n \in \mathbb{N}$, choose $x_n \in E^{\#}$ with

$$\|u_n x_n\| \geq \frac{1}{2},$$

and set

$$y_n := \frac{1}{\|u_n x_n\|} u_n x_n .$$

Then

$$0 = \lim_{n\to\infty} \frac{1}{\|u_n x_n\|}(\alpha 1 - u)u_n x_n = \lim_{n\to\infty}(\alpha 1 - u)y_n ,$$

contradicting the above inequality. Hence α is an eigenvalue of u. ■

Example 3.1.4.8 *Let E be a normed space and u a projection in E.*

a) $\sigma_p(u) = \sigma(u) \subset \{0,1\}$.

b) $\operatorname{Im} u$ *is the eigenspace of u corresponding to* 1.

c) $\operatorname{Ker} u$ *is the eigenspace of u corresponding to* 0.

Now

$$\sigma(u) \subset \{0,1\}$$

follows from Corollary 2.1.3.7. Furthermore,

$$ux = x \Longleftrightarrow x \in \operatorname{Im} u ,$$

$$ux = 0 \Longleftrightarrow x \in \operatorname{Ker} u$$

for each $x \in E$ and all the assertions now follow. ■

Example 3.1.4.9 *If $p \in \{0\} \cup [1, \infty]$ and if u_r is the right shift operator of ℓ^p, then*

$$\sigma_p(u) = \emptyset .$$

The same assertion holds when ℓ^p is replaced by c.

By Example 1.2.2.9 e), $\alpha 1 - u_r$ is injective for any $\alpha \in \mathbb{K}$. ■

Example 3.1.4.10 *Take $E := c$ or $E := \ell^p$ with $p \in [1, \infty] \cup \{0\}$ and let u_ℓ be the left shift of E.*

a) $$\sigma_p(u_\ell) = \begin{cases} \{\alpha \in \mathbb{K} \mid |\alpha| < 1\} & \text{if } E = \ell^p \text{ with } p \in [1, \infty[\cup \{0\} \\ \mathbb{K}^\# & \text{if } E = \ell^\infty \\ \{\alpha \in \mathbb{K} \mid |\alpha| < 1\} \cup \{1\} & \text{if } E = c . \end{cases}$$

b) *Given $\alpha \in \sigma_p(u_\ell)$, the eigenspace of u_ℓ corresponding to α is generated by $(\alpha^n)_{n \in \mathbb{N}}$.*

Take $\alpha \in \mathbb{K}$ and $x \in E$ such that

$$u_\ell x = \alpha x .$$

Then

$$x_{n+1} = \alpha x_n$$

for every $n \in \mathbb{N}$, so that

$$x = (\alpha^{n-1} x_1)_{n \in \mathbb{N}} .$$

a) and b) follow now immediately (Proposition 3.1.4.2 b)). ■

Example 3.1.4.11 *Let T be a completely regular space, $y \in C(T)$, and*

$$u : C(T) \longrightarrow C(T), \quad x \longmapsto xy .$$

Then for every $\alpha \in \mathbb{K}$, $\mathrm{Dim}\{x \in C(T) \mid \{x \neq 0\} \subset \overbrace{y^{-1}}^{\circ}(\alpha)\}$ is the multiplicity of α with respect to u, and

$$\sigma_p(u) := \{\alpha \in \mathbb{K} \mid \overbrace{y^{-1}}^{\circ}(\alpha) \neq \emptyset\} .$$

Given $x \in C(T)$,

$$x \text{ is an eigenvector of } u \text{ corresponding to } \alpha \Longleftrightarrow$$

$$\Longleftrightarrow x \text{ vanishes on } T \backslash \overset{-1}{y}(\alpha) \Longleftrightarrow$$

$$\Longleftrightarrow x \text{ vanishes on } \overline{T \backslash \overset{-1}{y}(\alpha)} = T \backslash \overset{\circ}{\overset{-1}{y}}(\alpha) \Longleftrightarrow \{x \neq 0\} \subset \overset{\circ}{\overset{-1}{y}}(\alpha)$$

(Proposition 3.1.4.2 b)). ■

Example 3.1.4.12 *Take* $\mathbb{K} = \mathbb{C}$ *and* $q \in [1, \infty] \cup \{0\}$. *Let* T *be a set,* $f : T \to T$ *a bijection and take*

$$u : \ell^q(T) \longrightarrow \ell^q(T), \quad x \longmapsto x \circ f.$$

We define f^n *and* T_n *for every* $n \in \mathbb{N}$ *inductively by setting*

$$f^1 := f, \quad f^{n+1} := f \circ f^n,$$

$$T_n := \{t \in T | f^n(t) = t\} \backslash \bigcup_{m=1}^{n-1} T_m$$

and we assume that

$$T = \bigcup_{n \in \mathbb{N}} T_n.$$

a) $\sigma_p(u) = \{\alpha \in \mathbb{C} | \exists n \in \mathbb{N}, \alpha^n = 1, T_n \neq \emptyset\}$.

b) $\sigma(u) = \overline{\sigma_p(u)}$.

a) Take $\alpha \in \sigma_p(u)$. Then, for some $x \in \ell^q(T) \backslash \{0\}$,

$$ux = \alpha x$$

(Proposition 3.1.4.2 b)). Choose $t \in T$ such that $x(t) \neq 0$ and $n \in \mathbb{N}$ such that $t \in T_n$. Then

$$\alpha^n x(t) = u^n x(t) = x(f^n(t)) = x(t),$$

$$\alpha^n = 1.$$

Hence

$$\sigma_p(u) \subset \{\alpha \in \mathbb{C} | \exists n \in \mathbb{N}, \alpha^n = 1, T_n \neq \emptyset\}.$$

Now take $\alpha \in \mathbb{C}$ for which there is an $n \in \mathbb{N}$ with $\alpha^n = 1$ and $T_n \neq \emptyset$. Choose $t \in T_n$. Define

$$x : T \longrightarrow \mathbb{C}, \quad s \longmapsto \begin{cases} \alpha^m & \text{if } s = f^m(t) \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

Then $x \in \ell^q(T)$ and

$$ux = \alpha x$$

so that $\alpha \in \sigma_p(u)$ (Proposition 3.1.4.2 b)). Hence

$$\{\alpha \in \mathbb{C} | \exists n \in \mathbb{N}, \alpha^n = 1, T_n \neq \emptyset\} \subset \sigma_p(u).$$

b) Since u is an isometry,

$$\sigma(u) \subset \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$$

(Corollary 2.2.4.8). If $\{n \in \mathbb{N} | T_n \neq \emptyset\}$ is infinite, then, by a),

$$\overline{\sigma_p(u)} = \{\alpha \in \mathbb{C} \mid |\alpha| = 1\}$$

and so

$$\sigma(u) = \overline{\sigma_p(u)}$$

(Proposition 3.1.4.2 a), Corollary 2.2.4.5).

Now suppose that $\{x \in \mathbb{N} | T_n \neq \emptyset\}$ is finite. Then, for some $n \in \mathbb{N}$, f^n is the identical map. Hence

$$u^n - 1 = 0$$

and therefore

$$\sigma_p(u) = \sigma(u)$$

by Corollary 3.1.4.6. Hence, by Corollary 2.2.4.5,

$$\overline{\sigma_p(u)} = \sigma(u). \qquad \blacksquare$$

Proposition 3.1.4.13 (1) (5) *Let $(T, \mathfrak{T}, \mu)$ be a measure space and take $u \in \mathcal{L}(L^2(\mu))$. Define*

$$\langle x|y\rangle := \int x\overline{y}d\mu$$

for $x, y \in L^2(\mu)$ and

$$f : L^2(\mu) \times L^2(\mu) \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \langle ux|y\rangle .$$

Then f is a sequilinear form and if f is Hermitian (positive), then

$$\sigma_p(u) \subset \mathbb{R} \quad (\sigma_p(u) \subset \mathbb{R}_+) .$$

It is easy to see that f is a sesquilinear form. Take $\alpha \in \sigma_p(u)$. Then

$$ux = \alpha x$$

for some $x \in L^2(\mu)\backslash\{0\}$ (Proposition 3.1.4.2 b)). Hence

$$\alpha\langle x|x\rangle = \langle \alpha x|x\rangle = \langle ux|x\rangle = f(x, x) \in \mathbb{R} \quad (\in \mathbb{R}_+)$$

(Proposition 2.3.3.8), so that $\alpha \in \mathbb{R}$ $(\alpha \in \mathbb{R}_+)$ and

$$\sigma_p(u) \subset \mathbb{R} \quad (\sigma_p(u) \subset \mathbb{R}_+) . \qquad \blacksquare$$

Example 3.1.4.14 *Let T be a locally compact group and λ a left invariant Haar measure on T. Given $\mu \in \mathcal{M}_b(T)$, define*

$$\widetilde{\mu} : L^2(\lambda) \longrightarrow L^2(\lambda), \quad x \longmapsto \mu * x$$

(Proposition 2.2.2.14 a)). Then

$$\sigma_p(\widetilde{\mu}) \subset \mathbb{R}$$

for every $\mu \in \mathrm{Re}\mathcal{M}_b(T)$ and

$$\sigma_p(\widetilde{\mu^* * \mu}) \subset \mathbb{R}_+$$

for every $\mu \in \mathcal{M}_b(T)$ (Proposition 2.3.2.28 a)).

Define

$$\langle x \mid y\rangle := \int x\overline{y}d\lambda$$

for $x, y \in L^2(\lambda)$ and

$$f_\mu : L^2(\lambda) \times L^2(\lambda) \longrightarrow \mathbb{K}, \quad (x, y) \longmapsto \langle \mu * x|y\rangle$$

for $\mu \in \mathcal{M}_b(T)$. By Example 2.3.3.10 and Proposition 3.1.4.13, if $\mu \in \mathrm{Re}\mathcal{M}_b(T)$, then f_μ is Hermitian, whence $\sigma_p(\widetilde{u}) \subset \mathbb{R}$, and if $\mu \in \mathcal{M}_b(T)$ then $f_{\mu^**\mu}$ is positive, whence $\sigma_p(\widetilde{\mu^* * \mu}) \subset \mathbb{R}_+$. $\blacksquare$

Example 3.1.4.15 *Let* $(T, \mathfrak{T}, \mu)$ *be a measure space. Take* $p \in [1, \infty]$, $x \in L^\infty(\mu)$ *and put*

$$u : L^p(\mu) \longrightarrow L^p(\mu), \quad y \longmapsto xy.$$

Then $u \in \mathcal{L}(L^p(\mu))$ *(Example 2.2.2.22) and*

$$\sigma_p(u) = \{\alpha \in \mathbb{K} | \overset{-1}{x}(\alpha) \text{ is not a } \mu\text{-null set}\}.$$

Take $\alpha \in \sigma_p(u)$. Then there is a $y \in L^p(\mu) \backslash \{0\}$ with

$$xy = \alpha y.$$

Since

$$\{y \neq 0\} \subset \overset{-1}{x}(\alpha),$$

it follows that $\overset{-1}{x}(\alpha)$ is not a μ-null set.

Now suppose that $\alpha \in \mathbb{K}$, so that $\overset{-1}{x}(\alpha)$ is not a μ-null set. There is a μ-integrable set A with

$$A \subset \overset{-1}{x}(\alpha), \quad \int e_A d\mu > 0.$$

We have

$$ue_A = xe_A = \alpha e_A,$$

so that α is an eigenvalue of u. ■

3.1.5 Spectrum of a Compact Operator

Theorem 3.1.5.1 (0) (F. Riesz, 1918) *Let E be a Banach space. Take $u \in \mathcal{K}(E)$ and $\alpha \in \mathbb{K}\backslash\{0\}$.*

a) $\operatorname{Dim}\operatorname{Ker}(\alpha 1 - u) = \operatorname{Dim}\operatorname{Ker}(\alpha 1 - u') = \operatorname{Dim}\operatorname{Coker}(\alpha 1 - u) =$
$= \operatorname{Dim}\operatorname{Coker}(\alpha 1 - u') < \infty$,

$$\operatorname{Im}(\alpha 1 - u) = {}^{\circ}(\operatorname{Ker}(\alpha 1 - u')).$$

b) *The following are equivalent:*

b$_1$) $\alpha \in \sigma(u)$,

b$_2$) $\alpha 1 - u$ *is not injective (i.e.* $\alpha \in \sigma_p(u)$*),*

b$_3$) $\alpha 1 - u$ *is not surjective,*

b$_4$) α *is an isolated point of* $\sigma(u)$.

c) *Either* $\sigma(u)$ *is finite or there is a null sequence* $(\alpha_n)_{n\in\mathbb{N}}$ *in* $\mathbb{K}$ *such that*

$$\sigma(u) = \{\alpha_n | n \in \mathbb{N}\} \cup \{0\}.$$

d) *If E is infinite-dimensional, then 0 belongs to* $\sigma(u)$.

e) $\sigma_p(u)\backslash\{0\} = \sigma_p(u')\backslash\{0\}$ *and the multiplicities of any* $\alpha \in \mathbb{K}\backslash\{0\}$ *with respect to u and u' coincide.*

By Corollary 3.1.3.12, $\alpha 1 - u$ is Fredholm and

$$\operatorname{Ind}(\alpha 1 - u) = \operatorname{Ind}\alpha 1 = 0.$$

a) follows from Proposition 3.1.3.4.

b$_1$ $\Leftrightarrow$ b$_2$ $\Leftrightarrow$ b$_3$ follows from the Proposition 3.1.3.20 c $\Leftrightarrow$ d $\Leftrightarrow$ e.

b$_4$ $\Rightarrow$ b$_1$ is trivial.

b$_1$ $\Rightarrow$ b$_4$. Assume that α is not an isolated point of $\sigma(u)$. There is a sequence $(\alpha_n)_{n\in\mathbb{N}}$ in $\sigma(u)\backslash\{\alpha\}$ converging to α. We may assume that

$$\alpha_m \neq \alpha_n \neq 0$$

for distinct $m, n \in \mathbb{N}$. Given $n \in \mathbb{N}$, let x_n be a non-zero eigenvector of u corresponding to α_n (b$_1$ $\Rightarrow$ b$_2$). By Proposition 3.1.4.3, $(x_n)_{n\in\mathbb{N}}$ is linearly independent. Given $n \in \mathbb{N}$, let E_n be the subspace of E generated by $(x_k)_{k\in\mathbb{N}_n}$. Then E_n is closed (Corollary 1.1.3.6) and

$$(\alpha_n 1 - u)(E_n) \subset E_{n-1} \qquad (E_o := \{0\})$$

$$E_n \subsetneq E_{n+1}$$

for every $n \in \mathbb{N}$. Hence there is a sequence $(y_n)_{n\in\mathbb{N}}$ in E such that

$$y_n \in E_n\,, \quad \|y_n\| = 1\,, \quad d_{E_{n-1}}(y_n) \geq \frac{1}{2}$$

for every $n \in \mathbb{N}$ (Proposition 1.1.4.3). Take $m, n \in \mathbb{N}$ with $m < n$. Then

$$u\left(\frac{1}{\alpha_m} y_m\right) - u\left(\frac{1}{\alpha_n} y_n\right) = \frac{1}{\alpha_m}(uy_m - \alpha_m y_m) - \frac{1}{\alpha_n}(uy_n - \alpha_n y_n) + y_m - y_n =$$

$$= -\frac{1}{\alpha_m}(\alpha_m 1 - u)y_m + \frac{1}{\alpha_n}(\alpha_n 1 - u)y_n + y_m - y_n\,,$$

$$(\alpha_m 1 - u)y_m \in E_{m-1} \subset E_{n-1}\,,$$

$$(\alpha_n 1 - u)y_n \in E_{n-1}\,,$$

$$y_m \in E_m \subset E_{n-1}\,,$$

and therefore

$$-\frac{1}{\alpha_m}(\alpha_m 1 - u)y_m + \frac{1}{\alpha_n}(\alpha_n 1 - u)y_n + y_m \in E_{n-1}\,,$$

$$\left\| u\left(\frac{1}{\alpha_m} y_m\right) - u\left(\frac{1}{\alpha_n} y_n\right)\right\| \geq d_{E_{n-1}}(y_n) \geq \frac{1}{2}\,.$$

It follows from

$$\lim_{n\to\infty} \left\|\frac{1}{\alpha_n} y_n\right\| = \lim_{n\to\infty} \frac{1}{|\alpha_n|} = \frac{1}{|\alpha|}$$

that $\left(\frac{1}{\alpha_n} y_n\right)_{n\in\mathbb{N}}$ is a bounded sequence in E. Thus $\left(u\left(\frac{1}{\alpha_n} y_n\right)\right)_{n\in\mathbb{N}}$ has a convergent subsequence (Proposition 3.1.1.19 a $\Rightarrow$ b), which contradicts the above inequality. Hence α is an isolated point of $\sigma(u)$.

c) $\sigma(u)$ is compact (Corollary 2.2.4.5) and by $b_1 \Rightarrow b_4$,

$$\{\alpha \in \sigma(u) \mid |\alpha| > \varepsilon\}$$

is finite for every $\varepsilon > 0$. The conclusion now follows.

d) Assume that $0 \notin \sigma(u)$. Then u is invertible and it follows from

$$1 = u^{-1}u$$

that $1 \in \mathcal{K}(E)$ (Proposition 3.1.1.11). By Proposition 3.1.1.10, E is finite-dimensional.

e) follows from a). ■

Remark. a) 0 may belong to $\sigma(u)\backslash\sigma_p(u)$ or to $\sigma_p(u)$.

b) Example 3.1.5.5 a) shows that any null sequence in $\mathbb{K}$ can be realized as the point-spectrum of a compact operator.

Corollary 3.1.5.2 *Let E be a complex Banach space. Take $u \in \mathcal{K}(E)$ and $\alpha, \beta \in \sigma(u)\backslash\{0\}$ with $\alpha \neq \beta$. Then*

$$E_a(1 - \frac{1}{\alpha}u) \subset E_b(1 - \frac{1}{\beta}u).$$

Let v (resp. w) be the residue of the resolvent of u in α (resp. β). By Corollary 3.1.3.19, v (resp. w) is the projection of E onto $E_a(1 - \frac{1}{\alpha}u)$ (resp. $E_a(1 - \frac{1}{\beta}u)$. Hence

$$\operatorname{Ker} v = E_b(1 - \frac{1}{\alpha}u) \qquad (\text{resp. } \operatorname{Ker} w = E_b(1 - \frac{1}{\beta}u)).$$

We have $wv = 0$ (Theorem 3.1.5.1 c), Corollary 2.2.6.4), so that

$$wx = wvx = 0, \quad x \in E_b(1 - \frac{1}{\beta}u)$$

whenever $x \in E_a(1 - \frac{1}{\alpha}u)$. Hence

$$E_a(1 - \frac{1}{\alpha}u) \subset E_b(1 - \frac{1}{\beta}u).$$ ■

Corollary 3.1.5.3 (4) *Let E be a Banach algebra. Given $x \in E$, define*

$$\widetilde{x} : E \longrightarrow E, \quad y \longmapsto xy$$

and assume that there is an approximate unit $\mathfrak{F}$ of E such that

$$\{x \in E | \widetilde{x} \in \mathcal{K}(E)\} \in \mathfrak{F}.$$

Take $x \in E$ and $\alpha \in \sigma(x)\backslash\{0\}$. Then α is an isolated point of $\sigma(x)$ and

$$F := \{y \in E | xy = \alpha y\}$$

is a non-trivial finite-dimensional right ideal of E.

By Corollary 3.1.1.15, $\widetilde{x} \in \mathcal{K}(E)$ and by Proposition 2.2.1.17,

$$\sigma(x)\backslash\{0\} \subset \sigma_{\mathcal{L}(E)}(\widetilde{x}) \subset \sigma(x)\,.$$

By Theorem 3.1.5.1 a), b), α is an isolated point of $\sigma(x)$ and

$$\{y \in E | xy = \alpha y\}$$

is a non-trivial finite-dimensional vector subspace of E. It is easy to check that it is a right ideal of E. ■

Example 3.1.5.4 *Let T be a set. Take $p \in \{0\} \cup [1, \infty]$. Put $E := \ell^p(T)$. Take $y \in \ell^\infty(T)$ and define*

$$u : E \longrightarrow E\,, \quad x \longmapsto xy\,.$$

a) *$u \in \mathcal{L}(E)$ and $\|u\| = \|y\|_\infty$.*

b) *$\sigma_p(u) = y(T)$.*

c) *Given $\alpha \in \sigma_p(u)$, the eigenspace of u corresponding to α is*

$$\{x \in E | x(\alpha e_T - y) = 0\}\,.$$

d) *$\sigma(u) = \overline{y(T)}$.*

e) *u is compact iff $y \in c_o(T)$.*

a) is easy to see.

b) and c). Take $\alpha \in \mathbb{K}$ and $x \in E$. Then

$$(ux = \alpha x) \Longleftrightarrow (xy = \alpha x) \Longleftrightarrow (x(\alpha e_T - y) = 0)$$

and b) and c) follow (Proposition 3.1.4.2 b)).

d) By b), Proposition 3.1.4.2 a), and Corollary 2.2.4.5,

$$\overline{y(T)} \subset \sigma(u)\,.$$

Take

$$\alpha \in \mathbb{K}\backslash\overline{y(t)}\,,$$

and define

$$v : E \longrightarrow E\,, \quad x \longmapsto \frac{x}{\alpha e_T - y}\,.$$

Then v is the inverse of $\alpha 1 - u$ in $\mathcal{L}(E)$, so that $\alpha \notin \sigma(u)$ and

$$\sigma(u) = \overline{y(T)}.$$

e) Take $y \in c_o(T)$. Given a finite subset A of T, define

$$u_A : E \longrightarrow E, \quad x \longmapsto (ux)e_A^T.$$

Then

$$u_A \in \mathcal{L}_f(E)$$

and

$$\|u_A - u\| = \|y(1 - e_A)\|_\infty$$

for every $A \in \mathfrak{P}_f(T)$. Hence

$$u \in \overline{\mathcal{L}_f(E)} \subset \mathcal{K}(E)$$

(Proposition 3.1.1.4).

The converse follows from b) and Theorem 3.1.5.1 a), c). ■

Example 3.1.5.5 *Take $p \in \{0\} \cup [1, \infty]$, $(\alpha_n)_{n \in \mathbb{N}}$ a null sequence in $\mathbb{K}$ such that*

$$\alpha_m \neq \alpha_n \neq 0$$

whenever $m, n \in \mathbb{N}$ are distinct, and $f \in \mathbb{N}^{\mathbb{N}}$. Take $k \in \mathbb{N} \cup \{0, \infty\}$. Then there is a compact operator u on ℓ^p satisfying:

a) $\sigma(u) = \{\alpha_n | n \in \mathbb{N}\} \cup \{0\}$.

b) *$f(n)$ is the multiplicity of α_n with respect to u for every $n \in \mathbb{N}$.*

c) *k is the multiplicity of 0 with respect to u.*

Take $y \in \mathbb{K}^{\mathbb{N}}$ for which

$$y(\mathbb{N}) \subset \{\alpha_n | n \in \mathbb{N}\} \cup \{0\},$$

$$\text{Card } y^{-1}(\alpha_n) = f(n)$$

for every $n \in \mathbb{N}$ and

$$\text{Card } y^{-1}(0) = k.$$

By Example 3.1.5.4, the operator

$$\ell^p \longrightarrow \ell^p, \quad x \longmapsto xy$$

has the required properties. ■

Example 3.1.5.6 *Take $p \in \{0\} \cup [1, \infty]$. Let v be the right shift on ℓ^p and $(\alpha_n)_{n\in\mathbb{N}}$ a null sequence, with $\alpha_n \neq 0$ for every $n \in \mathbb{N}$. Take*

$$w : \ell^p \longrightarrow \ell^p, \quad x \longmapsto (\alpha_n x_n)_{n\in\mathbb{N}},$$

and define $u := v \circ w$. Then u is compact and

$$\sigma_p(u) = \emptyset, \quad \sigma(u) = \{0\}.$$

By Example 3.1.5.4 e), w is compact, whence u is also compact (Proposition 3.1.1.11). Take $\alpha \in \mathbb{K}$ and $x \in \ell^p$ with

$$ux = \alpha x.$$

Then

$$\alpha x_1 = 0$$

and

$$\alpha x_{n+1} = \alpha_n x_n$$

for every $n \in \mathbb{N}$. Thus $x = 0$ so that α is not an eigenvalue of u and

$$\sigma_p(u) = \emptyset.$$

It is obvious that u is not invertible, so that $0 \in \sigma(u)$. By Theorem 3.1.5.1 $b_1 \Rightarrow b_2$,

$$\sigma(u) = \{0\}. \qquad \blacksquare$$

Remark. Take $x \in \ell^p$ and $\alpha \in \mathbb{K}\backslash\{0\}$. Define

$$y : \mathbb{N} \longrightarrow \mathbb{K}, \quad n \longmapsto \sum_{k=1}^{n} \frac{x_k}{\alpha^{n+1-k}} \prod_{i=k}^{n-1} \alpha_i.$$

Then $y \in \ell^p$. In fact, $\alpha \notin \sigma(u)$ and it is easy to see that

$$y = (\alpha 1 - u)^{-1} x.$$

Example 3.1.5.7 *Let K be a nonempty compact set of $\mathbb{K}$. Take $A \subset K$, $f : A \to \mathbb{N} \cup \{\infty\}$ and $p \in \{0\} \cup [1, \infty]$. Then there is a set T and an operator u on $\ell^p(T)$ such that:*

a) $\sigma(u) = K$.

b) $\sigma_p(u) = A$.

c) $f(\alpha)$ *is the multiplicity of* α *with respect to* u *for every* $\alpha \in A$.

If A *is countable, then* T *may be chosen to be countable.*

If $p = 2$ *and* $\overline{A} = K$ *then the above* u *may be taken normal (Example 2.3.2.27 c)); if in addition* $K \subset \mathbb{R}$ $(K \subset \{\alpha \in \mathbb{K} \mid |\alpha| = 1\})$ *then the above* u *may be taken selfadjoint (unitary).*

Let B be a countable subset of $K \backslash A$ for which

$$\overline{A \cup B} = K.$$

Define

$$T := \{(\alpha, n) \in A \times \mathbb{N} \mid n \leq f(\alpha)\} \cup (B \times \mathbb{N}).$$

Let v_1 be the right shift on ℓ^p, define

$$v_2 : \ell^p \longrightarrow \ell^p, \quad x \longmapsto \left(\frac{x_n}{n}\right)_{n \in \mathbb{N}}$$

and set $v := v_1 \circ v_2$. Define

$$ux : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} \alpha x(\alpha, n) & \text{if } t = (\alpha, n) \in A \times \mathbb{N} \\ ((v + \beta 1)x(\beta, \cdot))(n) & \text{if } t = (\beta, n) \in B \times \mathbb{N} \end{cases}$$

for $x \in \ell^p(T)$. It is easy to see that u is an operator on $\ell^p(T)$.

Take $\alpha \in \mathbb{K}$ and $x \in \ell^p(T)$ such that

$$ux = \alpha x.$$

Then x vanishes on

$$\{(\beta, n) \in (A \backslash \{\alpha\}) \times \mathbb{N} \mid n \leq f(\beta)\}$$

and, by Example 3.1.5.6, it vanishes on $B \times \mathbb{N}$ as well. Hence $\alpha \in \sigma_p(u)$ iff $\alpha \in A$, and in this case $f(\alpha)$ is the multiplicity of α with respect to u.

Take $\beta \in B$. By Example 3.1.5.6, $\beta \in \sigma(v + \beta 1)$, so that $\beta \in \sigma(u)$. Hence, by Proposition 3.1.4.2 a) and Corollary 2.2.4.5,

$$K \subset \sigma(u).$$

Take $\gamma \in \mathbb{K}\backslash K$ and $x \in \ell^p(T)$. Define

$$y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \begin{cases} \frac{1}{\gamma-\alpha}x(\alpha,n) & \text{if } t=(\alpha,n) \in A \times \mathbb{N} \\ ((\gamma-\beta)1-v)^{-1}x(\beta,\cdot))(n) & \text{if } t=(\beta,n) \in B \times \mathbb{N} \end{cases}$$

(Example 3.1.5.6). Then $y \in \ell^p(T)$ and

$$(\gamma 1 - u))y = x .$$

Hence $\gamma 1 - u$ is surjective. By the above, $\gamma \notin \sigma_p(u)$, so $\gamma \notin \sigma(u)$ (Proposition 3.1.4.2 c)) and

$$\sigma(u) = K .$$

Assume now $p = 2$ and $\overline{A} = K$. Then we may take $B = \emptyset$ and the assertion follows from Example 2.3.2.32 by taking the counting measure as μ. ■

Proposition 3.1.5.8 (1) (5) *Let $(T, \mathfrak{T}, \mu)$ be a measure space and take $u \in \mathcal{K}(L^2(\mu))$. Define*

$$\langle x|y\rangle = \int x\overline{y}d\mu$$

for $x, y \in L^2(\mu)$ and

$$f : L^2(\mu) \times L^2(\mu) \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \langle ux|y\rangle .$$

If the sequilinear form f is Hermitian (positive), then

$$\sigma(u) \subset \mathbb{R} \qquad (\sigma(u) \subset \mathbb{R}_+) .$$

By Theorem 3.1.5.1 $b_1 \Leftrightarrow b_2$,

$$\sigma(u)\backslash\{0\} = \sigma_p(u)\backslash\{0\}$$

and the assertion now follows from Proposition 3.1.4.13. ■

Proposition 3.1.5.9 *Let T be a set and $\ell^{2,2}(T,T)$ the involutive Banach algebra defined in Example 2.3.2.5.*

a) *Given $k \in \ell^{2,2}(T,T)$ and $u \in \mathcal{L}(\ell^2(T))$ there is an $h \in \ell^{2,2}(T,T)$ with*

$$\overset{\cap}{h} = u\overset{\cap}{k} .$$

b) $\sigma(k) = \sigma(\overset{\cap}{k})$ *for every* $k \in \ell^{2,2}(T,T)$.

c) $\ell^{2,2}(T,T)$ *is strongly symmetric.*

a) Define

$$h : T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto (\overset{\cap}{u k e_t})(s).$$

Then

$$\sum_{s,t\in T} |h(s,t)|^2 = \sum_{t\in T} \sum_{s\in T} |(\overset{\cap}{u k e_t})(s)|^2 = \sum_{t\in T} \|\overset{\cap}{u k e_t}\|^2 \leq$$

$$\leq \|u\|^2 \sum_{t\in T} \|\overset{\cap}{k e_t}\|^2 = \|u\|^2 \sum_{t\in T} \sum_{s\in T} |k(s,t)|^2 = \|u\|^2 \|k\|^2,$$

whence $h \in \ell^{2,2}(T,T)$. Since

$$(\overset{\cap}{h e_t})(s) = h(s,t) = (\overset{\cap}{u k e_t})(s)$$

for every $s,t \in T$,

$$\overset{\cap}{h e_t} = \overset{\cap}{u k e_t}$$

for every $t \in T$ and

$$\overset{\cap}{h} = \overset{\cap}{u k}.$$

b) follows from Example 2.2.2.5 c).

c) Take $k \in \ell^{2,2}(T,T)$. Define

$$\langle x|y\rangle = \sum_{t\in T} x(t)\overline{y(t)}$$

for $x,y \in \ell^2(T)$ and

$$f : \ell^2(T) \times \ell^2(T) \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \langle \overset{\cap}{(k^* \circ k)} x|y\rangle.$$

By Proposition 2.3.3.11 d), f is a positive sesquilinear form and by Proposition 3.1.2.3 $\overset{\cap}{(k^* \circ k)}$ is compact. Hence, by Proposition 3.1.5.8,

$$\sigma(\overset{\cap}{(k^* \circ k)}) \subset \mathbb{R}_+,$$

whence $\ell^{2,2}(T,T)$ is strongly symmetric. ∎

Theorem 3.1.5.10 (Lomonosov, 1973) *Let E be a Banach space and take $u \in \mathcal{K}(E) \backslash \mathbb{K}1$. Then E has a $\{u\}^c$-invariant closed non-trivial proper vector subspace.*

Case 1 $\quad \sigma(u) \neq \{0\}$

Take $\alpha \in \sigma(u) \backslash \{0\}$ and put

$$F := \mathrm{Ker}\,(\alpha 1 - u)\,.$$

By Theorem 3.1.5.1 $\mathrm{b}_1 \Rightarrow \mathrm{b}_2$, $F \neq \{0\}$. Since $u \notin \mathbb{K}1$, $F \neq E$. Take $v \in \{u\}^c$ and $x \in F$. Then

$$vx = v\frac{1}{\alpha}ux = \frac{1}{\alpha}uvx\,.$$

Thus $vx \in F$ and hence

$$v(F) \subset F\,.$$

Thus F is $\{u\}^c$-invariant.

Case 2 $\quad \sigma(u) = \{0\}$

We may assume that $\|u\| = 1$. Take $x \in E$ with $\|ux\| > 1$. Then

$$\|x\| = \|u\|\,\|x\| \geq \|ux\| > 1\,,$$

so that

$$0 \notin \overline{U_1(x)}\,.$$

If $y \in \overline{U_1(x)}$, then

$$\|ux - uy\| = \|u(x-y)\| \leq \|u\|\,\|x-y\| \leq 1\,.$$

Thus

$$uy \in \overline{U_1(ux)}\,,$$

$$K := \overline{u(U_1(x))} \subset \overline{U_1(ux)}\,,$$

$$0 \notin K\,.$$

Since u is compact, K is compact.

Assume that the assertion of the theorem does not hold. Take $y \in E\backslash\{0\}$. Since $\{u\}^c$ is a unital subalgebra of $\mathcal{L}(E)$ (Proposition 2.1.1.17 a)), $\{vy|v \in \{u\}^c\}$ is a vector subspace of E. By our assumption it is dense in E. Hence there is a $v_y \in \{u\}^c$ such that

$$v_y y \in U_1(x).$$

Then $\left(\overline{v}_y^{-1}(U_1(x))\right)_{y\in E\backslash\{0\}}$ is an open covering of K. Hence there is a finite subset A of $E\backslash\{0\}$ such that

$$K \subset \bigcup_{y\in A} \overline{v}_y^{-1}(U_1(x)).$$

Put

$$\mathcal{F} := \{v_y | y \in A\},$$

$$\alpha := \sup_{v\in\mathcal{F}} \|v\| < \infty.$$

Now construct a sequence $(v_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ inductively such that

$$v_n v_{n-1} \dots v_1 u^n x \in U_1(x)$$

for every $n \in \mathbb{N}$. Since $ux \in K$, there is a $v_1 \in \mathcal{F}$ with

$$v_1 ux \in U_1(x).$$

Take $n \in \mathbb{N}$ and suppose that the sequence has been constructed up to the n-th element. Since

$$v_n v_{n-1} \dots v_1 u^{n+1} x = u(v_n v_{n-1} \dots v_1 u^n x) \in K,$$

there is a $v_{n+1} \in \mathcal{F}$ such that

$$v_{n+1} v_n \dots v_1 u^{n+1} x \in U_1(x).$$

This completes the inductive construction.

We have

$$\|v_n v_{n-1} \dots v_1 u^n\| \leq \alpha^n \|u^n\|,$$

$$\|x\| - 1 < \|v_n v_{n-1} \dots v_1 u^n x\|^{\frac{1}{n}} \leq \|v_n v_{n-1} \dots v_1 u^n\|^{\frac{1}{n}} \|x\|^{\frac{1}{n}} \leq \alpha \|u^n\|^{\frac{1}{n}} \|x\|^{\frac{1}{n}},$$

$$0 < \|x\| - 1 \leq \alpha \lim_{n\to\infty} \|u^n\|^{\frac{1}{n}} = \alpha r(u) = 0$$

(Proposition 2.2.5.4) which is a contradiction. ■

Corollary 3.1.5.11 *Let E be an infinite-dimensional Banach space and take $u \in \mathcal{L}(E)$. If there is a $P \in \mathbb{K}[t]$ such that $P(u)$ is compact, then E has a u-invariant closed non-trivial proper vector subspace.*

If $P(u) \notin \mathbb{K}1$, then the assertion follows immediately from Lomonosov Theorem since $u \in \{P(u)\}^c$. If $P(u) \in \mathbb{K}1$, then we may take an $x \in E\backslash\{0\}$ and define F as the vector subspace of E generated by $\{u^k x \mid k \in \mathbb{N}_n \cup \{0\}\}$, where n is the degree of P. ■

3.1.6 Integral Operators

Definition 3.1.6.1 (0) *Let S be a set, $(T, \mathfrak{T}, \nu)$ a measure space, and take*

$$k : S \times T \longrightarrow \mathbb{K}\,.$$

We write $\mathcal{D}_o(k,\nu)$ for the set of functions $x : T \to \mathbb{K}$ for which $k(s,\cdot)x$ is ν-integrable for every $s \in S$ and put

$$k^\nu x := kx : S \longrightarrow \mathbb{K}\,, \quad s \longmapsto \int k(s,t)x(t)d\nu(t)$$

for every $x \in \mathcal{D}_o(k,\nu)$.

Proposition 3.1.6.2 (0) *Let S be a compact space, $(T, \mathfrak{T}, \nu)$ a measure space, p, q conjugate exponents, and*

$$k : S \times T \longrightarrow \mathbb{K}$$

a function for which $k(s,\cdot) \in \mathcal{L}^q(\nu)$ whenever $s \in S$ and the map

$$S \longrightarrow L^q(\nu)\,, \quad s \longmapsto k(s,\cdot)$$

is continuous. Then x belongs to $\mathcal{D}_o(k,\nu)$ and kx to $C(S)$ whenever $x \in \mathcal{L}^p(\nu)$ and the map

$$L^p(\nu) \longrightarrow C(S)\,, \quad x \longmapsto kx$$

is a compact operator of norm at most $\sup\limits_{s\in S} \|k(s,\cdot)\|_q$.

Take $x \in \mathcal{L}^p(\nu)$. By Hölder's Theorem, $k(s,\cdot)x \in \mathcal{L}^1(\nu)$ and

$$\|k(s,\cdot)x\|_1 \leq \|k(s,\cdot)\|_q \|x\|_p$$

for every $s \in S$. Hence x belongs to $\mathcal{D}_o(k,\nu)$ and by the hypothesis on k, kx is continuous. Since

$$|kx(s)| \leq \|k(s,\cdot)x\|_1$$

for every $s \in S$,

$$\|ks\| \leq \alpha \|x\|_p\,,$$

where

$$\alpha := \sup_{s \in S} \|k(s, \cdot)\|_q .$$

Hence the map

$$L^p(\nu) \longrightarrow C(S), \quad x \longmapsto kx$$

is an operator of norm at most α.

Let $\mathcal{F}$ be a bounded set of $L^p(\nu)$. Put

$$\beta := \sup_{x \in \mathcal{F}} \|x\|_p < \infty .$$

Since

$$\sup_{x \in \mathcal{F}} \|kx\| \leq \alpha\beta < \infty ,$$

$\{kx | x \in \mathcal{F}\}$ is bounded.

Take $s \in S$ and $\varepsilon > 0$. There is a neighbourhood U of s, such that

$$\|k(s, \cdot) - k(s', \cdot)\|_q < \frac{\varepsilon}{\beta + 1}$$

for every $s' \in U$. By Hölder's Theorem,

$$|kx(s) - kx(s')| = |\int k(s,t)x(t)d\nu(t) - \int k(s',t)x(t)d\nu(t)| \leq$$

$$\leq \int |(k(s,t) - k(s',t))x(t)|d\nu(t) \leq$$

$$\leq \|k(s, \cdot) - k(s', \cdot)\|_q \|x\|_p \leq \frac{\varepsilon}{\beta + 1}\beta < \varepsilon$$

for $x \in \mathcal{F}$ and $s' \in U$. Hence $\{kx | x \in \mathcal{F}\}$ is equicontinuous. By Ascoli's Theorem (Theorem 1.1.2.16), $\{kx | x \in \mathcal{F}\}$ is a relatively compact set of $C(S)$ and so the operator

$$L^p(\nu) \longrightarrow C(S), \quad x \longmapsto kx$$

is compact. ■

Lemma 3.1.6.3 (0) *Let S be a topological space, T a compact space, and*

$$k : S \times T \longrightarrow \mathbb{K}$$

a continuous function. Then $\{k(\cdot, t) \mid t \in T\}$ is equicontinuous.

Choose $s \in S$ and $\varepsilon > 0$. Given $a \in T$, there is a neighbourhood U_a of s and an open neighbourhood V_a of a such that

$$|k(s', b) - k(s, a)| < \frac{\varepsilon}{2}$$

for every $(s', b) \in U_a \times V_a$. $(V_a)_{a \in T}$ is a open covering of T. Since T is compact, there is a finite subset A of T such that

$$T = \bigcup_{a \in A} V_a .$$

Put

$$U := \bigcap_{a \in A} U_a .$$

U is a neighbourhood of s. Take $s' \in U$ and $t \in T$. There is an $a \in A$ with $t \in V_a$. Then

$$|k(s', t) - k(s, t)| \leq |k(s', t) - k(s, a)| + |k(s, a) - k(s, t)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Hence $\{k(\cdot, t) \mid t \in T\}$ is equicontinuous. ■

Corollary 3.1.6.4 (0) *Let S, T be compact spaces, ν a positive Radon measure on T, and take $k \in \mathcal{C}(S \times T)$. Then x belongs to $\mathcal{D}_0(k, \nu)$ and kx belongs to $\mathcal{C}(S)$ whenever $x \in \mathcal{L}^1(\nu)$ and the map*

$$u : L^1(\nu) \longrightarrow \mathcal{C}(S), \quad x \longmapsto kx$$

is a compact operator. Identifying $L^1(\nu)'$ canonically with $L^\infty(\nu)$, $\operatorname{Im} u' \subset \mathcal{C}(T)$ and

$$u'\mu(t) = \int k(\cdot, t) d\mu$$

for every Radon measure μ on S and $t \in T$.

Take $s \in S$ and $\varepsilon > 0$. By Lemma 3.1.6.3, there is a neighbourhood U of s such that

$$|k(s, t) - k(s', t)| < \varepsilon$$

for every $s' \in U$ and $t \in T$. Hence

$$\|k(s, \cdot) - k(s', \cdot)\|_\infty < \varepsilon$$

for every $s' \in U$, i.e. the map

$$S \longrightarrow L^\infty(\nu), \quad s \longmapsto k(s,\cdot)$$

is continuous. By Proposition 3.1.6.2, x belongs to $\mathcal{D}_o(k,\nu)$ and kx to $C(S)$ for every $x \in \mathcal{L}^1(\nu)$ and u is a compact operator.

Let μ be a Radon measure on $C(S)$ and put

$$y : T \longrightarrow \mathbb{K}, \quad t \longmapsto \int k(\cdot,t)d\mu .$$

Then y is continuous and

$$\langle x, u'\mu\rangle = \langle ux, \mu\rangle = \iint k(s,t)x(t)d\nu(t))d\mu(s) =$$

$$= \int (\int k(s,t)d\mu(s))x(t)d\nu(t) = \int yxd\nu = \langle x, y\rangle$$

for every $x \in L^1(\nu)$ (Fubini and Example 1.2.2.10), so that

$$u'\mu = y \in C(T)$$

and $\operatorname{Im} u' \subset C(T)$ (Example 1.2.2.10). ■

Definition 3.1.6.5 (0) *Let S,T be sets and take $k : S \times T \to \mathbb{K}$. We define*

$$k' : T \times S \longrightarrow \mathbb{K}, \quad (t,s) \longmapsto k(s,t)$$

and

$$k^* := \overline{k'}$$

*and call k' **the transposed kernel of** k and k^* **the adjoint kernel of** k.*

Corollary 3.1.6.6 (5) *Let S,T be compact spaces and μ (resp. ν) a positive Radon measure on S (resp. T). Take $k \in C(S,T)$ and*

$$u : C(T) \longrightarrow C(S), \quad y \longmapsto k^\nu y$$

(Corollary 3.1.6.4). Identify $C(S)'$ and $C(T)'$ with the Banach space of Radon measures on S and T, respectively (Example 1.2.2.10), and define

$$i : C(S) \longrightarrow C(S)', \quad x \longmapsto x \cdot \mu,$$

$$j : C(T) \longrightarrow C(T)', \quad y \longmapsto y \cdot \nu .$$

Then u is compact and

$$u'ix = jk'^\mu x$$

for every $x \in C(S)$.

By Corollary 3.1.6.4 (and Proposition 3.1.1.11), u is compact. Given $y \in C(T)$,

$$\langle y, u'ix\rangle = \langle uy, ix\rangle = \int \left(\int k(s,t)y(t)d\nu(t)\right) x(s)d\mu(s) =$$

$$= \int \left(\int k(s,t)x(s)d\mu(s)\right) y(t)d\nu(t) = \langle y, jk'^{\mu}x\rangle$$

(Corollary 3.1.6.4 and Fubini's Theorem), so that

$$u'ix = jk'^{\mu}x\,. \qquad \blacksquare$$

Corollary 3.1.6.7 (5) *Let T be a compact space and μ a positive Radon measure on T. Take $k \in C(T \times T)$,*

$$u : C(T) \longrightarrow C(T)\,, \quad x \longmapsto kx$$

(Corollary 3.1.6.4), and $\alpha \in \mathbb{K}\backslash\{0\}$. Then

$$\operatorname{Dim}\operatorname{Ker}(\alpha 1 - u) = \operatorname{Dim}\{y \in C(T) | k'y = \alpha y\}$$

and the following are equivalent for $x \in C(T)$:

a) $x \in \operatorname{Im}(\alpha 1 - u)$.

b) *For every $y \in C(T)$ such that $k'y = \alpha y$,*

$$\int xyd\mu = 0\,.$$

Identify $C(T)'$ with the Banach space of Radon measures on T (Example 1.2.2.10) and define

$$i : C(T) \longrightarrow C(T)'\,, \quad y \longmapsto y \cdot \mu\,.$$

We want to prove first

$$i(\{y \in C(T) | k'y = \alpha y\}) = \operatorname{Ker}(\alpha 1 - u')\,.$$

Let $y \in C(T)$ with

$$k'y = \alpha y\,.$$

Then, by Corollary 3.1.6.6,

$$u'iy = ik'y = \alpha iy\,,$$

$$iy \in \mathrm{Ker}\,(\alpha 1 - u')\,.$$

This proves the inclusion from left to the right.

Take now $\nu \in \mathrm{Ker}\,(\alpha 1 - u')$. Then $\nu = \frac{1}{\alpha}u'\nu$. By Corollary 3.1.6.4, $\nu = iy$ for some $y \in \mathcal{C}(T)$. Then

$$ik'y = u'iy = u'\nu = \alpha\nu = i\alpha y$$

(Corollary 3.1.6.6), so that

$$k'y = \alpha y\,.$$

This proves the inclusion from right to the left.

From the above relation and Theorem 3.1.5.1 a) (and Corollary 3.1.6.6) we get

$$\mathrm{Dim}\,\{y \in \mathcal{C}(T) | k'y = \alpha y\} = \mathrm{Dim}\,\mathrm{Ker}\,(\alpha 1 - u') = \mathrm{Dim}\,\mathrm{Ker}\,(\alpha 1 - u)\,.$$

a $\Rightarrow$ b. Let $y \in \mathcal{C}(T)$ such that

$$k'y = \alpha y\,.$$

By the above result,

$$iy \in \mathrm{Ker}(\alpha 1 - u')\,,$$

so by Proposition 1.3.5.8,

$$\int xyd\mu = \langle x, iy\rangle = 0\,.$$

b $\Rightarrow$ a. Let $\nu \in \mathrm{Ker}\,(\alpha 1 - u')$. By the above result, there is a $y \in \mathcal{C}(T)$ such that

$$k'y = \alpha y\,, \quad iy = \nu\,.$$

We get

$$\langle x, \nu\rangle = \int xd\nu = \int xyd\mu = 0\,,$$

by b). Hence

$$x \in {}^{\circ}(\mathrm{Ker}\,(\alpha 1 - u'))\,.$$

The conclusion now follows from Theorem 3.1.5.1 a) (and Corollary 3.1.6.6). ■

Corollary 3.1.6.8 (4) *If T is a compact group, then for each $x \in L^1(T)$, the operator*

$$L^1(T) \longrightarrow L^1(T), \quad y \longmapsto x * y$$

is compact.

Given $x \in L^1(T)$, define

$$\widetilde{x} : L^1(T) \longrightarrow L^1(T), \quad y \longmapsto x * y.$$

Take $x \in C(T)$. Then

$$T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto x(st^{-1})$$

is continuous, whence

$$L^1(T) \longrightarrow C(T), \quad y \longmapsto x * y$$

is compact by Corollary 3.1.6.6. Since the inclusion map $C(T) \to L^1(T)$, is continuous, $\widetilde{x}$ is compact (Proposition 3.1.1.11).

Now take $x \in L^1(T)$. There is a sequence $(x_n)_{n\in\mathbb{N}}$ in $C(T)$ which converges to x in $L^1(T)$. Then $(\widetilde{x}_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{K}(L^1(T))$ converging to $\widetilde{x}$ (Proposition 3.1.6.2). By Theorem 3.1.1.3, $\widetilde{x}$ is compact. ■

Remark. Let T be a unimodular locally compact group. If the above conclusion holds, then T is compact (Example 3.1.2.7).

Corollary 3.1.6.9 (4) *Let T be a compact group. Take $x \in L^1(T)$, $\alpha \in \sigma(x)\backslash\{0\}$ (Theorem 2.2.2.10 e)), and put*

$$E = \{y \in L^1(T) | x * y = \alpha y\}.$$

Then α is an isolated point of $\sigma(x)$ and E is a finite-dimensional right ideal of $L^1(T)$ satisfying

$$\{0\} \neq E \subset C(T).$$

By Proposition 2.2.2.13, there is an approximate unit $\mathfrak{F}$ of $L^1(T)$ with $C(T) \in \mathfrak{F}$. By Corollary 3.1.6.8,

$$L^1(T) \longrightarrow L^1(T), \quad y \longmapsto x * y$$

is a compact operator whenever $x \in L^1(T)$. Hence, by Corollary 3.1.5.3, α is an isolated point of $\sigma(x)$ and E is a non-trivial finite-dimensional right ideal of $L^1(T)$.

Take $y \in E$. Then $y * z \in E \cap C(T)$ for $z \in C(T)$ (Proposition 2.2.2.14 g)) and

$$y = \lim_{z,\mathfrak{F}} y * z .$$

Since E is finite-dimensional, $y \in C(T)$. ■

Lemma 3.1.6.10 (4) *Let T be a compact group, λ the Haar measure on T, and $x \in \mathcal{L}^1(\lambda)$. Then the function*

$$y : T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto x(st^{-1})$$

belongs to $\mathcal{L}^1(\lambda \otimes \lambda)$ and

$$\|y\|_{1,\lambda\otimes\lambda} = \|x\|_{1,\lambda}\lambda(T) .$$

Since

$$T \times T \longrightarrow T, \quad (s,t) \longmapsto st^{-1}$$

is continuous, y is $\lambda \otimes \lambda$-measurable. By Tonelli's Theorem,

$$\int^* |y| d(\lambda \otimes \lambda) = \int^* \left(\int^* |x(st^{-1})| d\lambda(s) \right) d\lambda(t) =$$

$$= \int^* \|x\|_{1,\lambda} d\lambda(t) = \|x\|_{1,\lambda}\lambda(T) ,$$

whence $y \in \mathcal{L}^1(\lambda \otimes \lambda)$ and

$$\|y\|_{1,\lambda\otimes\lambda} = \|x\|_{1,\lambda}\lambda(T) .$$ ■

Theorem 3.1.6.11 (4) *If T is a compact group, then the involutive Banach algebra $L^1(T)$ (Proposition 2.3.2.29 a)) is symmetric.*

It is enough to consider $\mathbb{K} = \mathbb{C}$. Take $x \in \operatorname{Re} L^1(T)$ and $\alpha \in \sigma(x) \setminus \{0\}$. By Corollary 3.1.6.9,

$$x * y = \alpha y$$

for some $y \in \mathcal{L}(T) \setminus \{0\}$. Then

$$\alpha\|y\|_2^2 = \int (\alpha y)\overline{y}d\lambda = \int (x * y)\overline{y}d\lambda = \int \left(\int x(st^{-1})y(t)d\lambda(t)\right)\overline{y}(s)d\lambda(s) =$$

$$= \int y(t)\left(\int x(st^{-1})\overline{y}(s)d\lambda(s)\right)d\lambda(t) = \int y(t)\left(\int \overline{x}(ts^{-1})\overline{y}(s)d\lambda(s)\right)d\lambda(t) =$$

$$= \int y\ \overline{x * y}\ d\lambda = \int y\ \overline{\alpha}\,\overline{y}\ d\lambda = \overline{\alpha}\|y\|_2^2,$$

where λ is Haar measure (Lemma 3.1.6.10). It follows, in succession, that

$$\alpha = \overline{\alpha}, \quad \alpha \in \mathbb{R}, \quad \sigma(x) \subset \mathbb{R},$$

and $L^1(T)$ is symmetric (Corollary 2.4.2.4 b $\Rightarrow$ a). ■

Theorem 3.1.6.12 **(4)** *Let T be a compact group and λ its Haar measure. Take $p \in [1, \infty]$ and $x \in L^1(T)$. Then the operators*

$$L^p(T) \longrightarrow L^p(T), \quad y \longmapsto (x \cdot \lambda) * y$$

and

$$C(T) \longrightarrow C(T), \quad y \longmapsto (x \cdot \lambda) * y,$$

(Proposition 2.2.2.14 a),b),c)) are compact.

First suppose that x is continuous. By Proposition 2.2.2.14 f) and Corollary 3.1.6.4, the operator

$$L^1(T) \longrightarrow C(T), \quad y \longmapsto (x \cdot \lambda) * y$$

is compact. Since the inclusion maps

$$C(T) \longrightarrow L^p(T) \longrightarrow L^1(T)$$

are linear and continuous, the assertion follows from Proposition 3.1.1.11.

Now take arbitrary x. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $C(T)$ converging to x in $L^1(T)$. Then the associated sequence of operators converges to the given operator (Proposition 2.2.2.14 c)) and the conclusion follows from Theorem 3.1.1.3. ■

Example 3.1.6.13 *Let* $\mathbb{K} = \mathbb{C}$. *Take* $(\alpha_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{Z})$. *Define*

$$k : \mathbf{T} \times \mathbf{T} \longrightarrow \mathbb{C}, \quad (s,t) \longmapsto \sum_{n\in\mathbb{Z}} \alpha_n e^{in(s-t)}.$$

Let λ *be the Lebesgue measure on* $\mathbf{T}$ *and define*

$$u : \mathcal{C}(\mathbf{T}) \longrightarrow \mathcal{C}(\mathbf{T}), \quad x \longmapsto k^{\lambda} x.$$

Then u *is a compact operator,*

$$\sigma_p(u) = \{2\pi\alpha_n | n \in \mathbb{Z}\},$$

and for $\alpha \in \mathbb{C}$,

$$\text{Card}\,\{n \in \mathbb{Z} | 2\pi\alpha_n = \alpha\}$$

is the multiplicity of α *with respect to* u.

u is compact by Corollary 3.1.6.6. Given $k \in \mathbb{Z}$, define

$$x_k : \mathbf{T} \longrightarrow \mathbb{C}, \quad t \longmapsto e^{ikt}.$$

Then

$$ux_k(s) = \sum_{n\in\mathbb{Z}} \alpha_n e^{ins} \int_{-\pi}^{\pi} e^{-int} e^{ikt} dt = 2\pi\alpha_k x_k(s)$$

for every $s \in \mathbf{T}$, whence

$$ux_k = 2\pi\alpha_k x_k$$

for every $k \in \mathbb{Z}$.

Take $x \in \mathcal{C}(\mathbf{T})$. It follows from the theory of Fourier series that

$$\sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} x(t) dt \right) x_n = x$$

in $L^1(\lambda)$, so that

$$ux = \sum_{n\in\mathbb{Z}} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-int} x(t) dt \right) ux_n = \sum_{n\in\mathbb{Z}} \alpha_n \left(\int_{-\pi}^{\pi} e^{-int} x(t) dt \right) x_n$$

in $\mathcal{C}(\mathbf{T})$ (Corollary 3.1.6.4). Take $\alpha \in \mathbb{C}$ and assume that

$$ux = \alpha x.$$

Then

$$\sum_{n\in\mathbb{Z}}(2\pi\alpha_n-\alpha)\left(\int_{-\pi}^{\pi}e^{-\mathrm{int}}x(t)dt\right)x_n=0$$

in $L^1(\lambda)$, so that

$$2\pi\alpha_n\neq\alpha\Longrightarrow\int_{-\pi}^{\pi}e^{-\mathrm{int}}x(t)dt=0$$

for every $n\in\mathbb{Z}$ and

$$x=\sum_{\substack{n\in\mathbb{Z}\\2\pi\alpha_n=\alpha}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-\mathrm{int}}x(t)dt\right)x_n\,.$$

Hence

$$\mathrm{Card}\,\{n\in\mathbb{N}|2\pi\alpha_n=\alpha\}$$

is the multiplicity of α with respect to u and

$$\sigma_p(u)=\{2\pi\alpha_n|n\in\mathbb{Z}\}\,.$$ ■

Definition 3.1.6.14 (0) *A measure space $(T,\mathfrak{T},\mu)$ is called σ-finite if there is a sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathfrak{T}$ with*

$$\bigcup_{n\in\mathbb{N}}A_n=T\,.$$

Definition 3.1.6.15 (0) *Let $(S,\mathfrak{S},\mu)$ and $(T,\mathfrak{T},\nu)$ be measure spaces and take*

$$k:S\times T\longrightarrow\mathbb{K}\,.$$

Then $\mathcal{D}(k,\mu,\nu)$ denotes the set of functions $x:T\to\mathbb{K}$ for which $k(s,\cdot)x$ is ν-integrable for μ-almost every $s\in S$ and for $x\in\mathcal{D}(k,\mu,\nu)$ define

$$k^\nu x:=kx:S\longrightarrow\mathbb{K}\,,\quad s\longmapsto\begin{cases}\int k(s,t)x(t)d\nu(t) & \text{if}\quad k(s,\cdot)x\in L^1(\nu)\\ 0 & \text{if}\quad k(s,\cdot)x\notin L^1(\nu)\,.\end{cases}$$

The notation $k^\nu x=kx$ is consistent with the notation introduced in Definition 3.1.6.1.

Proposition 3.1.6.16 (0) *Let $(S,\mathfrak{S},\mu)$ and $(T,\mathfrak{T},\nu)$ be σ-finite measure spaces and p,q conjugate exponents. Take $x\in\mathcal{L}^p(\nu)$ and $k\in\mathcal{L}^q(\mu\otimes\nu)$. Then x belongs to $\mathcal{D}(k,\mu,\nu)$, kx belongs to $\mathcal{L}^q(\mu)$, and*

$$\|kx\|_{q,\mu}\leq\|k\|_{q,\mu\otimes\nu}\|x\|_{p,\nu}\,.$$

Define

$$A := \{s \in S | k(s,\cdot) \notin \mathcal{L}^q(\nu)\}.$$

By Fubini's Theorem, A is a μ–null set and by Hölder's Theorem,

$$k(s,\cdot)x \in \mathcal{L}^1(\nu)$$

and

$$\int |k(s,\cdot)x| d\nu \leq \|k(s,\cdot)\|_{q,\nu} \|x\|_{p,\nu}$$

for $s \in S \backslash A$. Thus $x \in \mathcal{D}(k,\mu,\nu)$.

Take $B \in \mathfrak{S}$ and $C \in \mathfrak{T}$ (resp. $C = T$ if $p = \infty$). Then $e_B^S \otimes x e_C^T \in \mathcal{L}^p(\mu \otimes \nu)$ and so $k(e_B^S \otimes x e_C^T) \in \mathcal{L}^1(\mu \otimes \nu)$ (Hölder). Hence

$$S \longrightarrow \mathbb{K}, \quad s \longmapsto e_B^S(s) \int_C k(s,\cdot)x d\nu$$

is μ–integrable (Fubini) and therefore $\mathfrak{S}$–measurable. Since B and C are arbitrary and since $(T,\mathfrak{T},\nu)$ is σ–finite, it follows that kx is μ–measurable. By Fubini's Theorem, the above inequality implies that

$$\left(\int \left|\int k(s,\cdot)x d\nu\right|^q d\mu(s)\right)^{\frac{1}{q}} \leq \left(\int \left(\int |k(s,\cdot)|^q d\nu\right) d\mu(s)\right)^{\frac{1}{q}} \|x\|_{p,\nu} =$$

$$= \|k\|_{q,\mu\otimes\nu} \|x\|_{p,\nu},$$

and so $kx \in \mathcal{L}^q(\mu)$ and

$$\|kx\|_{q,\mu} \leq \|k\|_{q,\mu\otimes\nu} \|x\|_{p,\nu}$$

if $q \neq \infty$. These assertions are trivial if $q = \infty$. ■

Proposition 3.1.6.17 (0) *Let p,q be conjugate exponents and $(S,\mathfrak{S},\mu)$ and $(T,\mathfrak{T},\nu)$ measure spaces such that $(T,\mathfrak{T},\nu)$ is σ-finite if $q = \infty$. Take $k \in L^q(\mu \otimes \nu)$ and*

$$\widetilde{k} : L^p(\nu) \longrightarrow L^q(\mu), \quad x \longmapsto k^\nu x$$

(Proposition 3.1.6.16).

a) *$\widetilde{k}$ is linear and continuous and*

$$\|\widetilde{k}\| \leq \|k\|_{q,\mu\otimes\nu}.$$

b) *If* $q \neq \infty$ *or if* μ *and* ν *are bounded, then* $\tilde{k}$ *is compact.*

c) *Given* $(x, y) \in \mathcal{L}^p(\nu) \times \mathcal{L}^p(\mu)$,

$$\int (y \otimes x) k d(\mu \otimes \nu) = \int y k^{\nu} x d\mu = \int x k'^{\mu} y d\nu .$$

d) *Given* $y \in L^p(\mu)$,

$$(\tilde{k})' y = k'^{\mu} y .$$

e) *If* $p \neq \infty$ *then* $\tilde{k} = 0$ *implies that* $k = 0 \quad \mu \otimes \nu$*–a.e.*

f) *If* $(R, \mathfrak{R}, \lambda)$ *is a measure space,* $p = q = 2$ *and* $u \in \mathcal{L}(L^2(\mu), L^2(\lambda))$, *then there is a unique* $h \in L^2(\lambda \otimes \nu)$ *such that*

$$u\tilde{k}x = h^{\nu} x$$

for every $x \in L^2(\nu)$. *Furthermore*

$$\|h\|_{2,\lambda\otimes\nu} \leq \|u\| \, \|k\|_{2,\mu\otimes\nu} .$$

We may assume $(S, \mathfrak{S}, \mu)$ and $(T, \mathfrak{T}, \nu)$ to be σ–finite.

a) follows from Proposition 3.1.6.16.

b) If there is a finite family $(x_\iota, y_\iota)_{\iota \in I}$ in $L^q(\mu) \times L^q(\nu)$ with

$$k = \sum_{\iota \in I} x_\iota \otimes y_\iota ,$$

then $\tilde{k} \in \mathcal{L}_f(L^p(\nu), L^q(\mu))$. Since the set of such functions is dense in $L^q(\mu \otimes \nu)$ it follows from a) and Proposition 3.1.1.4 that $\tilde{k}$ is compact.

c) It follows from $y \otimes x \in \mathcal{L}^p(\mu \otimes \nu)$ that

$$(y \otimes x) k \in \mathcal{L}^1(\mu \otimes \nu)$$

(Hölder). By Fubini's Theorem,

$$\int y k^{\nu} x d\mu = \int y(s) \Big(\int k(s,t) x(t) d\nu(t) \Big) d\mu(s) =$$

$$= \int (y \otimes x) k d(\mu \otimes \nu) = \int \Big(\int k(s,t) y(s) d\mu(s) \Big) x(t) d\nu(t) =$$

$$= \int \Big(\int k'(t,s) y(s) d\mu(s) \Big) x(t) d\nu(t) = \int x k'^{\mu} y d\nu .$$

d) Given $x \in L^p(\nu)$,

$$\langle x, (\widetilde{k})'y\rangle = \langle \widetilde{k}x, y\rangle = \int yk^{\nu}xd\mu = \int xk'^{\mu}yd\nu = \langle x, k'^{\mu}y\rangle$$

by c) and so

$$(\widetilde{k})'y = k'^{\mu}y\,.$$

e) By c),

$$\int (y \otimes x)kd(\mu \otimes \nu) = 0$$

for $(x, y) \in L^p(\nu) \times L^p(\mu)$. Since the vector subspace of $L^p(\mu \otimes \nu)$ generated by

$$\{y \otimes x | (x, y) \in L^p(\nu) \times L^p(\mu)\}$$

is dense in $L^p(\mu \otimes \nu)$,

$$k = 0 \quad \mu \otimes \nu\text{–a.e.}$$

f) Let $((x_\iota, y_\iota))_{\iota \in I}$ be a finite family in $L^2(\mu) \times L^2(\nu)$. By Fubini's Theorem,

$$\left\| \sum_{\iota \in I} x_\iota \otimes y_\iota \right\|^2_{2,\mu\otimes\nu} = \int \left(\int \left| \sum_{\iota \in I} x_\iota(s) y_\iota(t) \right|^2 d\mu(s) \right) d\nu(t) =$$

$$= \int \left\| \sum_{\iota \in I} y_\iota(t) x_\iota \right\|^2_{2,\mu} d\nu(t)\,,$$

so that

$$\left\| \sum_{\iota \in I} (ux_\iota) \otimes y_\iota \right\|^2_{2,\lambda\otimes\nu} = \int \left\| \sum_{\iota \in I} y_\iota(t) u x_\iota \right\|^2_{2,\lambda} d\nu(t) =$$

$$= \int \left\| u \left(\sum_{\iota \in I} y_\iota(t) x_\iota \right) \right\|^2_{2,\lambda} d\nu(t) \leq$$

$$\leq \|u\|^2 \int \left\| \sum_{\iota \in I} y_\iota(t) x_\iota \right\|^2_{2,\mu} d\nu(t) =$$

$$= \|u\|^2 \left\| \sum_{\iota \in I} x_\iota \otimes y_\iota \right\|_{2,\mu\otimes\nu}^2 .$$

Let $\mathcal{F}$ be the subspace of $L^2(\mu \otimes \nu)$ generated by

$$\{x \otimes y | (x, y) \in L^2(\mu) \times L^2(\nu)\} .$$

By the above, there is a continuous linear map

$$v : \mathcal{F} \longrightarrow L^2(\lambda \otimes \nu)$$

such that

$$v(x \otimes y) = (ux) \otimes y$$

for every $(x, y) \in L^2(\mu) \times L^2(\nu)$ and

$$\|v\| \leq \|u\| .$$

By continuity, v may be extended to an operator

$$L^2(\mu \otimes \nu) \longrightarrow L^2(\lambda \otimes \nu) .$$

Put

$$h := vk .$$

Then

$$\|h\|_{2,\lambda\otimes\nu} \leq \|v\| \, \|k\|_{2,\mu\otimes\nu} \leq \|u\| \, \|k\|_{2,\mu\otimes\nu} .$$

If $k \in \mathcal{F}$, then

$$\widetilde{uk}x = hx$$

for every $x \in L^2(\nu)$. By continuity, the above holds for arbitrary k. The uniqueness of h is obvious. ■

Remark. The inequality of a) cannot be improved to an equality as shown by the following. Take

$$S := T := \{0, 1\} ,$$

$$\mu := \nu := \text{counting measure on } \{0, 1\} ,$$

$$p := q := 2$$

and

$$k(s,t) := \begin{cases} 1 & \text{if } (s,t) = (0,0) \\ 2 & \text{if } (s,t) = (1,1) \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\|\widetilde{k}\| = 2 < \sqrt{5} = \|k\|_{2,\mu\otimes\nu}.$$

In fact, $\|k\|_{2,\mu\otimes\nu}$ is equal to the Hilbert–Schmidt norm of $\widetilde{k}$ as we will see in Proposition 6.1.4.9 a).

b) does not hold in general if $q = \infty$ and μ and ν are not bounded as Example 3.1.6.20 shows.

Proposition 3.1.6.18 (5) *Let $(T, \mathfrak{T}, \mu)$ be a measure space and $\mathcal{S}$ the involutive vector space of sesquilinear forms on $L^2(\mu)$ (Proposition 2.3.3.3 c)). Define*

$$\langle x|y\rangle := \int x\overline{y}d\mu$$

for $x, y \in L^2(\mu)$ and

$$f_k : L^2(\mu) \times L^2(\mu) \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \langle kx|y\rangle,$$

$$\widetilde{k} : L^2(\mu) \longrightarrow L^2(\mu), \quad x \longmapsto kx$$

for $k \in L^2(\mu \otimes \mu)$ (Proposition 3.1.6.16).

a) *$\langle kx|y\rangle = \langle x|k^*y\rangle$ for $x, y \in L^2(\mu)$ and $k \in L^2(\mu \otimes \mu)$.*

b) *$L^2(\mu \otimes \mu)$ endowed with the involution*

$$L^2(\mu \otimes \mu) \longrightarrow L^2(\mu \otimes \mu), \quad k \longmapsto k^*$$

is an involutive Banach space.

c) *$f_k \in \mathcal{S}$ for every $k \in L^2(\mu \otimes \mu)$ and the map*

$$L^2(\mu \otimes \mu) \longrightarrow \mathcal{S}, \quad k \longmapsto f_k$$

is injective, linear, and involutive, where $L^2(\mu \otimes \mu)$ is endowed with the involution of b).

d) *For $k \in L^2(\mu \otimes \mu)$, the sesquilinear form f_k is Hermitian iff k is selfadjoint.*

e) *Given $k \in L^2(\mu \otimes \mu)$, if f_k is Hermitian (positive), then*

$$\sigma(\widetilde{k}) \subset \mathbb{R} \quad (\sigma(\widetilde{k}) \subset \mathbb{R}_+).$$

We may assume $(T, \mathfrak{T}, \mu)$ to be σ–finite and complete.

a) By Proposition 3.1.6.17 c),

$$\langle kx|y\rangle = \int (kx)\overline{y}d\mu = \int x(k'\overline{y})d\mu = \int x\overline{k^*y}d\mu = \langle x|k^*y\rangle .$$

b) is easy to see.

c) It is obvious that $f_k \in \mathcal{S}$ for $k \in L^2(\mu \otimes \mu)$ and that the map

$$u : L^2(\mu \otimes \mu) \longrightarrow \mathcal{S}, \quad k \longmapsto f_k$$

is linear. Take $k \in L^2(\mu \otimes \mu)$. If $f_k = 0$, then

$$\langle kx|y\rangle = 0$$

for every $x, y \in L^2(\mu)$ and so

$$kx = 0$$

for every $x \in L^2(\mu)$. Hence $\widetilde{k} = 0$ and $k = 0$ by Proposition 3.1.6.17 e). Thus u is injective. Moreover, by a),

$$f_k^*(x,y) = \overline{f_k(y,x)} = \overline{\langle ky|x\rangle} = \overline{\langle y|k^*x\rangle} = \langle k^*x, y\rangle = f_{k^*}(x,y)$$

for every $x, y \in L^2(\mu)$ whence

$$f_k^* = f_{k^*}$$

and u is involutive.

d) By c), if k is selfadjoint, then f_k is Hermitian. Now suppose that f_k is Hermitian. Then

$$f_{k-k^*} = f_k - f_{k^*} = f_k - f_k^* = 0,$$

so that

$$k = k^*$$

by c).

e) Take $\alpha \in \sigma(\widetilde{k})\backslash\{0\}$. By Proposition 3.1.6.17 b), $\widetilde{k}$ is compact. Thus, by Theorem 3.1.5.1 $b_1 \Rightarrow b_2$,

$$\widetilde{k}x = \alpha x .$$

for some $x \in L^2(\mu)\backslash\{0\}$. Then, by d),

$$\alpha f_k(x,x) = \alpha\langle kx|x\rangle = \alpha\langle x|kx\rangle = \alpha\langle x|\alpha x\rangle = \langle \alpha x|\alpha x\rangle \in \mathbb{R}_+ .$$

By Proposition 2.3.3.8 a $\Rightarrow$ b, $\alpha \in \mathbb{R}$ $(\alpha \in \mathbb{R}_+)$ and

$$\sigma(\widetilde{k}) \subset \mathbb{R} \quad (\sigma(\widetilde{k}) \subset \mathbb{R}_+) . \qquad \blacksquare$$

Proposition 3.1.6.19 *Let $(T, \mathfrak{T}, \mu)$ be a σ-finite measure space. We set*

$$\widetilde{k} : L^2(\mu) \longrightarrow L^2(\mu) , \quad x \longmapsto kx$$

for all $k \in L^2(\mu \otimes \mu)$ (Proposition 3.1.6.16).

a) *For all $h, k \in L^2(\mu \otimes \mu)$ there is a unique element $h \circ k \in L^2(\mu \otimes \mu)$ such that*

$$(h \circ k)(s,t) = \int h(s,r)k(r,t)d\mu(r)$$

for $\mu \otimes \mu$-almost every $(s,t) \in T \times T$. Moreover,

$$\|h \circ k\|_2 \leq \|h\|_2 \|k\|_2 .$$

b) *$L^2(\mu \otimes \mu)$ is a symmetric involutive Banach algebra with respect to the multiplication*

$$L^2(\mu \otimes \mu) \times L^2(\mu \otimes \mu) \longrightarrow L^2(\mu \otimes \mu) , \quad (h,k) \longmapsto h \circ k$$

and the involution

$$L^2(\mu \otimes \mu) \longrightarrow L^2(\mu \otimes \mu) , \quad k \longmapsto k^* ,$$

the map

$$L^2(\mu \otimes \mu) \longrightarrow \mathcal{L}(L^2(\mu)) , \quad k \longmapsto \widetilde{k}$$

(Proposition 3.1.6.17 a)) is an injective, continuous, involutive algebra homomorphism (Example 2.3.2.27 c)),

$$\sigma(k) = \sigma(\widetilde{k}) ,$$

and $f_{k^* \circ k}$ (defined in Proposition 3.1.6.18) is positive for every $k \in L^2(\mu \otimes \mu)$.

c) *The map*

$$g : L^2(\mu \otimes \mu) \times L^2(\mu \otimes \mu) \longrightarrow \mathbb{K}, \quad (h,k) \longmapsto \int (h \circ k^*)(t,t)d\mu(t)$$

is a positive sesquilinear form such that

$$g(k,k) = \|k\|_2^2$$

for every $k \in L^2(\mu \otimes \mu)$.

a) Given $s,t \in T$,

$$\left(\int^* |h(s,r)k(r,t)|d\mu(r)\right)^2 \leq \left(\int^* |h(s,r)|^2 d\mu(r)\right)\left(\int^* |k(r,t)|^2 d\mu(r)\right)$$

(Hölder), so that

$$\int^* \left(\int^* |h(s,r)k(r,t)|d\mu(r)\right)^2 d(\mu \otimes \mu)(s,t) \leq \|h\|_2^2 \|k\|_2^2$$

(Fubini). Thus $h(s,\cdot)k(\cdot,t) \in \mathcal{L}^1(\mu)$ for $\mu \otimes \mu$-almost every $(s,t) \in T \times T$, the map

$$h \circ k : T \times T \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto \int h(s,r)k(r,t)d\mu(r)$$

(which is defined $\mu \otimes \mu$-a.e.) belongs to $\mathcal{L}^2(\mu \otimes \mu)$, and

$$\|h \circ k\|_2 \leq \|h\|_2 \|k\|_2 .$$

The uniqueness is obvious.

b) The associativity of the multiplication follows from Fubini's Theorem. By a), $L^2(\mu \otimes \mu)$ is a Banach algebra. Take $h,k \in L^2(\mu \otimes \mu)$. Then

$$(h \circ k)^*(s,t) = \overline{(h \circ k)(t,s)} = \overline{\int h(t,r)k(r,s)d\mu(r)} =$$

$$= \int \overline{h(t,r)}\,\overline{k(r,s)}d\mu(r) = \int k^*(s,r)h^*(r,t)d\mu(r) = (k^* \circ h^*)(s,t)$$

for $\mu \otimes \mu$-almost every $(s,t) \in T \times T$. Thus

$$(h \circ k)^* = k^* \circ h^* .$$

Hence $L^2(\mu \otimes \mu)$ is an involutive Banach algebra. Using Fubini's Theorem, once again, we see that the map

$$u : L^2(\mu \otimes \mu) \longrightarrow \mathcal{L}(L^2(\mu)), \quad k \longmapsto \widetilde{k}$$

is an algebra homomorphism, which is continuous and injective by Proposition 3.1.6.17 a), e).

Take $k \in L^2(\mu \otimes \mu)$. By the above, $\operatorname{Im} u$ is a subalgebra of $\mathcal{L}(L^2(\mu))$ and

$$\sigma(k) = \sigma_{\operatorname{Im} u}(\widetilde{k}).$$

By Proposition 3.1.6.17 f),

$$(v, w) \in (\operatorname{Im} u) \times \mathcal{L}(L^2(\mu)), \, vw = wv \Longrightarrow vw \in \operatorname{Im} u$$

and so, by Corollary 2.1.3.15,

$$\sigma_{\operatorname{Im} u}(\widetilde{k}) = \sigma(\widetilde{k}).$$

Hence

$$\sigma(k) = \sigma(\widetilde{k}).$$

By Proposition 3.1.6.17 d) (and the definition of $(\widetilde{k})^*$, given in Example 2.3.2.27),

$$(\widetilde{k})^* x = \overline{(\widetilde{k})' \overline{x}} = \overline{\widetilde{k'} \overline{x}} = \widetilde{\widetilde{k'}} x = \widetilde{k^*} x$$

for $x \in L^2(\mu)$, whence

$$(\widetilde{k})^* = \widetilde{k^*}$$

and u is thus involutive. By a),

$$f_{k^* \circ k}(x, x) = \langle (k^* \circ k)x | x \rangle = \langle k^* k x | x \rangle = \langle kx | kx \rangle \in \mathbb{R}_+$$

for every $x \in L^2(\mu)$. $f_{k^* \circ k}$ thus is positive. By Proposition 3.1.6.17 b), $\widetilde{k^* \circ k}$ is compact. Hence, by Proposition 3.1.5.8,

$$\sigma(k^* \circ k) \subset \mathbb{R}_+.$$

$L^2(\mu \otimes \mu)$ is thus symmetric.

c) is easy to see. ■

Example 3.1.6.20 (4) *The Fourier transform (Theorem 2.4.6.2) is not compact.*

Let x be the characterstic function of $[0, 2\pi]^n$. Given $k \in \mathbb{N}$, define

$$x_k : \mathbb{R}^n \longrightarrow \mathbb{C}, \quad t \longmapsto e^{-\imath kt_1} x .$$

Then

$$\|x_k\|_1 = (2\pi)^n$$

for $k \in \mathbb{N}$, whence $(x_n)_{k \in \mathbb{N}}$ is a bounded sequence in $L^1(\lambda)$. Put

$$e_1 := (1, 0, 0, \ldots, 0) \in \mathbb{R}^n .$$

Then

$$\widehat{x}_k(je_1) = (2\pi)^{n-1} \int_0^{2\pi} e^{\imath jt_1} e^{-\imath kt_1} dt_1 = \begin{cases} (2\pi)^n & \text{if} \quad j = k \\ 0 & \text{if} \quad j \neq k \end{cases}$$

for every $j, k \in \mathbb{N}$. Hence

$$\|\widehat{x}_j - \widehat{x}_k\|_\infty \geq (2\pi)^n$$

for distinct $j, k \in \mathbb{N}$. Thus $(\widehat{x}_k)_{k \in \mathbb{N}}$ possesses no Cauchy subsequence and the Fourier transform is not compact (Proposition 3.1.1.19 a $\Rightarrow$ b). ■

Proposition 3.1.6.21 (5) *Let E, F, G be Banach spaces such that $E \subset F \subset G$ and the inclusions are linear and continuous. Take $u \in \mathcal{L}(G, E)$ and*

$$v : E \longrightarrow E, \quad x \longmapsto ux ,$$

$$w : F \longrightarrow F, \quad x \longmapsto ux .$$

a) $\operatorname{Ker}(\alpha 1 - u) = \operatorname{Ker}(\alpha 1 - w)$ *for* $\alpha \in \mathbb{K} \backslash \{0\}$.

b) *If u is compact, then v and w are compact and*

$$\sigma(v) \backslash \{0\} = \sigma(w) \backslash \{0\} .$$

If, in addition, E is infinite-dimensional, then

$$\sigma(v) = \sigma(w) .$$

a) Given $x \in G$,

$$(x \in \operatorname{Ker}(\alpha 1 - u)) \Longleftrightarrow (\alpha x = ux) \Longleftrightarrow (x \in F \text{ and } \alpha x = ux) \Longleftrightarrow$$

$$\Longleftrightarrow (x \in \operatorname{Ker}(\alpha 1 - w)).$$

b) It is easy to see that v and w are compact (e.g. by Proposition 3.1.1.11). Moreover, for $\alpha \in \mathbb{K}\backslash\{0\}$,

$$\alpha \in \sigma(v) \Longleftrightarrow \operatorname{Ker}(\alpha 1 - v) \neq \{0\},$$

$$\alpha \in \sigma(w) \Longleftrightarrow \operatorname{Ker}(\alpha 1 - w) \neq \{0\},$$

(Theorem 3.1.5.1 $b_1 \Leftrightarrow b_2$) and so, by a),

$$\alpha \in \sigma(v) \Longleftrightarrow \alpha \in \sigma(w).$$

Hence

$$\sigma(v)\backslash\{0\} = \sigma(w)\backslash\{0\}.$$

If, in addition, E is infinite-dimensional, then, by Theorem 3.1.5.1 d),

$$0 \in \sigma(v) \cap \sigma(w). \qquad \blacksquare$$

Corollary 3.1.6.22 (5) *Let T be a compact space and μ a positive Radon measure on T. Take $k \in C(T \times T)$, $p \in [1, \infty]$,*

$$u : L^p(\mu) \longrightarrow L^p(\mu), \quad x \longmapsto kx,$$

$$v : C(T) \longrightarrow C(T), \quad x \longmapsto kx,$$

(Corollary 3.1.6.4), and $\alpha \in \mathbb{K}\backslash\{0\}$. Then u and v are compact and the eigenvectors of u corresponding to α are continuous.

The operator

$$L^1(\mu) \longrightarrow C(T), \quad x \longmapsto kx$$

is compact (Corollary 3.1.6.4). By Proposition 3.1.6.21, u and v are compact and

$$\operatorname{Ker}(\alpha 1 - u) = \operatorname{Ker}(\alpha 1 - v).$$

Hence the eigenvectors of u corresponding to α are continuous. $\blacksquare$

Theorem 3.1.6.23 (The Fredholm Alternative, 1903) *Let* $(T, \mathfrak{T}, \mu)$ *be a measure space. Take* $k \in L^2(\mu \otimes \mu)$,

$$\widetilde{k} : L^2(\mu) \longrightarrow L^2(\mu), \quad x \longmapsto kx$$

(Proposition 3.1.6.16), and $\alpha \in \mathbb{K}\backslash\{0\}$.

a) *The following are equivalent:*

a_1) $\alpha \notin \sigma(\widetilde{k})$.

a_1) *Given* $y \in L^2(\mu)$, *there is a unique* $x \in L^2(\mu)$ *such that*

$$\alpha x - \int k(\cdot, t)x(t)d\mu(t) = y.$$

b) *If* α *belongs to the spectrum of* $\widetilde{k}$, *then*

$$0 < \mathrm{Dim}\{x \in L^2(\mu) \mid kx = \alpha x\} = \mathrm{Dim}\{x \in L^2(\mu) \mid k'x = \alpha x\} =$$

$$= \mathrm{Dim}\, L^2/\{\alpha x - kx \mid x \in L^2\} = \mathrm{Dim}\, L^2/\{\alpha x - k'x \mid x \in L^2\} < \infty.$$

c) *The following are equivalent for* $y \in L^2(\mu)$:

c_1) *There is an* $x \in L^2(\mu)$ *such that*

$$\alpha x - \int k(\cdot, t)x(t)d\mu(t) = y.$$

c_2) *If* $z \in L^2(\mu)$ *such that*

$$\int k(t, \cdot)z(t)d\mu(t) = \alpha z,$$

then

$$\int yzd\mu = 0.$$

$\widetilde{k}$ is well–defined and compact and

$$(\widetilde{k})'x = k'x$$

whenever $x \in L^2(\mu)$ (Proposition 3.1.6.17 a),b),d). The assertions now all follow from Theorem 3.1.5.1. ■

Proposition 3.1.6.24 *Let $(S, \mathfrak{S}, \mu)$, $(T, \mathfrak{T}, \nu)$ be measure spaces, $p, q \in [1, \infty]$, p' the conjugate exponent of p, and $k : S \times T \to \mathbb{K}$ be a measurable function such that $k(s, \cdot) \in \mathcal{L}^{p'}(\nu)$ for μ-almost all $s \in S$ and $kx \in \mathcal{L}^q(\mu)$ for all $x \in \mathcal{L}^p(\nu)$. Then the map*

$$L^p(\nu) \longrightarrow L^q(\mu), \quad x \longmapsto kx$$

is linear and continuous.

We first remark that, by Hölder's Theorem, $\mathcal{L}^p(\nu) \subset \mathcal{D}(k, \mu, \nu)$, so that kx is well defined for every $x \in \mathcal{L}^p(\nu)$. It is clear that the above map is linear. We prove that it has a closed graph G. Take $(x, y) \in \overline{G}$. There is a sequence $(x_n, y_n)_{n \in \mathbb{N}}$ in G converging to (x, y). Replacing this sequence with a subsequence, if necessary, we may assume that $(y_n)_{n \in \mathbb{N}}$ converges ν-almost everywhere to y. Then

$$y(s) = \lim_{n \to \infty} y_n(s) = \lim_{n \to \infty} \int k(s, \cdot) x_n d\nu = \int k(s, \cdot) x d\nu$$

for μ-almost all $s \in S$ (Hölder). Thus $(x, y) \in G$, proving that G is closed. By the Closed Graph Theorem, the above map is continuous. ■

Proposition 3.1.6.25 (3) (Grothendieck) *Let $p \in [1, \infty[$ and $q := \frac{p}{p-1}$. Let $(S, \mathfrak{S}, \mu)$, $(T, \mathfrak{T}, \nu)$ be measure spaces such that $(T, \mathfrak{T}, \nu)$ is σ-finite if $p = 1$ and*

$$u \in \mathcal{L}^1(L^p(\nu), L^q(\mu)).$$

Then there is a unique $k \in L^q(\mu \otimes \nu)$ such that

$$ux = k^\nu x$$

whenever $x \in L^p(\nu)$ (Proposition 3.1.6.16). Moreover

$$\|k\|_{q, \mu \otimes \nu} \leq \|u\|_1.$$

Let $(x'_\iota, y_\iota)_{\iota \in I}$ be a family in $L^p(\nu)' \times L^q(\mu)$ such that

$$\sum_{\iota \in I} \|x'_\iota\| \, \|y_\iota\| < \infty$$

and

$$ux = \sum_{\iota \in I} \langle x, x'_\iota \rangle y_\iota$$

for every $x \in L^p(\nu)$. Identifying $L^p(\nu)'$ canonically with $L^q(\nu)$, we get a family $(y_\iota \otimes x'_\iota)_{\iota \in I}$ in $L^q(\mu \otimes \nu)$ with

$$\sum_{\iota \in I} \|y_\iota \otimes x'_\iota\|_{q,\mu\otimes\nu} = \sum_{\iota \in I} \|y_\iota\|_{q,\mu} \|x'_\iota\|_{q,\nu} =$$

$$= \sum_{\iota \in I} \|x'_\iota\| \, \|y_\iota\| < \infty .$$

Set

$$k := \sum_{\iota \in I} y_\iota \otimes x'_\iota \in L^q(\mu \otimes \nu)$$

(Corollary 1.1.6.10 a $\Rightarrow$ b). Then

$$\|k\|_{q,\mu\otimes\nu} \leq \sum_{\iota \in I} \|y_\iota \otimes x'_\iota\|_{q,\mu\otimes\nu} = \sum_{\iota \in I} \|x'_\iota\| \, \|y_\iota\| ,$$

so that

$$\|k\|_{q,\mu\otimes\nu} \leq \|u\|_1 .$$

Define

$$\tilde{k} : L^p(\nu) \longrightarrow L^q(\nu) , \quad x \longmapsto kx$$

(Proposition 3.1.6.17 a)) and

$$u_J : L^p(\nu) \longrightarrow L^q(\mu) , \quad x \longmapsto \sum_{\iota \in J} \langle x, x'_\iota \rangle y_\iota$$

for $J \in \mathfrak{P}_f(I)$. Then

$$\|u - u_J\| \leq \|u - u_J\|_1 \leq \sum_{\iota \in I \setminus J} \|x'_\iota\| \, \|y_\iota\| ,$$

$$\|\tilde{k} - u_J\| \leq \left\| k - \sum_{\iota \in J} y_\iota \otimes x'_\iota \right\|_{q,\mu\otimes\nu} \leq \|u - u_J\|_1 \leq \sum_{\iota \in I \setminus J} \|x'_\iota\| \, \|y_\iota\| ,$$

so

$$\|u - \tilde{k}\| \leq \|u - u_J\| + \|\tilde{k} - u_J\| \leq 2 \sum_{\iota \in I \setminus J} \|x'_\iota\| \, \|y_\iota\|$$

for every $J \in \mathfrak{P}_f(I)$ (Theorem 1.6.1.3 a), Proposition 3.1.6.17 a)). Hence

$$ux = \tilde{k}x = kx$$

for $x \in L^p(\nu)$.

The uniqueness of k follows from Proposition 3.1.6.17 e)). ■

Remark. It is not the case in general, that

$$\|k\|_{q,\mu\otimes\nu} = \|u\|_1 .$$

For instance, in the case of the example in the Remark to Proposition 3.1.6.17,

$$\|k\|_{2,\mu\otimes\nu} = \sqrt{5} < 3 = \|u\|_1 .$$

Proposition 3.1.6.26 *Let $(S, \mathfrak{S}, \mu)$, $(T, \mathfrak{T}, \nu)$ be σ-finite measure spaces and k an $\mathfrak{S}\otimes\mathfrak{T}$-measurable function on $S \times T$ such that*

$$\alpha := \sup_{s\in S} \int |k(s,t)| d\nu(t) < \infty ,$$

$$\beta := \sup_{t\in T} \int |k(s,t)| d\mu(s) < \infty .$$

Then

$$\mathcal{L}^2(\nu) \subset \mathcal{D}(k,\mu,\nu) ,$$

$$k^\nu(\mathcal{L}^2(\nu)) \subset \mathcal{L}^2(\mu) ,$$

and

$$k^\nu : L^2(\nu) \longrightarrow L^2(\mu) , \quad x \longmapsto k^\nu x$$

is continuous with

$$\|k^\nu\|^2 \leq \alpha\beta .$$

Take $x \in \mathcal{L}^2(\nu)$. Then

$$|k^\nu x(s)|^2 = \left| \int k(s,t)x(t)d\nu(t) \right|^2 \leq \left(\int |k(s,t)|\,|x(t)| d\nu(t) \right)^2 \leq$$

$$\leq \left(\int |k(s,t)|d\nu(t)\right)\left(\int |k(s,t)|\,|x(t)|^2 d\nu(t)\right) \leq \alpha \int |k(s,t)|\,|x(t)|^2 d\nu(t)$$

for every $s \in S$ so that

$$\int |k^\nu x(s)|^2 d\mu(s) \leq \alpha \int \left(\int |k(s,t)|\,|x(t)|^2 d\nu(t)\right) d\mu(s) =$$

$$= \alpha \int \left(\int |k(s,t)| d\mu(s)\right) |x(t)|^2 d\nu(t) \leq \alpha\beta \|x\|_{2,\nu}^2 ,$$

by Tonelli's Theorem. ■

Proposition 3.1.6.27 *Take $a, b \in \mathbb{R}$, $a < b$. Let λ be Lebesgue measure on $[a,b]$ and h a bounded Borel function on $[a,b] \times [a,b]$. Take $\alpha \in]0,1[$, and*

$$k : [a,b] \times [a,b] \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto \frac{h(s,t)}{(s-t)^\alpha} .$$

Then

$$\mathcal{L}^2(\lambda) \subset \mathcal{D}(k,\lambda,\lambda) ,$$

$$k^\lambda(\mathcal{L}^2(\lambda)) \subset \mathcal{L}^2(\lambda) ,$$

and

$$k^\lambda : L^2(\lambda) \longrightarrow L^2(\lambda), \quad x \longmapsto k^\lambda x$$

is a compact operator.

Put

$$\beta := \sup\{|h(s,t)| \mid s,t \in [a,b]\} .$$

Then

$$\int_a^b |k(s,t)|dt \leq \beta \int_a^b \frac{dt}{|s-t|^\alpha} \leq \frac{2\beta(b-a)^{1-\alpha}}{1-\alpha}$$

for every $s \in [a,b]$. By Proposition 3.1.6.26,

$$\mathcal{L}^2(\lambda) \subset \mathcal{D}(k,\lambda,\lambda) , \qquad k^\lambda(\mathcal{L}^2(\lambda)) \subset \mathcal{L}^2(\lambda)$$

and

$$k^\lambda : \mathcal{L}^2(\lambda) \longrightarrow L^2(\lambda), \quad x \longmapsto k^\lambda x$$

is continuous and

$$\|k^\nu\| \leq \frac{2\beta(b-a)^{1-\alpha}}{1-\alpha}.$$

Take $\varepsilon > 0$. Define

$$k_1 : [a,b] \times [a,b] \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto \begin{cases} k(s,t) & \text{if} \quad |s-t| > \varepsilon \\ 0 & \text{if} \quad |s-t| \leq \varepsilon \end{cases},$$

$$k_2 := k - k_1.$$

By Proposition 3.1.6.17 b), k_1^ν is compact. By the above, k_2^ν is continuous and

$$\|k_2^\nu\| \leq \frac{2\beta\varepsilon^{1-\alpha}}{1-\alpha}.$$

Since ε is arbitrary, k^ν is compact (Theorem 3.1.1.3). ■

Exercises

E 3.1.1 Take $\mathbb{K}^{(\mathbb{N})}$ with the $\|\cdot\|_\infty$–norm. Define

$$y := (\frac{1}{n})_{n\in\mathbb{N}},$$

$$u : \mathbb{K}^{(\mathbb{N})} \longrightarrow \mathbb{K}^{(\mathbb{N})}, \quad x \longmapsto xy,$$

and

$$u_p : \mathbb{K}^{(\mathbb{N})} \longrightarrow \mathbb{K}^{(\mathbb{N})}, \quad x \longmapsto xye_{\mathbb{N}_p}$$

for every $p \in \mathbb{N}$. Prove:

a) $u \in \mathcal{L}(\mathbb{K}^{(\mathbb{N})}) \backslash \mathcal{K}(\mathbb{K}^{(\mathbb{N})})$.

b) $u_p \in \mathcal{K}(\mathbb{K}^{(\mathbb{N})})$ for every $p \in \mathbb{N}$.

c) $\lim\limits_{p\to\infty} u_p = u$.

Remark. This example shows that if E is a non–complete normed space, then $\mathcal{K}(E)$ need not be closed in $\mathcal{L}(E)$.

E 3.1.2 Let E be a normed space. Take $u \in \mathcal{L}(E)$ and $\alpha \in \sigma_p(u)$. Let F be the eigenspace of u corresponding to α and take $P, Q \in \mathbb{K}[t]$, with $Q(u)$ invertible in $\mathcal{L}(E)$. Prove:

a) $Q(\alpha) \neq 0$.

b) $\frac{P(\alpha)}{Q(\alpha)} \in \sigma_p(P(u)Q(u)^{-1})$.

c) F is contained in the eigenspace of $P(u)Q(u)^{-1}$ corresponding to $\frac{P(\alpha)}{Q(\alpha)}$.

3.2 Linear Differential Equations

3.2.1 Boundary Value Problems for Differential Equations

Through this subsection:

a) a and b denote real numbers and $a < b$.

b) $\mathcal{C}$ stands for $\mathcal{C}([a,b])$.

c) $n \in \mathbb{N}\backslash\{1\}$.

d) $(\alpha_{ij})_{i,j\in\mathbb{N}_n}$ and $(\beta_{ij})_{i,j\in\mathbb{N}_n}$ are families in $\mathbb{K}$. For each $i \in \mathbb{N}_n$, we define

$$f_{a_i}(x) := \sum_{j=1}^{n} \alpha_{ij} x^{(j-1)}(a),$$

$$f_{b_i}(x) := \sum_{j=1}^{n} \beta_{ij} x^{(j-1)}(b),$$

$$f_i(x) := f_{a_i}(x) + f_{b_i}(x)$$

for those $x \in \mathcal{C}$ for which the above expressions are defined ($x^{(j-1)}$ denotes the derivative of x of order $j-1$);

e) $\mathcal{D}$ stands for $\left\{ x \in \mathcal{C} \;\middle|\; \begin{array}{l} x \text{ is } n\text{-times continuously differentiable} \\ \text{and } f_i(x) = 0 \text{ for all } i \in \mathbb{N}_n \end{array} \right\}$.

f) $(p_i)_{i\in\mathbb{N}_n\cup\{0\}}$ is a family in $\mathcal{C}$ such that p_n vanishes nowhere.

g) $u : \mathcal{D} \longrightarrow \mathcal{C}, \quad x \longmapsto \sum_{i=0}^{n} p_i x^{(i)}$.

h) $(x_i)_{i\in\mathbb{N}_n}$ is a fundamental system of solutions of the differential equation

$$\sum_{i=0}^{n} p_i x^{(i)} = 0.$$

Except in Lemma 3.2.1.1, u is assumed to be injective.

Lemma 3.2.1.1 (5) *u is injective iff*

$$\operatorname{Det}[f_i(x_j)]_{i,j\in\mathbb{N}_n} \neq 0.$$

Assume that u is injective and that

$$\operatorname{Det}[f_i(x_j)]_{i,j\in\mathbb{N}_n} = 0.$$

There is a family $(\alpha_j)_{j\in\mathbb{N}_n}$ in $\mathbb{K}$, so that not every α_j $(j \in \mathbb{N}_n)$ vanishes and

$$f_i\left(\sum_{j=1}^{n} \alpha_j x_j\right) = 0$$

for every $i \in \mathbb{N}_n$. Then $\sum_{j=1}^{n} \alpha_j x_j$ belongs to $\operatorname{Ker} u$. Thus

$$\sum_{j=1}^{n} \alpha_j x_j = 0$$

contradicting the fact that $(x_j)_{j\in\mathbb{N}_n}$ are linearly independent.

Now suppose that

$$\operatorname{Det}[f_i(x_j)]_{i,j\in\mathbb{N}_n} \neq 0,$$

and take $x \in \operatorname{Ker} u$. There is a family $(\alpha_j)_{j\in\mathbb{N}_n}$ in $\mathbb{K}$ such that

$$x = \sum_{j=1}^{n} \alpha_j x_j.$$

Then

$$\sum_{j=1}^{n} f_i(x_j)\alpha_j = f_i(x) = 0$$

for every $i \in \mathbb{N}_n$. Hence each α_j $(j \in \mathbb{N}_n)$ vanishes. Thus $x = 0$ and so u is injective. ■

Remark. If u is injective, then by the above, the boundary conditions $f_i = 0$ $(i \in \mathbb{N}_n)$ are linearly independent. But it may happen that the boundary conditions are linearly independent and for no $\alpha \in \mathbb{K}$ is $u - \alpha 1$ injective: Consider the equation $x'' = 0$ on $[-1, 1]$ with the following boundary conditions

$$\begin{cases} x(1) + x(-1) = 0 \\ x'(1) - x'(-1) = 0. \end{cases}$$

Take $\alpha \in \mathbb{C}\backslash\{0\}$ and define

$$x := e^{\alpha t} - e^{-\alpha t},$$

$$y : [-1,1] \longrightarrow \mathbb{C}, \quad t \longmapsto t.$$

Then

$$x' = \alpha(e^{\alpha t} + e^{-\alpha t}),$$

$$x'' = \alpha^2(e^{\alpha t} - e^{-\alpha t}),$$

$$x'' - \alpha^2 x = 0,$$

$$\begin{cases} x(1) + x(-1) = e^{\alpha} - e^{-\alpha} + e^{-\alpha} - e^{\alpha} = 0, \\ x'(1) - x'(-1) = \alpha(e^{\alpha} + e^{-\alpha}) - \alpha(e^{-\alpha} + e^{\alpha}) = 0, \end{cases}$$

$$y' = 1, \quad y'' = 0,$$

$$\begin{cases} y(1) + y(-1) = 1 - 1 = 0, \\ y'(1) - y'(-1) = 1 - 1 = 0, \end{cases}$$

Hence $u^2 - \alpha^2 1$ and u are not injective.

Definition 3.2.1.2 (5) *A* ***Green function*** *(for the given boundary problem) is a continuous map*

$$g : [a,b] \times [a,b] \longrightarrow \mathbb{K},$$

such that for every $t \in]a,b[$*:*

a) $g(\cdot,t)$ *is* $n-2$*-times continuously differentiable;*

b) $g(\cdot,t)$ *is* n*-times continuously differentiable on* $[a,b]\backslash\{t\}$ *and*

$$\lim_{\substack{s \to t \\ s<t}} g^{(n-1)}(s,t) - \lim_{\substack{s \to t \\ s>t}} g^{(n-1)}(s,t) = \frac{1}{p_n(t)}.$$

c) $\sum_{i=0}^{n} p_i g^{(i)}(\cdot,t) = 0$ *on* $[a,b]\backslash\{t\}$.

d) $f_i(g(\cdot,t)) = 0$, *whenever* $i \in \mathbb{N}_n$.

The derivatives of the Green function are always taken with respect to the first variable. We adhere to this convention throughout the entire subsection. Moreover, we define

$$gx : [a,b] \longrightarrow \mathbb{K}, \quad s \longmapsto \int g(s,t)x(t)dt,$$
$$g'x : [a,b] \longrightarrow \mathbb{K}, \quad s \longmapsto \int g(t,s)x(t)dt,$$

for $x \in \mathcal{C}$. *This is consistent with the notation introduced in the Definitions 3.1.6.1 and 3.1.6.5.*

Theorem 3.2.1.3 (5) *There is exactly one Green function for the given boundary value problem. There are two families* $(a_i)_{i\in\mathbb{N}_n}$, $(b_i)_{i\in\mathbb{N}_n}$ *in* $\mathcal{C}$ *such that*

$$g(s,t) = \sum_{i=1}^{n} a_i(t)x_i(s)$$

for $s,t \in [a,b]$ *with* $s \leq t$, *and*

$$g(s,t) = \sum_{i=1}^{n} b_i(t)x_i(s)$$

for $s,t \in [a,b]$ *with* $s \geq t$, *where* g *denotes the Green function.* $g^{(i)}$ *is continuous for every* $i \in \mathbb{N}_{n-2}$. *Moreover,*

$$\sum_{i=1}^{n}(a_i(t) - b_i(t))x_i^{(j-1)}(t) = 0,$$

$$\sum_{i=1}^{n}(a_i(t) - b_i(t))x_i^{(n-1)}(t) = \frac{1}{p_n(t)}$$

for every $j \in \mathbb{N}_{n-1}$ *and* $t \in [a,b]$.

Let g be a Green function and take $t \in]a,b[$. By c) (and by the continuity of g)), there are two uniquely determined families $(a_i(t))_{i\in\mathbb{N}_n}$ and $(b_i(t))_{i\in\mathbb{N}_n}$ in $\mathbb{K}$ such that

$$g(s,t) = \sum_{i=1}^{n} a_i(t)x_i(s)$$

for $s \in [a,t]$ and

$$g(s,t) = \sum_{i=1}^{n} b_i(t)x_i(s)$$

for $s \in [t,b]$. By a),

$$\sum_{i=1}^{n}(a_i(t) - b_i(t))x_i^{(j-1)}(t) =$$

$$= \sum_{i=1}^{n} a_i(t)x_i^{(j-1)}(t) - \sum_{i=1}^{n} b_i(t)x_i^{(j-1)}(t) =$$

$$= g^{(j-1)}(t,t) - g^{(j-1)}(t,t) = 0$$

for every $j \in \mathbb{N}_{n-1}$ and by b),

$$\sum_{i=1}^{n}(a_i(t) - b_i(t))x_i^{(n-1)}(t) = \sum_{i=1}^{n} a_i(t)x_i^{(n-1)}(t) - \sum_{i=1}^{n} b_i(t)x_i^{(n-1)}(t) =$$

$$= \lim_{\substack{s \to t \\ s<t}} g^{(n-1)}(s,t) - \lim_{\substack{s \to t \\ s<t}} g^{(n-1)}(s,t) = \frac{1}{p_n(t)}.$$

Since the Wronski determinant of $(x_i)_{i\in\mathbb{N}_n}$ does not vanish, the functions $a_i - b_i$ $(i \in \mathbb{N}_n)$ do not depend on g. By Cramer's rule, $a_i - b_i$ can also be defined for $t = a$ and for $t = b$ and the extended functions $a_i - b_i$ $(i \in \mathbb{N}_n)$ are continuous. From d),

$$\sum_{j=1}^{n} f_i(x_j)a_j(t) = \sum_{j=1}^{n} f_{a_i}(x_j)a_j(t) + \sum_{j=1}^{n} f_{b_i}(x_j)a_j(t) =$$

$$= f_{a_i}\left(\sum_{j=1}^{n} a_j(t)x_j\right) + f_{b_i}\left(\sum_{j=1}^{n} b_j(t)x_j\right) + \sum_{j=1}^{n} f_{b_i}(x_j)(a_j(t) - b_j(t)) =$$

$$= f_i(g(\cdot,t)) + \sum_{j=1}^{n} f_{b_i}(x_j)(a_j(t) - b_j(t)) =$$

$$= \sum_{j=1}^{n} f_{b_i}(x_j)(a_j(t) - b_j(t))$$

for every $i \in \mathbb{N}_n$. Since

$$\operatorname{Det}[f_i(x_j)]_{i,j\in\mathbb{N}_n} \neq 0$$

(Lemma 3.2.1.1), it follows again by the above considerations that the functions a_i $(i \in \mathbb{N}_n)$ do not depend on g. By Cramer's rule, once more, we may define them for $t = a$ and $t = b$ and the extended functions are continuous. But then the functions b_i $(i \in \mathbb{N}_n)$ do not depend on g and may be extended continuously to $t = a$ and $t = b$. Since g is continuous,

$$g(s,t) = \sum_{i=1}^{n} a_i(t)x_i(s)$$

for $s, t \in [a,b]$ with $s \leq t$ and

$$g(s,t) = \sum_{i=1}^{n} b_i(t)x_i(s)$$

for $s, t \in [a,b]$ with $s \geq t$. This proves the uniqueness of the Green function and the fact that $g^{(i)}$ is continuous for every $i \in \mathbb{N}_{n-2}$.

Now we establish the existence of the Green function. Let $(a_i)_{i\in\mathbb{N}_n}$ and $(b_i)_{i\in\mathbb{N}_n}$ be the families in C defined by the above equalities. Define

$$g : [a,b] \times [a,b] \longrightarrow \mathbb{K}, \quad (s,t) \longmapsto \begin{cases} \sum_{i=1}^{n} a_i(t)x_i(s) & \text{if } s \leq t \\ \sum_{i=1}^{n} b_i(t)x_i(s) & \text{if } s \geq t. \end{cases}$$

It is easy to see that g is a Green function. ■

Proposition 3.2.1.4 (5) *Let g be the Green function and take $x \in C$. Then $gx \in \mathcal{D}$ and*

$$ugx = -x.$$

By Theorem 3.2.1.3, there are two families $(a_i)_{n\in\mathbb{N}_n}$ and $(b_i)_{i\in\mathbb{N}_n}$ in C such that

$$g(s,t) = \sum_{i=1}^{n} a_i(t)x_i(s)$$

for all $s, t \in [a,b]$ with $s \leq t$ and

$$g(s,t) = \sum_{i=1}^{n} b_i(t)x_i(s)$$

for all $s, t \in [a,b]$ with $s \geq t$.

Take $s \in [a,b]$. Then

$$(gx)(s) = \int_a^b g(s,t)x(t)dt = \int_a^s g(s,t)x(t)dt + \int_s^b g(s,t)x(t)dt =$$

$$= \int_a^s \left(\sum_{i=1}^n b_i(t)x_i(s)\right) x(t)dt + \int_s^b \left(\sum_{i=1}^n a_i(t)x_i(s)\right) x(t)dt =$$

$$= \sum_{i=1}^n \left(\int_a^s b_i(t)x(t)dt + \int_s^b a_i(t)x(t)dt\right) x_i(s).$$

Using complete induction on j, we prove that

$$(gx)^{(j)}(s) = \sum_{i=1}^n \left(\int_a^s b_i(t)x(t)dt + \int_s^b a_i(t)x(t)dt\right) x_i^{(j)}(s) - \delta_{n,j}\frac{x(s)}{p_n(s)} =$$

$$= \int_a^b g^{(j)}(s,t)x(t)dt - \delta_{n,j}\frac{x(s)}{p_n(s)}$$

holds for every $j \in \mathbb{N}_n \cup \{0\}$. Take $j \in \mathbb{N}_n$ and assume the above equalities hold for $j-1$. Then

$$(gx)^{(j)}(s) = \sum_{i=1}^n (b_i(s)x(s) - a_i(s)x(s))x_i^{(j-1)}(s) +$$

$$+ \sum_{i=1}^n \left(\int_a^s b_i(t)x(t)dt + \int_s^b a_i(t)x(t)dt\right) x_i^{(j)}(s) =$$

$$= \left(\sum_{i=1}^n (b_i(s) - a_i(s))\, x_i^{(j-1)}(s)\right) x(s) +$$

$$+ \sum_{i=1}^n \left(\int_a^s b_i(t)x(t)dt + \int_s^b a_i(t)x(t)dt\right) x_i^{(j)}(s).$$

Then, by Theorem 3.2.1.3,

$$(gx)^{(j)}(s) = -\delta_{n,j}\frac{x(s)}{p_n(s)} + \sum_{i=1}^n \left(\int_a^s b_i(t)x(t)dt + \int_s^b a_i(t)x(t)dt\right) x_i^{(j)}(s) =$$

$$= \int_a^s \left(\sum_{i=1}^n b_i(t) x_i^{(j)}(s) \right) x(t)dt + \int_s^b \left(\sum_{i=1}^n a_i(t) x_i^{(j)}(s) \right) x(t)dt - \delta_{n,j} \frac{x(s)}{p_n(s)} =$$

$$= \int_a^s g^{(j)}(s,t)x(t)dt + \int_s^b g^{(j)}(s,t)x(t)dt - \delta_{n,j} \frac{x(s)}{p_n(s)} =$$

$$= \int_a^b g^{(j)}(s,t)x(t)dt - \delta_{n,j} \frac{x(s)}{p_n(s)} .$$

This completes the induction.

It follows from the above result that gx is n–times continuously differentiable and that

$$f_i(gx) = \sum_{j=1}^n \alpha_{ij} (gx)^{(j-1)}(a) + \sum_{j=1}^n \beta_{ij} (gx)^{(j-1)}(b) =$$

$$= \int_a^b \left(\sum_{j=1}^n \alpha_{ij} (g)^{(j-1)}(a,t) \right) x(t)dt + \int_a^b \left(\sum_{j=1}^n \beta_{ij} g^{(j-1)}(b,t) \right) x(t)dt =$$

$$= \int_a^b f_i(g(\cdot,t))x(t)dt = 0$$

for every $i \in \mathbb{N}_n$. Hence $gx \in \mathcal{D}$. Moreover

$$(ugx)(s) = \sum_{i=0}^n p_i(s)(gx)^{(i)}(s) =$$

$$= \int_a^b \left(\sum_{i=0}^n p_i(s) g^{(i)}(s,t)x(t)dt - x(s) \right) = -x(s)$$

for every $s \in [a,b]$. ■

Corollary 3.2.1.5 (5) *u is bijective,*

$$u^{-1}x = -gx$$

for $x \in \mathcal{C}$ and

$$gux = -x$$

for $x \in \mathcal{D}$, where g is the Green function.

The assertions follow immediately from Proposition 3.2.1.4. ■

Corollary 3.2.1.6 (5) *Let g be the Green function. Take $\alpha \in \mathbb{K}$ and put*

$$\mathcal{F} := \{x \in \mathcal{D} \mid ux = \alpha x\},$$

$$\mathcal{G} := \{\alpha x - ux \mid x \in \mathcal{D}\}.$$

a) $\mathcal{F} = \{x \in C \mid x + \alpha g x = 0\}$.

b) $\mathcal{G} = \{x + \alpha g x \mid x \in C\}$.

c) $\operatorname{Dim} \mathcal{F} = \operatorname{Dim} C/\mathcal{G} \leq n$.

d) *We have*

$$\operatorname{Dim} \mathcal{F} = \operatorname{Dim} \{z \in C \mid z + \alpha g' z = 0\}$$

and the following are equivalent for every $x \in C$:

d_1) $x = \alpha y - uy$ *for some* $y \in \mathcal{D}$.

d_2) *If* $z \in C$ *such that* $z + \alpha g' z = 0$, *then*

$$\int_a^b x(t)z(t)dt = 0.$$

a) Take $x \in \mathcal{F}$. By Proposition 3.2.1.4,

$$x + \alpha g x \in \mathcal{D},$$

$$u(x + \alpha g x) = ux - \alpha x = 0,$$

so that

$$x + \alpha g x = 0.$$

Hence

$$\mathcal{F} \subset \{x \in C \mid x + \alpha g x = 0\}.$$

On the other hand (Proposition 3.2.1.4)

$$x = -\alpha g x \in \mathcal{D},$$

$$ux = -\alpha ugx = \alpha x$$

for every $x \in \mathcal{C}$ for which $x + \alpha gx = 0$ and this proves the reverse inclusion.

b) Take $x \in \mathcal{D}$. By Corollary 3.2.1.5, there is a $y \in \mathcal{C}$ such that

$$gy = x\,.$$

Hence

$$\alpha x - ux = \alpha gy - ugy = y + \alpha gy$$

(Proposition 3.2.1.4) which establishes the inclusion

$$\mathcal{G} \subset \{x + \alpha gx \mid x \in \mathcal{C}\}\,.$$

Now take $x \in \mathcal{C}$. By Corollary 3.2.1.5,

$$uy = -x$$

for some $y \in \mathcal{D}$. Hence

$$x + \alpha gx = -uy - \alpha guy = \alpha y - uy$$

(Corollary 3.2.1.5), which proves the reverse inclusion.

c) We may assume that $\alpha \neq 0$, since, by b), the assertion is trivial for $\alpha = 0$. The operator

$$\mathcal{C} \longrightarrow \mathcal{C}\,, \quad x \longmapsto gx$$

is compact (Corollary 3.1.6.22) and so

$$\operatorname{Dim}\{x \in \mathcal{C} \mid x + \alpha gx = 0\} = \operatorname{Dim}\mathcal{C}/\{x + \alpha gx \mid x \in \mathcal{C}\} < \infty$$

(Theorem 3.1.5.1 a)). By a) and b), it follows that

$$\operatorname{Dim}\mathcal{F} = \operatorname{Dim}\mathcal{C}/\mathcal{G}\,.$$

$$\operatorname{Dim}\mathcal{F} \leq n$$

is a consequence of the general theory of differential equations.

d) For $\alpha = 0$ the assertion is trivial (Proposition 3.2.1.4), so we may assume $\alpha \neq 0$. By a) and Corollary 3.1.6.7,

$$\operatorname{Dim}\mathcal{F} = \operatorname{Dim}\operatorname{Ker}(1 + \alpha g) = \operatorname{Dim}\operatorname{Ker}(1 + \alpha g') = \operatorname{Dim}\{z \in \mathcal{C} \mid z + \alpha g'z = 0\}\,.$$

$d_1 \Rightarrow d_2$. By b), there is a $y \in \mathcal{C}$ such that

$$x = y + \alpha g y = -\alpha \left(-\frac{1}{\alpha} y - g y \right) .$$

Since

$$g' z = -\frac{1}{\alpha} z ,$$

$$\int x(t) z(t) dt = 0$$

by Corollary 3.1.6.7 a $\Rightarrow$ b.

$d_2 \Rightarrow d_1$. Now

$$\int x(t) z(t) dt = 0$$

for every $z \in \mathcal{C}$ with

$$g'(z) = -\frac{1}{\alpha} z .$$

Hence by Corollary 3.1.6.7 b $\Rightarrow$ a, there is a $y_0 \in \mathcal{C}$ with

$$-\frac{1}{\alpha} x = -\frac{1}{\alpha} y_0 - g y_0 ,$$

$$x = y_0 + \alpha g y_0 .$$

By b), there is a $y \in \mathcal{D}$ with

$$x = \alpha y - u y . \qquad \blacksquare$$

Corollary 3.2.1.7 (5) *Take* $\alpha \in \mathbb{K}$ *and*

$$v : \mathcal{D} \longrightarrow \mathcal{C} , \quad x \longmapsto \alpha x - u x .$$

The following are equivalent:

a) v *is injective.*

b) v *is surjective.*

c) *For each* $y \in \mathcal{C}$ *and each family* $(\alpha_i)_{i \in \mathbb{N}_n}$ *in* $\mathbb{K}$, *there is a unique* n-*times continuously differentiable function* x *on* $[a, b]$ *such that*

$$\sum_{i=0}^{n} p_i x^{(i)} - \alpha x = y$$

and

$$f_i(x) = \alpha_i$$

for every $i \in \mathbb{N}_n$.

a $\Leftrightarrow$ b follows from Corollary 3.2.1.6 c).

a & b $\Rightarrow$ c. The uniqueness follows from a). By Lemma 3.2.1.1, there is a family $(\beta_j)_{j\in\mathbb{N}_n}$ in $\mathbb{K}$ such that

$$f_i\left(\sum_{j=1}^{n}\beta_j x_j\right) = \alpha_i$$

for all $i \in \mathbb{N}_n$. By b), there is a $z \in \mathcal{D}$ such that

$$uz - \alpha z = y + \alpha\sum_{j=1}^{n}\beta_j x_j .$$

The function

$$x := z + \sum_{j=1}^{n}\beta_j x_j$$

has the requested properties.

c $\Rightarrow$ a is trivial. ■

Proposition 3.2.1.8 (5) *Given $x, y \in \mathcal{C}$, define*

$$\langle x|y\rangle := \int_a^b x(t)\overline{y(t)}dt .$$

Then the following are equivalent:

a) *The Green function is selfadjoint.*

b) $x, y \in \mathcal{D} \Rightarrow \langle ux|y\rangle = \langle x|uy\rangle$.

c) *The sesquilinear form*

$$\mathcal{C}\times\mathcal{C} \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \langle gx|y\rangle$$

is Hermitian.

a $\Rightarrow$ b. By Corollary 3.2.1.5 and Proposition 3.1.6.18 a),

$$\langle ux|y\rangle = -\langle ux|guy\rangle = -\langle gux|uy\rangle = \langle x|uy\rangle .$$

b $\Rightarrow$ c. Let $x, y \in \mathcal{C}$. By Corollary 3.2.1.4,

$$\langle gx|y\rangle = -\langle gx|ugy\rangle = -\langle ugx|gy\rangle = \langle x|gy\rangle = \overline{\langle gy|x\rangle} .$$

c $\Rightarrow$ a. Let λ be Lebesgue measure on $[a,b]$. Since $\mathcal{C}$ is dense in $L^2(\lambda)$, it follows that the sesquilinear form

$$L^2(\lambda)\times L^2(\lambda) \longrightarrow \mathbb{K}, \quad (x,y) \longmapsto \int (gx)\overline{y}d\lambda$$

is Hermitian. By Proposition 3.1.6.18 d), g is selfadjoint. ■

3.2.2 Supplementary Results

In this subsection we use the notation from Subsection 3.2.1, but do not assume u injective. Given $x, y \in \mathcal{C}$, define

$$\langle x|y\rangle = \int_a^b x(t)\overline{y(t)}dt\,.$$

Proposition 3.2.2.1 (5) *Given $\alpha, \beta \in \mathbb{K}$, define*

$$u_\alpha : \mathcal{D} \longrightarrow \mathcal{C}\,, \quad x \longmapsto ux + \alpha x\,,$$

$$\mathcal{F}_{\alpha,\beta} := \{x \in \mathcal{D} \mid u_\alpha x = \beta x\}\,,$$

$$A_\alpha := \{\beta \in \mathbb{K} \mid \mathcal{F}_{\alpha,\beta} \neq \{0\}\}\,.$$

Take $\alpha, \beta \in \mathbb{K}$.

a) $\mathcal{F}_{\alpha,\beta} = \mathcal{F}_{0,\beta-\alpha}$.

b) $A_\alpha = \alpha + A_0$.

c) *Either $A_0 = \mathbb{K}$ or A_0 has no points of adherence (i.e. A_0 is finite or it consists of a sequence which converges to ∞).*

d) *If $\mathbb{K} = \mathbb{C}$ and*

$$\langle ux|x\rangle = \langle x|ux\rangle$$

for every $x \in \mathcal{D}$, then A_0 is contained in $\mathbb{R}$ and has no points of adherence.

a) Given $x \in \mathcal{D}$,

$$(x \in \mathcal{F}_{\alpha,\beta}) \Longleftrightarrow (u_\alpha x = \beta x) \Longleftrightarrow$$

$$\Longleftrightarrow (ux + \alpha x = \beta x) \Longleftrightarrow (ux = (\beta - \alpha)x) \Longleftrightarrow (x \in \mathcal{F}_{0,\beta-\alpha})\,.$$

b) By a), if $\beta \in \mathbb{K}$, then

$$(\beta \in A_\alpha) \Longleftrightarrow (\mathcal{F}_{\alpha,\beta} \neq \{0\}) \Longleftrightarrow (\mathcal{F}_{0,\beta-\alpha} \neq \{0\}) \Longleftrightarrow (\beta - \alpha \in A_0)\,.$$

c) Assume that $A_0 \neq \mathbb{K}$. Take $\alpha \in \mathbb{K}$ such that $-\alpha \notin A_0$. By b), $0 \notin A_\alpha$. Hence u_α is injective. Let g be the Green function associated to u_α (Theorem 3.2.1.3) and $\tilde{g}$ the compact operator

$$\mathcal{C} \longrightarrow \mathcal{C}\,, \quad x \longmapsto gx$$

(Corollary 3.1.6.22). We have

$$A_\alpha = \{\beta \in \mathbb{K} \mid -\frac{1}{\beta} \in \sigma(\tilde{g})\}$$

(Corollary 3.2.1.6 a) and Theorem 3.1.5.1 $b_1 \Leftrightarrow b_2$). By Theorem 3.1.5.1 c), A_α has no points of adherence. By b), A_0 has no points of adherence.

d) Take $\beta \in A_0$. Then there is an $x \in \mathcal{D}\backslash\{0\}$ such that

$$ux = \beta x\,.$$

Then

$$\beta\langle x|x\rangle = \langle \beta x|x\rangle = \langle ux|x\rangle = \langle x|ux\rangle = \langle x|\beta x\rangle = \overline{\beta}\langle x|x\rangle\,.$$

Since

$$\langle x|x\rangle = \int_a^b |x(t)|^2 dt \neq 0\,,$$

it follows that

$$\beta = \overline{\beta}\,.$$

Hence $\beta \in \mathbb{R}$ and $A_0 \subset \mathbb{R}$. By c), A_0 has no points of adherence. ■

Proposition 3.2.2.2 (5) *Let p be a real continuous function on $[a,b]$ and define*

$$v : \mathcal{D} \longrightarrow \mathcal{C}\,, \quad x \longmapsto ux + px\,.$$

Then the following are equivalent:

a) $x, y \in \mathcal{D} \Rightarrow \langle ux|y\rangle = \langle x|uy\rangle$.

b) $x, y \in \mathcal{D} \Rightarrow \langle vx|y\rangle = \langle x|vy\rangle$.

Hence, if u and v are injective and g, h are the Green functions associated to u and v, respectively, then g is selfadjoint iff h is selfadjoint.

a $\Rightarrow$ b. We have

$$\langle vx|y\rangle = \langle ux + px|y\rangle = \langle ux|y\rangle + \langle px|y\rangle =$$

$$= \langle x|uy\rangle + \langle x|py\rangle = \langle x|uy + py\rangle = \langle x|vy\rangle .$$

b $\Rightarrow$ a follows from a $\Rightarrow$ b.

The last assertion follows from a $\Leftrightarrow$ b and Proposition 3.2.1.8 a $\Leftrightarrow$ b. ■

Definition 3.2.2.3 (5) *If p_i is i-times continuously differentiable for each $i \in \mathbb{N}_n$, then*

$$u^* : \mathcal{D} \longrightarrow \mathcal{C}, \quad x \longmapsto \sum_{i=0}^{n} (-1)^i (\overline{p}_i x)^{(i)}$$

is called ***the adjoint differential operator of*** *u. u is called* ***selfadjoint*** *if $u = u^*$.*

If x and y are Lebesgue integrable functions on $[a,b]$ then we say that $ux = y$ ***in the sense of L^2-distributions*** *if*

$$\langle y|\psi\rangle = \langle x|u^*\psi\rangle$$

for all C^∞-functions ψ on $]a,b[$ with compact carrier.

If $\mathbb{K} = \mathbb{R}$ then $ux = y$ in the sense of L^2-distributions conincides with $ux = y$ in the classical sense of distribution theory.

Take

$$v : \mathcal{D} \longrightarrow \mathcal{C}, \quad x \longmapsto px^{(n)},$$

where p is n-times continuous differentiable. Then for $x \in \mathcal{D}$,

$$v^*x = (-1)^n(\overline{p}x)^{(n)} = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \overline{p}^{(n-i)} x^{(i)},$$

$$(v^*)^*x = (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} (p^{n-i})x)^{(i)} =$$

$$= (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} \sum_{j=0}^{i} \binom{i}{j} p^{(n-j)} x^{(j)} =$$

$$= (-1)^n \sum_{j=0}^{n} p^{(n-j)} x^{(i)} \sum_{i=j}^{n} (-1)^i \frac{n!}{i!(n-i)!} \frac{i!}{j!(i-j)!} =$$

$$= (-1)^n \sum_{j=0}^{n} p^{(n-j)} x^{(j)} \sum_{k=0}^{n-j} (-1)^{j+k} \frac{n!}{(n-j-k)!j!k!} =$$

$$= (-1)^n \sum_{j=0}^{n} (-1)^j \binom{n}{j} p^{(n-j)} x^{(j)} \sum_{k=0}^{n-j} (-1)^k \frac{(n-j)!}{k!(n-j-k)!} =$$

$$= px^{(n)} + (-1)^n \sum_{j=0}^{n-1} (-1)^j \binom{n}{j} p^{(n-j)} x^{(j)} \sum_{k=0}^{n-j} (-1)^k \binom{n-j}{k} = px^{(n)}.$$

Hence

$$(u^*)^* = u.$$

Example 3.2.2.4 (5) *Take $n = 2$. If p_i is real and i-times continuously differentiable for each $i \in \mathbb{N}_2 \cup \{0\}$, then u is selfadjoint iff*

$$p_1 = p_2'.$$

Take $x \in \mathcal{D}$. Then

$$u^*x = (p_0x) - (p_1x)' + (p_2x)'' = p_0x - p_1'x - p_1x' + p_2''x + 2p_2'x' + p_2x'' =$$

$$= (p_0 - p_1' + p_2'')x + (2p_2' - p_1)x' + p_2x'',$$

$$u^*x - ux = (p_2'' - p_1')x + 2(p_2' - p_1)x',$$

$$x(u^*x - ux) = (p_2'' - p_1')x^2 + 2(p_2' - p_1)xx' = ((p_2' - p_1)x^2)'.$$

Since x is arbitrary, it follows that $u = u^*$ iff $p_1 = p_2'$. ■

Proposition 3.2.2.5 (5) *If p_i is i-times continuously differentiable for each $i \in \mathbb{N}_n$, then given any n-times continuously differentiable functions x, y on $[a, b]$ we have*

$$\left(\sum_{i=0}^{n} p_i x^{(i)} \right) \overline{y} - x \left(\sum_{i=0}^{n} (-1)^i (p_i \overline{y})^{(i)} \right) =$$

$$= \frac{d}{dt} \sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right),$$

$$\langle ux|y\rangle - \langle x|u^* y\rangle = \left(\sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right)(b) -$$

$$- \left(\sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right)(a).$$

We have, for each $i \in \mathbb{N}_n$,

$$\frac{d}{dt} \left(\sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right) =$$

$$= \sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j+1)} x^{(i-j-1)} + \sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j)} =$$

$$= - \sum_{j=1}^{i} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j)} + \sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j)} =$$

$$= -(-1)^i (p_i \overline{y})^{(i)} x + p_i \overline{y} x^{(i)}.$$

It follows that

$$\left(\sum_{i=0}^{n} p_i x^{(i)} \right) \overline{y} = p_0 x \overline{y} + \sum_{i=1}^{n} (p_i \overline{y}) x^{(i)} =$$

$$= p_0 x \overline{y} + \sum_{i=1}^{n} \left((-1)^i (p_i \overline{y})^{(i)} x + \frac{d}{dt} \left(\sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right) \right),$$

$$\left(\sum_{i=0}^{n} p_i x^{(i)} \right) \overline{y} - x \sum_{i=0}^{n} (-1)^i \left(p_i \overline{y} \right)^{(i)} \Big) =$$

$$= \frac{d}{dt} \left(\sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right) \right).$$

Thus

$$\langle ux|y\rangle - \langle x|u^*y\rangle = \int_a^b \left(\left(\sum_{i=0}^{n} p_i x^{(i)} \right) \overline{y} - x \left(\sum_{i=0}^{n} (-1)^i (p_i \overline{y})^{(i)} \right) \right) dt =$$

$$= \left(\sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right) \right) (b) -$$

$$- \left(\sum_{i=1}^{n} \left(\sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right) \right) (a) . \qquad \blacksquare$$

Corollary 3.2.2.6 (5) *If p_i is i-times continuously differentiable for each $i \in \mathbb{N}_n$, then the following are equivalent for all $x, y \in \mathcal{D}$:*

a)

$$\left(\sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right) (a) = \left(\sum_{i=1}^{n} \sum_{j=0}^{i-1} (-1)^j (p_i \overline{y})^{(j)} x^{(i-j-1)} \right) (b) .$$

b) $\langle ux|y\rangle = \langle x|u^*y\rangle$.

If in addition u is injective then the following are equivalent:

c) *u is selfadjoint.*

d) *The Green function for the given boundary problem is selfadjoint.*

a $\Leftrightarrow$ b follows from Proposition 3.2.2.5.

c $\Leftrightarrow$ d. Take C^∞-functions φ, ψ on $]a, b[$ with compact carrier. By a $\Rightarrow$ b,

$$\langle u\varphi \mid \psi\rangle = \langle \varphi | u^*\psi\rangle ,$$

so that the assertion follows from Proposition 3.2.1.8 a $\Leftrightarrow$ b. $\blacksquare$

Corollary 3.2.2.7 (5) *Suppose that u is injective and its coefficients are C^∞-functions. If x is a Lebesgue integrable function on $[a, b]$, then $gx \in C$ and*

$$ugx = -x$$

in the sense of L^2-distributions.

By Corollary 3.1.6.4, $gx \in \mathcal{C}$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that

$$\lim_{n\to\infty}\int_a^b |x_n(t)-x(t)|dt = 0\,.$$

Let φ be a $\mathcal{C}^\infty$-function on $]a,b[$ with compact carrier and u^* the adjoint of u. By Corollary 3.2.2.6 a $\Rightarrow$ b and Proposition 3.2.1.4,

$$\int_a^b (gx_n)\overline{(u^*\varphi)}dt = \int_a^b (ugx_n)\overline{\varphi}dt = -\int_a^b x_n\overline{\varphi}dt$$

for every $n \in \mathbb{N}$. It follows that

$$\langle gx|u^*\varphi\rangle = \int_a^b (gx)\overline{(u^*\varphi)}dt = \lim_{n\to\infty}\int_a^b (gx_n)\overline{(u^*\varphi)}dt =$$

$$= -\lim_{n\to\infty}\int_a^b x_n\overline{\varphi}dt = -\int_a^b x\overline{\varphi}dt = \langle -x|\varphi\rangle\,.$$

Hence

$$ugx = -x$$

in the sense of L^2-distributions. ■

Corollary 3.2.2.8 (5) *Take* $n = 2$, $\mathbb{K} = \mathbb{R}$, *and let* u *be selfadjoint. Put*

$$\alpha := \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\,, \quad \beta := \beta_{11}\beta_{22} - \beta_{12}\beta_{21}\,.$$

Let x, y *be twice continuously differentiable functions on* $[a,b]$ *and take*

$$D := xy' - x'y\,.$$

a) $\langle ux|y\rangle - \langle x|uy\rangle = p_2(a)D(a) - p_2(b)D(b)\,.$

b) $x, y \in \mathcal{D} \Longrightarrow \begin{cases} \alpha(\langle ux|y\rangle - \langle x|uy\rangle) = -(\alpha p_2(b) + \beta p_2(a))D(b)\,, \\ \beta(\langle ux|y\rangle - \langle x|uy\rangle) = (\alpha p_2(b) + \beta p_2(a))D(a)\,. \end{cases}$

c) *If* $x, y \in \mathcal{D}$ *and*

$$\alpha = \beta = 0$$

and if the boundary conditions $f_i = 0 \quad (i \in \mathbb{N}_n)$ *are linearly independent, then*

$$D(a) = D(b) = 0\,.$$

a) By Example 3.2.2.4,

$$\sum_{i=1}^{2}\sum_{j=0}^{i-1}(-1)^{(j)}(p_i y)^{(j)}x^{(i-j-1)} = (p_1 y)x + (p_2 y)x' - (p_2 y)'x =$$

$$= p_1 yx + p_2 yx' - p_2' yx - p_2 y'x = -p_2 D\,.$$

We deduce from Proposition 3.2.2.5

$$\langle ux|y\rangle - \langle x|uy\rangle = p_2(a)D(a) - p_2(b)D(b)\,.$$

b) We have

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} x(a) & y(a) \\ x'(a) & y'(a) \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}\begin{bmatrix} x(b) & y(b) \\ x'(b) & y'(b) \end{bmatrix} = 0\,.$$

It follows from this that

$$\alpha D(a) + \beta D(b) = 0\,.$$

From a) we deduce that

$$\alpha(\langle ux|y\rangle - \langle x|uy\rangle) = \alpha p_2(a)D(a) - \alpha p_2(b)D(b) =$$

$$= -\beta p_2(a)D(b) - \alpha p_2(b)D(b) = -(\alpha p_2(b) + \beta p_2(a))D(b)\,,$$

$$\beta(\langle ux|y\rangle - \langle x|uy\rangle) = \beta p_2(a)D(a) - \beta p_2(b)D(b) =$$

$$= \beta p_2(a)D(a) + \alpha p_2(b)D(a) = (\alpha p_2(b) + \beta p_2(a))D(a)\,.$$

c) Since the boundary conditions are linearly independent, it follows that the rank of the matrix

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} & \beta_{11} & \beta_{12} \\ \alpha_{21} & \alpha_{22} & \beta_{21} & \beta_{22} \end{bmatrix}$$

is 2. Hence there are $i, j \in \{1, 2\}$ such that

$$\operatorname{Det}\begin{bmatrix} \alpha_{1i} & \beta_{1j} \\ \alpha_{2i} & \beta_{2j} \end{bmatrix} \neq 0\,.$$

From

$$\begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} x(a) & y(a) \\ x'(a) & y'(a) \end{bmatrix} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} x(b) & y(b) \\ x'(b) & y'(b) \end{bmatrix} = 0$$

we deduce that

$$0 = [\alpha_{2i}, -\alpha_{1i}] \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} x(a) & y(a) \\ x'(a) & y'(a) \end{bmatrix} +$$

$$+ [\alpha_{2i}, -\alpha_{1i}] \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix} \begin{bmatrix} x(b) & y(b) \\ x'(b) & y'(b) \end{bmatrix} =$$

$$= [\alpha_{2i}\alpha_{11} - \alpha_{1i}\alpha_{21} \quad \alpha_{2i}\alpha_{12} - \alpha_{1i}\alpha_{22}] \begin{bmatrix} x(a) & y(a) \\ x'(a) & y'(a) \end{bmatrix} +$$

$$+ [\alpha_{2i}\beta_{11} - \alpha_{1i}\beta_{21} \quad \alpha_{2i}\beta_{12} - \alpha_{1i}\beta_{22}] \begin{bmatrix} x(b) & y(b) \\ x'(b) & y'(b) \end{bmatrix},$$

$$\begin{cases} (\alpha_{2i}\beta_{11} - \alpha_{1i}\beta_{21})x(b) + (\alpha_{2i}\beta_{12} - \alpha_{1i}\beta_{22})x'(b) = 0 \\ (\alpha_{2i}\beta_{11} - \alpha_{1i}\beta_{21})y(b) + (\alpha_{2i}\beta_{12} - \alpha_{1i}\beta_{22})y'(b) = 0 . \end{cases}$$

Since

$$\alpha_{2i}\beta_{1j} - \alpha_{1i}\beta_{2j} \neq 0$$

we see that

$$D(b) = x(b)y'(b) - x'(b)y(b) = 0 .$$

Similarly,

$$D(a) = 0 .$$ ■

Corollary 3.2.2.9 (5) *Take* $n = 2$, $\mathbb{K} = \mathbb{R}$, *and let* u *be selfadjoint. If the boundary conditions* $f_i = 0$ $(i \in \mathbb{N}_n)$ *are linearly independent, then the following are equivalent:*

a) $\langle ux|y\rangle = \langle x|uy\rangle$ *for all* $x, y \in \mathcal{D}$.

b) $(\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21})p_2(b) + (\beta_{11}\beta_{22} - \beta_{12}\beta_{21})p_2(a) = 0$.

If in addition u is injective, then the above assertions are equivalent to the following one:

c) *The Green function for the given boundary problem is selfadjoint.*

Put

$$\alpha := \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21}\,, \quad \beta := \beta_{11}\beta_{22} - \beta_{12}\beta_{21}\,.$$

a $\Rightarrow$ b. We may assume that

$$\alpha \neq 0 \text{ or } \beta \neq 0\,.$$

If $\alpha \neq 0$ (resp. $\beta \neq 0$) then there are $x, y \in \mathcal{D}$ such that

$$x(b)y'(b) - x'(b)y(b) \neq 0 \qquad (\text{resp. } x(a)y'(a) - x'(a)y(a) \neq 0)\,.$$

By a) and Corollary 3.2.2.8 b),

$$\alpha p_2(b) + \beta p_2(a) = 0\,.$$

b $\Rightarrow$ a. If

$$\alpha \neq 0 \text{ or } \beta \neq 0$$

then the assertion follows from Corollary 3.2.2.8 b). If

$$\alpha = \beta = 0\,,$$

then the assertion follows from Corollary 3.2.2.8 a), c).

a $\Leftrightarrow$ c follows from Proposition 3.2.1.8 a $\Leftrightarrow$ b. ■

Proposition 3.2.2.10 (5) *Assume that*

$$p_i(a) = p_i(b)$$

for every $i \in \mathbb{N}_n \cup \{0\}$ *and that*

$$f_i(x) = x^{(i-1)}(a) - x^{(i-1)}(b)$$

for every $i \in \mathbb{N}_n$. *Put*

$$\widetilde{\mathcal{D}} := \{x \in \mathcal{D} \mid x^{(n)}(a) = x^{(n)}(b)\}\,,$$

$$\widetilde{\mathcal{C}} := \{x \in \mathcal{C} \mid x(a) = x(b)\}\,,$$

and let x_0 *denote a function in* $\mathcal{D}$ *such that*

$$x_0^{(i-1)}(a) = x_0^{(i-1)}(b) = 0$$

for every $i \in \mathbb{N}_n$ *and*

$$x_0^{(n)}(a) = 0\,, \quad x_0^{(n)}(b) = 1\,.$$

a) $x \in \mathcal{D} \Rightarrow x + (x^{(n)}(a) - x^{(n)}(b))x_0 \in \widetilde{\mathcal{D}}$.

b) $x \in \mathcal{C} \Rightarrow x + (x(a) - x(b))x_0^{(n)} \in \widetilde{\mathcal{C}}$.

c) *$\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{C}}$ are vector subspaces of $\mathcal{D}$ and $\mathcal{C}$, respectively, of codimension one.*

d) $\bar{u}^1(\widetilde{\mathcal{C}}) = \widetilde{\mathcal{D}}$.

e) $\alpha \in \mathbb{K} \Rightarrow \{x \in \mathcal{D} \mid ux = \alpha x\} = \{x \in \widetilde{\mathcal{D}} \mid ux = \alpha x\}$.

f) *$\{\alpha \in \mathbb{K} \mid \exists x \in \mathcal{D}\backslash\{0\}, ux = \alpha x\}$ has no points of adherence.*

g) *If g is the corresponding Green function, then*

$$g^{(n)}(a,t) = g^{(n)}(b,t)$$

for every $t \in]a,b[$.

a) and b) are easy to see.

c) follows from a) and b).

d) Take $x \in \mathcal{D}$. Then

$$p_n(a)\,(x^{(n)}(a) - x^{(n)}(b)) = \sum_{j=0}^{n} p_j(a)x^{(j)}(a) - \sum_{j=0}^{n} p_j(b)x^{(j)}(b) = ux(a) - ux(b).$$

Hence $x \in \widetilde{\mathcal{D}}$ iff $ux \in \widetilde{\mathcal{C}}$.

e) Take $x \in \mathcal{D}$ with

$$ux = \alpha x.$$

Then

$$ux(a) - ux(b) = \alpha(x(a) - x(b)) = 0,$$

so that $ux \in \widetilde{\mathcal{C}}$. By d), $x \in \widetilde{\mathcal{D}}$.

f) follows from *M.A. Neumark, Lineare Differentialoperatoren, Akademie-Verlag, 1960, pages 54-56.*

g) We have

$$g^{(n)}(a,t) = -\sum_{i=0}^{n-1} \frac{p_i(a)}{p_n(a)} g^{(i)}(a,t) =$$

$$= -\sum_{i=0}^{n-1} \frac{p_i(b)}{p_n(b)} g^{(i)}(b,t) = g^{(n)}(b,t). \qquad \blacksquare$$

Remark. The above proposition shows how the results of this paragraph may be applied to the differential equations on $\mathbf{T}$.

Proposition 3.2.2.11 (5) *Let q_2, q_1, q_0 be continuous functions on $\mathbf{T}$ such that*

$$q_2(t) < 0$$

for every $t \in \mathbf{T}$. Given $k \in \mathbb{N} \cup \{0\}$, let $C^{(k)}$ be the set of k-times continuously differentiable functions on $\mathbf{T}$. Define

$$u : C^{(2)} \longrightarrow C^{(0)}, \quad x \longmapsto q_2 x'' + q_1 x' + q_0 x$$

and for each $\alpha \in \mathbb{R}$ put

$$\mathcal{F}_\alpha := \{x \in C^{(2)} \mid x \text{ is real and } ux = \alpha x\}.$$

Then

$$\{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\} \subset [\inf_{t \in \mathbf{T}} \operatorname{re} q_0(t), \infty[.$$

Take $\alpha \in \mathbb{R}$ with $\mathcal{F}_\alpha \neq \{0\}$ and $x \in \mathcal{F}_\alpha \backslash \{0\}$. Take $t_1, t_2 \in \mathbf{T}$ such that

$$x(t_1) = \sup_{t \in \mathbf{T}} x(t), \quad x(t_2) = \inf_{t \in \mathbf{T}} x(t).$$

Then

$$x'(t_1) = x'(t_2) = 0, \quad x''(t_1) \leq 0 \leq x''(t_2).$$

We get

$$\alpha x(t_1) = ux(t_1) = q_2(t_1)x''(t_1) + q_0(t_1)x(t_1) \geq q_0(t_1)x(t_1),$$

$$\alpha x(t_2) = ux(t_2) = q_2(t_2)x''(t_2) + q_0(t_2)x(t_2) \leq q_0(t_2)x(t_2).$$

If

$$x(t_1) > 0,$$

then $q_0(t_1) \in \mathbb{R}$ and

$$\alpha \geq q_0(t_1) \geq \inf_{t \in \mathbf{T}} \operatorname{re} q_0(t).$$

If

$$x(t_1) \leq 0,$$

then we may assume

$$x(t_2) < 0\,.$$

Therefore $q_0(t_2) \in \mathbb{R}$ and

$$\alpha > q_0(t_2) \geq \inf_{t\in\mathbb{T}} \operatorname{re} q_0(t)\,.$$

Hence

$$\{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\} \subset \left[\inf_{t\in\mathbb{T}} \operatorname{re} q_0(t)\,, \infty\right[\,. \qquad \blacksquare$$

Example 3.2.2.12 (5) *Assume* $\mathbb{K} = \mathbb{C}$. *Let* $(\alpha_k)_{k\in\mathbb{N}_n\cup\{0\}}$ *be a family in* $\mathbb{C}$ *with* $\alpha_n \neq 0$. *Given* $k \in \mathbb{N}\cup\{0\}$, *let* $C^{(k)}$ *denote the set of* k*-times continuously differentiable functions on* $\mathbb{T}$. *Define*

$$u : C^{(n)} \longrightarrow C^{(0)}\,, \quad x \longmapsto \sum_{k=0}^{n} \alpha_k x^{(k)}\,,$$

and let A *be the set of those* $\alpha \in \mathbb{C}$ *for which there is an* $x \in C^{(n)}\backslash\{0\}$ *such that*

$$ux = \alpha x\,.$$

If A *is infinite, then it has no points of adherence and*

$$\lim_{\substack{|\alpha|\to\infty \\ \alpha\in A}} \frac{\alpha}{|\alpha|} = i^n \frac{\alpha_n}{|\alpha_n|}$$

for n *even, and* $\pm i^n \frac{\alpha_n}{|\alpha_n|}$ *are the only possible points of adherence of the filter* $\left(\frac{\alpha}{|\alpha|}\right)_{\alpha\in A,|\alpha|\to\infty}$ *for* n *odd.*

By Proposition 3.2.2.1 c) (and Proposition 3.2.2.10 e),f)), A has no points of adherence.

Given $\alpha \in \mathbb{C}$, define

$$P_\alpha(t) := \sum_{k=0}^{n} \alpha_k t^k - \alpha \in \mathbb{C}[t]\,,$$

$$B_\alpha := \{\beta \in \mathbb{C} \mid P_\alpha(\beta) = 0\}\,,$$

$$C := \left\{\beta \in \mathbb{C} \;\middle|\; \sum_{k=1}^{n} k\alpha_k \beta^{k-1} = 0\right\}\,.$$

C is finite. Take $\alpha \in A\backslash C$. Then B_α has precisely n elements and $\left(e^{\beta\theta}\right)_{\beta\in B_\alpha}$ is a fundamental system of solutions of the differential equation

$$ux - \alpha x = 0\,.$$

Hence, if $x \in C^{(n)}\backslash\{0\}$ satisfies

$$ux = \alpha x\,,$$

then there is a family $(\gamma_\beta)_{\beta\in B_\alpha}$ in $\mathbb{C}$ such that

$$x(\theta) = \sum_{\beta\in B_\alpha} \gamma_\beta e^{\beta\theta}\,.$$

Thus, if $k \in \mathbb{N}_n$, then

$$\sum_{\beta\in B_\alpha} \beta^{k-1}\gamma_\beta = x^{(k-1)}(0) = x^{(k-1)}(1) = \sum_{\beta\in B_k} \beta^{k-1}e^\beta\gamma_\beta\,,$$

so that

$$\sum_{\beta\in B_\alpha} (1 - e^\beta)\beta^{k-1}\gamma_\beta = 0\,.$$

Since not every γ_β $(\beta \in B_\alpha)$ vanishes, the determinant of the above system of linear equations must vanish. Hence there is a $\beta_\alpha \in B_\alpha$ with

$$e^{\beta_\alpha} = 1\,.$$

There is therefore a $j_\alpha \in \mathbb{Z}$, so that

$$\beta_\alpha = 2\pi i j_\alpha\,.$$

It follows that

$$0 = P_\alpha(\beta_\alpha) = \sum_{k=0}^{n} \alpha_k(2\pi i j_\alpha)^k - \alpha\,.$$

Hence

$$\lim_{\substack{|\alpha|\to\infty\\ \alpha\in A}} |j_\alpha| = \infty\,,$$

and therefore

$$\lim_{\substack{|\alpha|\to\infty\\ \alpha\in A}} \frac{\alpha}{|\alpha|} = \lim_{\substack{|\alpha|\to\infty\\ \alpha\in A}} \left(\frac{j_\alpha}{|j_\alpha|}\right)^n \frac{\alpha_n}{|\alpha_n|} i^n = i^n \frac{\alpha_n}{|\alpha_n|}$$

if n is even. If n is odd, then $\pm i^n \frac{\alpha_n}{|\alpha_n|}$ are the only possible points of adherence of the filter $\left(\frac{\alpha}{|\alpha|}\right)_{\alpha\in A,|\alpha|\to\infty}$. ■

Proposition 3.2.2.13 (5) *Take $n \in \mathbb{N}$, define*

$$A := \{j \in \mathbb{N}_n \cup \{0\} \mid j \text{ is even}\}, \quad B := \{j \in \mathbb{N}_n \mid j \text{ is odd}\},$$

and denote for every $j \in \mathbb{N}_n \cup \{0\}$ by $C^{(j)}$ the set of j-times continuous differentiable functions on $\mathbf{T}$. Let $(p_j)_{j \in \mathbb{N}_n \cup \{0\}}$ be a family in $C^{(0)} \cap \mathbb{R}^{\mathbf{T}}$ such that $p_j \in C^{(j)}$ for every $j \in A$ and put

$$u : C^{(n)} \longrightarrow C^{(0)}, \quad x \longmapsto \sum_{j=0}^{n} p_j x^{(j)}.$$

Then for each $k \in A$ there are families $(q_j)_{j \in A}$ and $(r_j)_{j \in B}$ in $C^{(0)} \cap \mathbb{R}^{\mathbf{T}}$ such that $q_j \in C^{(j)}$ for every $j \in A$ and

$$ux = \sum_{\substack{j \in A \\ j > n-k}} \left(q_j x^{(\frac{j}{2})}\right)^{(\frac{j}{2})} + \sum_{\substack{j \in A \\ j \leq n-k}} q_j x^{(j)} + \sum_{j \in B} r_j x^{(j)}$$

for each $x \in C^{(n)}$.

We use induction on k. For $k = 0$ the assertion is trivial. So take $k \in A$, $k \leq n-2$, and assume the assertion holds for this k. We show that the assertion also holds for $k+2$.

Take $x \in C^{(n)}$. Let ℓ be the greatest element of A with $\ell \leq n-k$. Then

$$\left(q_\ell x^{(\frac{\ell}{2})}\right)^{(\frac{\ell}{2})} = \sum_{i=0}^{\frac{\ell}{2}} \binom{\frac{\ell}{2}}{i} q_\ell^{(\frac{\ell}{2}-i)} x^{(\frac{\ell}{2}+i)},$$

$$q_\ell x^{(\ell)} - \left(q_\ell x^{(\frac{\ell}{2})}\right)^{(\frac{\ell}{2})} = -\sum_{i=0}^{\frac{\ell}{2}-1} \binom{\frac{\ell}{2}}{i} q_\ell^{(\frac{\ell}{2}-i)} x^{(\frac{\ell}{2}+i)}$$

Given $j \in A$, define

$$\widetilde{q}_j := \begin{cases} q_j - \binom{\frac{\ell}{2}}{j-\frac{\ell}{2}} q_\ell^{(\ell-j)} & \text{if } \frac{\ell}{2} \leq j < \ell \\ q_j & \text{otherwise,} \end{cases}$$

and for each $j \in B$, define

$$\widetilde{r}_j := \begin{cases} r_j - \binom{\frac{\ell}{2}}{j-\frac{\ell}{2}} q_\ell^{(\ell-j)} & \text{if } \frac{\ell}{2} \leq j < \ell \\ r_j & \text{otherwise.} \end{cases}$$

Then $\widetilde{q}_j \in C^{(j)}$ for every $j \in A$ and

$$ux = \sum_{\substack{j\in A\\ j>n-k-2}} \left(q_j x^{(\frac{\ell}{2})}\right)^{(\frac{\ell}{2})} - \left(q_\ell x^{(\frac{\ell}{2})}\right)^{(\frac{\ell}{2})} + q_\ell x^{(\ell)} +$$

$$+ \sum_{\substack{j\in A\\ j\le n-k-2}} q_j x^{(j)} + \sum_{j\in B} r_j x^{(j)} =$$

$$= \sum_{\substack{j\in A\\ j>n-k-2}} \left(q_j x^{(\frac{\ell}{2})}\right)^{(\frac{\ell}{2})} - \sum_{i=0}^{\frac{\ell}{2}-1} \binom{\frac{\ell}{2}}{i} q_\ell^{(\frac{\ell}{2}-i)} x^{(\frac{\ell}{2}+i)} +$$

$$+ \sum_{\substack{j\in A\\ j\le n-k-2}} q_j x^{(j)} + \sum_{j\in B} r_j x^{(j)} =$$

$$= \sum_{\substack{j\in A\\ j>n-k-2}} \left(\tilde{q}_j x^{(\frac{\ell}{2})}\right)^{(\frac{\ell}{2})} + \sum_{\substack{j\in A\\ j\le n-k-2}} \tilde{q}_j x^{(j)} + \sum_{j\in B} \tilde{r}_j x^{(j)}. \qquad \blacksquare$$

Proposition 3.2.2.14 (5) *Take $n \in \mathbb{N}$ and denote for every $j \in \mathbb{N}_n \cup \{0\}$ by $C^{(j)}$ the set of j-times continuously differentiable functions on $\mathbf{T}$. Let*

$$(p_j)_{j\in\mathbb{N}_n\cup\{0\}} \in \prod_{j=0}^{n} (C^{(j)} \cap \mathbb{R}^{\mathbf{T}})$$

and consider

$$u : C^{(2n)} \longrightarrow C^{(0)}, \quad x \longmapsto \sum_{j=0}^{n} (p_j x^{(j)})^{(j)}.$$

Then

$$\int_0^{2\pi} (ux)(e^{it})\overline{y(e^{it})}dt = \int_0^{2\pi} x(e^{it})\overline{(uy)(e^{it})}dt$$

for all $x, y \in C^{(2n)}$.

By partial integration,

$$\int_0^{2\pi} (ux)(e^{it})\overline{y(e^{it})}dt = \sum_{j=0}^{n} \int_0^{2\pi} (p_j x^{(j)})^{(j)}(e^{it})\overline{y(e^{it})}dt =$$

$$= \sum_{j=0}^{n} (-1)^j \int_0^{2\pi} (p_j x^{(j)})(e^{it})\overline{y^{(j)}(e^{it})}dt =$$

$$= \sum_{j=0}^{n} (-1)^j \int_0^{2\pi} x^{(j)}(e^{it}) \overline{(p_j y^{(j)})(e^{it})} dt =$$

$$= \sum_{j=0}^{n} \int_0^{2\pi} x(e^{it}) \overline{(p_j y^{(j)})^{(j)}}(e^{it}) dt = \int_0^{2\pi} x(e^{it}) \overline{(uy)(e^{it})} dt . \quad \blacksquare$$

Theorem 3.2.2.15 (5) *Take $n \in \mathbb{N}$ and denote for every $j \in \mathbb{N}_n \cup \{0\}$ by $C^{(j)}$ the set of j-times continuously differentiable functions on $\mathbf{T}$. Let*

$$(p_j)_{j \in \mathbb{N}_n \cup \{0\}} \in \prod_{j=0}^{n} (C^{(j)} \cap \mathbb{R}^{\mathbf{T}})$$

and consider

$$u : C^{(n)} \longrightarrow C^{(0)}, \quad x \longmapsto \sum_{j=0}^{n} p_j x^{(j)} .$$

Put

$$\langle x|y \rangle := \int_0^{2\pi} x(e^{it}) \overline{y(e^{it})} dt$$

for all $x, y \in C^0$. Then the following are equivalent:

a) $\langle ux|y \rangle = \langle x|uy \rangle$ *for all* $x, y \in C^{(n)}$.

b) *n is even and there is a family*

$$(q_j)_{j \in \mathbb{N}_{\frac{n}{2}} \cup \{0\}} \in \prod_{j=0}^{\frac{n}{2}} (C^{(2j)} \cap \mathbb{R}^{\mathbf{T}})$$

such that

$$ux = \sum_{j=0}^{\frac{n}{2}} (q_j x^{(j)})^{(j)}$$

for every $x \in C^{(n)}$.

c) *u is selfadjoint (Definition 3.2.2.3).*

If p_n vanishes nowhere and u is injective, then the above assertions are equivalent to the following:

d) *The Green function for the given boundary problem (Theorem 3.2.1.3, Proposition 3.2.2.10 g)) is selfadjoint.*

a $\Rightarrow$ b. Put

$$A := \{j \in \mathbb{N}_n \cup \{0\} \mid j \text{ is even}\}, \quad B := \{j \in \mathbb{N}_n \mid j \text{ is odd}\}.$$

By Proposition 3.2.2.13, there are families $(q_j)_{j\in A}$ and $(r_j)_{j\in B}$ in $\mathcal{C}^{(0)} \cap \mathbb{R}^{\mathbf{T}}$ such that $q_j \in \mathcal{C}^{(j)}$ for every $j \in A$ and

$$ux = \sum_{j\in A} \left(q_j x^{(\frac{1}{2})}\right)^{(\frac{1}{2})} + \sum_{j\in B} r_j x^{(j)}$$

for every $x \in \mathcal{C}^{(n)}$. Define

$$v : \mathcal{C}^{(n)} \longrightarrow \mathcal{C}^{(0)}, \quad x \longmapsto \sum_{j\in A} \left(q_j x^{(\frac{1}{2})}\right)^{(\frac{1}{2})},$$

$$w : \mathcal{C}^{(n)} \longrightarrow \mathcal{C}^{(0)}, \quad x \longmapsto \sum_{j\in B} r_j x^{(j)}.$$

Take $x, y \in \mathcal{C}^{(n)}$. By Proposition 3.2.2.14,

$$\langle vx|y\rangle = \langle x|vy\rangle.$$

It follows that

$$\langle wx|y\rangle = \langle x|wy\rangle.$$

Take $k, \ell \in \mathbb{Z}$. Define

$$\begin{cases} x : \mathbf{T} \longrightarrow \mathbb{C}, & e^{it} \longmapsto e^{ikt} \\ y : \mathbf{T} \longrightarrow \mathbb{C}, & e^{it} \longmapsto e^{i\ell t}. \end{cases}$$

Then

$$\langle wx|y\rangle = \int_0^{2\pi} \sum_{j\in B} r_j(e^{it})(ik)^j e^{ikt} e^{-i\ell t} dt =$$

$$= \sum_{j\in B} (ik)^j \int_0^{2\pi} r_j(e^{it}) e^{i(k-\ell)t} dt,$$

$$\langle x|wy\rangle = \int_0^{2\pi} e^{ikt} \sum_{j\in B} r_j(e^{it})(-i\ell)^j e^{-i\ell t} dt =$$

$$= \sum_{j \in B} (-i\ell)^j \int_0^{2\pi} r_j(e^{it}) e^{i(k-\ell)t} dt,$$

so that

$$\sum_{j \in B} ((ik)^j - (-i\ell)^j) \int_0^{2\pi} r_j(e^{it}) e^{i(k-\ell)t} dt = 0.$$

Since k and ℓ are arbitrary,

$$\sum_{j \in B} i^j ((k+\ell)^j - (-1)^j \ell^j) \int_0^{2\pi} r_j(e^{it}) e^{ikt} dt = 0$$

for all $k, \ell \in \mathbb{Z}$. It follows that

$$\int_0^{2\pi} r_j(e^{it}) e^{ikt} dt = 0$$

for all $j \in B$ and $k \in \mathbb{Z}$. As is known, this implies $r_j = 0$ for every $j \in B$, i.e.

$$w = 0$$

and

$$u = v.$$

b $\Rightarrow$ a follows from Proposition 3.2.2.14.

a $\Leftrightarrow$ c follows from Corollary 3.2.2.6 a $\Rightarrow$ b.

a $\Leftrightarrow$ d follows from Proposition 3.2.1.8 b $\Leftrightarrow$ a (and Proposition 3.2.2.10 g)). ■

3.2.3 Linear Partial Differential Equations

Throughout this subsection, we assume $\mathbb{K} = \mathbb{R}$ and use the following notational conventions:

a) T is a Riemannian manifold of dimension d and λ is the volume measure on T.

b) u is a differential operator on T of the form

$$\sum_{i=1}^{k} X_i^2 + X_0 + a,$$

where a is a C^∞–function on T and $(X_i)_{i \in \mathbb{N}_k \cup \{0\}}$ is a family of first order C^∞–differential operators on T such that the coefficients of the principal part of u are not all identically zero on T and the rank of the Lie algebra generated by $\{X_i \mid i \in \mathbb{N}_k \cup \{0\}\}$ is d.

c) Given an open set U of T, $\mathcal{H}(U)$ is the set of C^∞–functions x on U such that $ux = 0$.

d) If x is a locally λ–integrable function on an open set of T, then ux is defined in the sense of distributions.

Throughout this subsection, we also use the notational conventions and terminology of *C. Constantinescu - A. Cornea, Potential Theory on Harmonic Spaces, Springer-Verlag, 1972* (which we abbreviate to [C-C]). T with the sheaf $\mathcal{H}$ is a Bauer space satisfying Doob's convergence axiom *(J.-M. Bony, Principe du maximum, inégalité de Harnack et unicité du problème de Cauchy pour les opérateurs elliptiques dégénérés, Ann. Inst. Fourier 19.1 (1969) 277–304, Théorème 8.2)* and every superharmonic function x on an open set U of T is locally λ–integrable and $-ux$ is a positive Radon measure on U (*P. Negrini - V. Scornazzani, Superharmonic Functions and Regularity of Boundary Points for a Class of Elliptic-Parabolic Partial Differential Operators, Bolletino della Unione Matemàtica Italiana, Serie 6, vol. 3 C (1984) 85–107, Theorem 1 i) $\Rightarrow$ ii)*).
Given a real function f on T, the expression "f converges to 0 at the Alexandroff point of T" means "if T is not compact then f converges to 0 at the Alexandroff point of T".

Proposition 3.2.3.1 (5) *Let S be a harmonic space, p a continuous potential on S, and q a continuous real function on S such that pq converges to 0 at the Alexandroff point of S. If $\mathcal{B}$ denotes the Banach space of bounded Borel functions on S (with the supremum norm), then the map*

$$\mathcal{B} \longrightarrow \mathcal{C}(S), \quad x \longmapsto (x \cdot p)q$$

([C-C] Theorem 8.2.1) is a compact operator.

Let $\overline{S}$ be the Alexandroff compactification of S. We identify $\mathcal{C}_0(S)$ with a subspace of $\mathcal{C}(\overline{S})$ by extending every function $x \in \mathcal{C}_0(S)$ to $\overline{S}$ by putting $x = 0$ at the Alexandroff point of S. Since

$$|(x \cdot p)q| \leq \|x\| \, |pq|$$

for every $x \in \mathcal{B}$, the above map is in fact an operator

$$\mathcal{B} \longrightarrow \mathcal{C}_0(S) \subset \mathcal{C}(\overline{S}).$$

By Ascoli's theorem (Theorem 1.1.2.16), it suffices to show that

$$\{(x \cdot p)q \mid x \in \mathcal{B}^{\#}\}$$

is equicontinuous on $\overline{S}$. Take $s_0 \in S$ and $\varepsilon > 0$. There is a resolutive set U of S such that $t_0 \in U$ and

$$0 \leq p - H_p^U < \frac{\varepsilon}{3}$$

on U. Given $x \in \mathcal{B}^{\#}$, we get successively

$$-p \preceq x \cdot p \preceq p,$$

$$-(p - H_p^U) \leq x \cdot p - H_{x \cdot p}^U \leq p - H_p^U,$$

$$|x \cdot p - H_{x \cdot p}^U| \leq p - H_p^U < \frac{\varepsilon}{3}$$

on U. Since $\{H_{x \cdot p}^U \mid x \in \mathcal{B}^{\#}\}$ is equicontinuous ([C-C] Theorem 11.1.1 b)), there is an open set V of S such that

$$s_0 \in V \subset U$$

and

$$|H^U_{x\cdot p}(s) - H^U_{x\cdot p}(s_0)| < \frac{\varepsilon}{3}$$

for every $x \in \mathcal{B}^\#$ and $s \in V$. Thus

$$|(x\cdot p)(s) - (x\cdot p)(s_0)| \leq$$

$$\leq |(x\cdot p)(s) - H^U_{x\cdot p}(s)| + |H^U_{x\cdot p}(s) - H^U_{x\cdot p}(s_0)| + |H^U_{x\cdot p}(s_0) - (x\cdot p)(s_0)| <$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

Hence $\{x\cdot p \mid x \in \mathcal{B}^\#\}$ is equicontinuous at s_0. It follows that $\{(x\cdot p)q \mid x \in \mathcal{B}^\#\}$ is equicontinuous at s_0.

Now let a be the Alexandroff point of S and take $\varepsilon > 0$. There is a neighbourhood U of a in $\overline{S}$ such that $|pq| < \varepsilon$ on U. We see that

$$|((x\cdot p)q)(s)| \leq |p(s)q(s)| < \varepsilon$$

for every $x \in \mathcal{B}^\#$ and $s \in U$. Hence $\{(x \cdot p)q \mid x \in \mathcal{B}^\#\}$ is equicontinuous at a. ■

Lemma 3.2.3.2 (5) *If T is compact and ue_T strictly negative, then $\mathcal{H}(T) = \{0\}$ and e_T is a potential.*

Let x be a harmonic function on T and take $t_1, t_2 \in T$ such that

$$x(t_1) = \sup_{t\in T} x(t), \quad x(t_2) = \inf_{t\in T} x(t).$$

Let

$$\sum_{i,j=1}^{n} a_{ij}D_{ij} + \sum_{i=1}^{n} b_iD_i + c$$

be u expressed in local coordinates with $a_{ij} = a_{ji}$ for all $i, j \in \mathbb{N}_n$. Then

$$0 = \sum_{i,j=1}^{n} a_{ij}(t_1)(D_{ij}x)(t_1) - (ue_T)(t_1)x(t_1),$$

$$x(t_1) = \frac{1}{(ue_T)(t_1)} \sum_{i,j=1}^{n} a_{ij}(t_1)(D_{ij}x)(t_1) \leq 0.$$

Similarly,

$$0 = \sum_{i,j=1}^{n} a_{ij}(t_2)(D_{ij}x)(t_2) - (ue_T)(t_2)x(t_2)\,,$$

$$x(t_2) = \frac{1}{(ue_T)(t_2)} \sum_{i,j=1}^{n} a_{ij}(t_2)(D_{ij}x)(t_2) \geq 0\,.$$

It follows that $x = 0$, i.e. $\mathcal{H}(T) = \{0\}$. Since e_T is superharmonic, it is a potential. ■

Lemma 3.2.3.3 (5) *Let x be a positive C^∞-function on T such that ux is strictly negative. Then x vanishes nowhere and there is a strictly positive C^∞-potential p on T such that $up = ux$.*

Take $s_0 \in T$ such that $x(s_0) = 0$. Then u takes the form

$$\sum_{i,j=1}^{n} a_{ij}D_{ij} + \sum_{i=1}^{n} b_i D_i + c$$

in local coordinates, where $a_{ij} = a_{ji}$ for all $i, j \in \mathbf{N}_n$. Thus

$$0 > (ux)(s_0) = \sum_{i,j=1}^{n} a_{ij}(s_0)(D_{ij}x)(s_0) \geq 0$$

which is a contradiciton. Hence x vanishes nowhere.

Let y be the greatest harmonic lower bound for x. Then $p := x - y$ is a C^∞-potential on T such that

$$up = u(x - y) = ux < 0$$

on T. By the above considerations, p vanishes nowhere on T. ■

Lemma 3.2.3.4 (5) *Let p be a potential on T such that*

$$r := -up$$

is a locally λ-integrable function and $x \in \overline{\mathbb{R}}^T$.

a) *If x is p-integrable ([C-C] Exercise 8.2.3) and x and rx are locally λ-integrable, then*

$$u(x \cdot p) = -rx\,.$$

b) *If* $\frac{1}{r}ux$ *is* p*-integrable,* ux *and* $\frac{1}{r}ux$ *are locally* λ*-integrable, and*

$$\mathcal{H}(T) = \{0\},$$

then

$$(\frac{1}{r}ux)\cdot p = -x.$$

a) Let $\mathcal{F}$ denote the set of p-integrable functions y on T such that y and ry are locally λ-integrable and put

$$\mathcal{G} := \{x \in \mathcal{F} \mid u(x\cdot p) = -rx\}.$$

Step 1 $\quad C(T) \subset \mathcal{G}$

Take $x \in C(T)$. Then $x \in \mathcal{F}$ and

$$2\|x\|p = (\|x\|e_T + x)\cdot p + (\|x\|e_T - x)\cdot p.$$

Since $(\|x\|e_T + x)\cdot p$ and $(\|x\|e_T - x)\cdot p$ are potentials, both $-u((\|x\|e_T + x)\cdot p)$ and $-u((\|x\|e_T - x)\cdot p)$ are positive Radon measures on T. Moreover,

$$-u((\|x\|e_T + x)\cdot p) - u((\|x\|e_T - x)\cdot p) = -2\|x\|up = 2\|x\|r$$

and so it follows that there are positive, locally λ-integrable functions x_1, x_2 on T such that

$$-u((\|x\|e_T + x)\cdot p) = x_1, \quad -u((\|x\|e_T - x)\cdot p) = x_2,$$

$$x_1 + x_2 = 2\|x\|r.$$

Thus

$$-u(x\cdot p) = \frac{1}{2}(-u((\|x\|e_T + x)\cdot p) + u((\|x\|e_T - x)\cdot p)) = \frac{1}{2}(x_1 - x_2) =: x_0.$$

Let U be an open set of T and take $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha \le x \le \beta$$

on U. Then $(x - \alpha e_T)\cdot p$, $(\beta e_T - x)\cdot p$ are superharmonic on U and so

$$x_0 - \alpha r = -u(x\cdot p) + \alpha up = u((\alpha e_T - x)\cdot p) \ge 0,$$

$$\beta r - x_0 = -\beta up + u(x \cdot p) = u((x - \beta e_T) \cdot p) \geq 0$$

and

$$\alpha r \leq x_0 \leq \beta r$$

on U. Since U, α, β are arbitrary, $x_0 = xr$. Thus

$$u(x \cdot p) = -rx$$

and so $x \in \mathcal{G}$.

Step 2 If $(x_n)_{n \in \mathbb{N}}$ is a monotone sequence in $\mathcal{F}$ converging pointwise to some $x \in \mathcal{F}$, then $x \in \mathcal{G}$

$(x_n \cdot p)_{n \in \mathbb{N}}$ is a monotone sequence converging pointwise to $x \cdot p$. Thus

$$u(x \cdot p) = \lim_{n \to \infty} u(x_n \cdot p) = \lim_{n \to \infty} (-rx_n) = -rx.$$

Hence $x \in \mathcal{G}$.

Step 3 Every Borel function in $\mathcal{F}$ belongs to $\mathcal{G}$

This follows immediately from the first two steps.

Step 4 Every bounded $x \in \mathcal{F}$, belongs to $\mathcal{G}$

There are bounded Borel functions y, z on T such that

$$y \leq x \leq z \qquad \text{and} \qquad z - y = 0 \quad \lambda\text{-a.e.}$$

By Step 3, $y, z, z - y \in \mathcal{G}$. Hence

$$u((z - y) \cdot p) = -r(z - y) = 0,$$

i.e. $(z - y) \cdot p$ is a positive harmonic function on T. Since it is dominated by the potential $\|z - y\|_\infty p$, it vanishes identically. We deduce successively that

$$y \cdot p \leq x \cdot p \leq z \cdot p = y \cdot p,$$

$$x \cdot p = y \cdot p,$$

$$u(x \cdot p) = u(y \cdot p) = -ry = -rx,$$

and finally

$$x \in \mathcal{G}.$$

Step 5 $\mathcal{F} = \mathcal{G}$

Let x be a positive function of $\mathcal{F}$. By Step 4, $x \wedge ne_T \in \mathcal{G}$, for every $n \in \mathbb{N}$. Hence, by Step 2, $x \in \mathcal{G}$.

If x is an arbitrary function in $\mathcal{F}$, then

$$x = x \vee 0 - (-x) \vee 0 \in \mathcal{G}.$$

b) By a),

$$u((\frac{1}{r}ux)\cdot p + x) = u((\frac{1}{r}ux)\cdot p) + ux = -ux + ux = 0.$$

Hence $(\frac{1}{r}ux)\cdot p + x$ is harmonic on T. By assumption,

$$(\frac{1}{r}ux)\cdot p + x = 0$$

i.e.

$$(\frac{1}{r}ux)\cdot p = -x. \qquad \blacksquare$$

Theorem 3.2.3.5 (5) *Let p be a strictly positive continuous potential on T such that*

$$r := -up$$

is a strictly positive continuous function (sufficient conditions, for the existence of such a p are given in the Lemmas 3.2.3.2 and 3.2.3.3). Let q be a strictly positive continuous function on T such that $\frac{p}{q}$ converges to 0 at the Alexandroff point of T. Given $\alpha \in \mathbb{R}$, define

$$v : C(T) \longrightarrow C(T), \quad x \longmapsto \frac{x\cdot p}{q},$$

$$C_p := \{x \in C(T) \mid \frac{qx}{p} \in C(T)\},$$

$$\mathcal{F}_\alpha := \{x \in C_p \mid u(qx) = -\alpha r x\}$$

and

$$A := \{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\}.$$

a) $u(qvx) = -rx$ *for every* $x \in \mathcal{C}(T)$.

b) v *is injective and compact.*

c) *If* $x \in \mathcal{C}_p$, *then*

$$\left(\frac{1}{r}u(qx) \in \mathcal{C}(T)\right) \Longleftrightarrow (x \in \operatorname{Im} v) \Longrightarrow \left(v(\frac{1}{r}u(qx)) = -x\right).$$

d) $\mathcal{F}_\alpha = \{x \in \mathcal{C}(T) \mid vx = \frac{1}{\alpha}x\}$ *for every* $\alpha \in \mathbb{R}\backslash\{0\}$.

e) A *is a subset of* $]0, \infty[$ *with no points of adherence in* $\mathbb{R}$.

f) *If* x *is a bounded Borel function on* T *vanishing outside a Lebesgue null-set, then*

$$x \cdot p = 0.$$

([C-C] Theorem 8.2.1).

g) *If* T *is compact and* $p = q = e_T$, *then*

$$v\left(\frac{ux}{ue_T}\right) = x$$

for every $\mathcal{C}^\infty$*-function* x *on* T.

a) follows immediately from Lemma 3.2.3.4 a).

b) It follows from Proposition 3.2.3.1, that v is compact. Take $x \in \operatorname{Ker} v$. Then by a),

$$x = -\frac{1}{r}u(qvx) = 0.$$

c) Assume that $\frac{1}{r}u(qx) \in \mathcal{C}(T)$. By a),

$$u\left(q\left(v\left(\frac{1}{r}u(qx)\right) + x\right)\right) = u\left(qv\left(\frac{1}{r}u(qx)\right)\right) + u(qx) = -u(qx) + u(qx) = 0.$$

Hence $q(v(\frac{1}{r}u(qx)) + x)$ is harmonic. We have

$$|qv(\frac{1}{r}u(qx))| \leq \|\frac{1}{r}u(qx)\|p,$$

$$|qx| \leq \|\frac{qx}{p}\|p.$$

Thus $|q(v(\frac{1}{r}u(qx))+x)|$ is dominate by a potential and so it vanishes identically. Thus

$$v(\frac{1}{r}u(qx)) = -x\,,$$

i.e.

$$x \in \operatorname{Im} v\,.$$

If $x \in \operatorname{Im} v$, then $\frac{1}{r}u(qx) \in \mathcal{C}(T)$ by a).

d) Take $x \in \mathcal{F}_\alpha$. By c),

$$vx = -\frac{1}{\alpha}v(\frac{1}{r}u(qx)) = \frac{1}{\alpha}x\,.$$

Now take $x \in \mathcal{C}(T)$ such that

$$vx = \frac{1}{\alpha}x\,.$$

Then $x \in \mathcal{C}_p$ and by a),

$$u(qx) = \alpha u(qvx) = -\alpha r x\,,$$

i.e. $x \in \mathcal{F}_\alpha$.

e) Take $x \in \mathcal{F}_0$. Then by c),

$$x = -v(\frac{1}{r}u(qx)) = 0\,.$$

Hence $0 \notin A$.

Take $\alpha > 0$ and $x \in \mathcal{F}_{-\alpha}$. Define

$$x^+ := x \vee 0 \quad \text{and} \quad x^- := (-x) \vee 0\,.$$

By d)

$$x^+ \cdot p - x^- \cdot p = x \cdot p = qvx = -\frac{q}{\alpha}x = -\frac{q}{\alpha}(x^+ - x^-)\,,$$

so that

$$x^+ \cdot p + \frac{q}{\alpha}x^+ = x^- \cdot p + \frac{q}{\alpha}x^-\,.$$

Since $x^- \cdot p$ is harmonic on $\{x > 0\}$ ([C-C] Propositions 8.1.6 and 8.2.1) and since x^- vanishes on this set, it follows that $x^+ \cdot p + \frac{q}{\alpha}x^+$ is harmonic on $\{x > 0\}$. Since $x^+ \cdot p$ is superharmonic, $-\frac{q}{\alpha}x^+$ is superharmonic on $\{x > 0\}$. Take $\beta \in \mathbb{R}_+$ such that $|qx| \leq \beta p$. Then for every compact set K of T

$$\beta R_p^{T\setminus K} - qx^+ \geq 0$$

on $\{x > 0\}\setminus K$ and

$$\beta R_p^{T\setminus K} - qx^+ \geq 0$$

on the boundary of $\{x > 0\}$. Since $\{x > 0\}$ is an MP-set ([C-C] Corollary 2.3.3),

$$\beta R_p^{T\setminus K} - qx^+ \geq 0\,,$$

$$qx^+ \leq \beta R_p^{T\setminus K}$$

on $\{x > 0\}$. Since K is arbitrary, it follows that $qx^+ = 0$ on $\{x > 0\}$. Hence x^+ vanishes identically. It follows that $x^- = 0$ and $x = 0$. Therefore $-\alpha \notin A$ and A is a subset of $]0, \infty[$.

Since v is compact, it follows from d) and Theorem 3.1.5.1 c) that A has no points of adherence $\mathbb{R}$.

f) Define

$$y := \frac{x \cdot p}{q}\,.$$

Then

$$\frac{qy}{p} = \frac{x \cdot p}{p} \in C_p$$

([C-C] Corollary 8.2.1) and by Lemma 3.2.3.4 a),

$$\frac{1}{r}u(qy) = \frac{1}{r}u(x \cdot p) = -x = 0\,.$$

By c)

$$y = -v(\frac{1}{r}(u(qy)) = 0\,,$$

hence

$$x \cdot p = 0\,.$$

g) follows from Lemma 3.2.3.2 and Lemma 3.2.3.4 b). ■

Corollary 3.2.3.6 (5) *Assume T to be compact and take $\gamma \in \mathbb{R}$ such that*

$$ue_T = \gamma e_T\,.$$

Given $\alpha \in \mathbb{R}$, *define*

$$\mathcal{F}_\alpha := \{x \in \mathcal{C}(T) \mid ux = -\alpha x\}.$$

Put

$$A := \{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\}.$$

Then $A \subset [-\gamma, \infty[$.

Given $\alpha, \beta \in \mathbb{R}$, put

$$u_\alpha := u + \alpha,$$

$$\mathcal{F}_{\alpha,\beta} := \{x \in \mathcal{C}(T) \mid u_\alpha x = -\beta x\},$$

$$A_\alpha := \{\beta \in \mathbb{R} \mid \mathcal{F}_{\alpha,\beta} \neq \{0\}\}.$$

Then

$$\mathcal{F}_{\alpha,\beta} = \mathcal{F}_{\alpha+\beta},$$

$$A_\alpha = A - \alpha$$

for all $\alpha, \beta \in \mathbb{R}$.

Take $\alpha < -\gamma$. Then

$$u_\alpha e_T = ue_T + \alpha e_T = (\alpha + \gamma)e_T < 0.$$

By Theorem 3.2.3.5 e), $A_\alpha \subset]0, \infty[$. It follows that $A \subset]\alpha, \infty[$. Since α is arbitrary, $A \subset [-\gamma, \infty[$. ■

Example 3.2.3.7 (5) *Assume* $\mathbb{K} = \mathbb{C}$. *Take* T *to be* $\mathbf{T}^n$ *with the canonical euclidean metric (equivalently* $T := \mathbb{R}^n/\mathbb{Z}^n$ *) and let* u *be the Laplace operator on* T. *For* $\alpha \in \mathbb{R}$, *define*

$$\mathcal{F}_\alpha := \{x \in \mathcal{C}(T) \mid ux = -\alpha x\}.$$

Put

$$A := \{\alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\}\}$$

and for each $p \in \mathbb{Z}^n$, *define*

$$x_p : T \longrightarrow \mathbb{C}, \quad t \longmapsto e^{2\pi i \sum_{k=1}^{n} p_k t_k}.$$

Then, given $\alpha \in \mathbb{R}$, $\mathcal{F}_\alpha$ *is the vector subspace of* $C(T)$ *generated by*

$$\left\{ x_p \;\middle|\; p \in \mathbb{Z}^n, \quad 4\pi^2 \sum_{k=1}^{n} p_k^2 = \alpha \right\}$$

and

$$A = \left\{ 4\pi^2 \sum_{k=1}^{n} p_k^2 \mid p \in \mathbb{Z}^n \right\}.$$

Take $\alpha \in \mathbb{R}$ and $x \in \mathcal{F}_\alpha$. Since x is a C^∞–function, there is a family $(\alpha_p)_{p \in \mathbb{Z}^n}$ in $\mathbb{C}$ such that

$$x = \sum_{p \in \mathbb{Z}^n} \alpha_p x_p$$

and

$$ux = \sum_{p \in \mathbb{Z}^n} \alpha_p u x_p = -4\pi^2 \sum_{p \in \mathbb{Z}^n} \alpha_p \left(\sum_{k=1}^{n} p_k^2 \right) x_p.$$

Thus

$$\sum_{p \in \mathbb{Z}^n} \alpha \alpha_p x_p = \alpha x = -ux = 4\pi^2 \sum_{p \in \mathbb{Z}^n} \alpha_p \left(\sum_{k=1}^{n} p_k^2 \right) x_p.$$

Hence, given $p \in \mathbb{Z}^n$, α_p vanishes whenever

$$4\pi^2 \sum_{k=1}^{n} p_k^2 \neq \alpha.$$

For each $\alpha \in \mathbb{R}$, $\mathcal{F}_\alpha$ is thus the vector subspace of $C(T)$ generated by

$$\left\{ x_p \;\middle|\; p \in \mathbb{Z}^n, 4\pi^2 \sum_{k=1}^{n} p_k^2 = \alpha \right\}$$

and

$$A = \left\{ 4\pi^2 \sum_{k=1}^{n} p_k^2 \;\middle|\; p \in \mathbb{Z}^n \right\}. \qquad \blacksquare$$

We generalize the above example.

Example 3.2.3.8 (5) *Assume* $\mathbb{K} = \mathbb{C}$. *Let* $(\omega_j)_{j \in \mathbb{N}_n}$ *be an algebraic basis of* $\mathbb{R}^n$,

$$T := \mathbb{R}^n / \left\{ \sum_{j=1}^{n} p_j \omega_j \;\middle|\; p \in \mathbb{Z}^n \right\},$$

φ *the linear map* $\mathbb{R}^n \to \mathbb{R}^n$ *such that*

$$\varphi \omega_j = e_j$$

for every $j \in \mathbb{N}_n$ *and* ψ *the map* $T \to \mathbb{T}^n$ *defined by* φ. *Let* ω *denote the matrix* $[\omega_{jk}]_{j,k}$ *and* ω' *the matrix* ω^{-1}. *Let* u *be the Laplace operator on* T *and for each* $p \in \mathbb{Z}^n$ *and* $\alpha \in \mathbb{R}$ *define*

$$x_p : \mathbb{T}^n \longrightarrow \mathbb{C}, \quad s \longmapsto e^{2\pi i \sum_{k=1}^{n} p_k s_k},$$

$$\mathcal{F}_\alpha := \{ x \in C(T) \mid ux = -\alpha x \}.$$

Finally, let

$$A := \{ \alpha \in \mathbb{R} \mid \mathcal{F}_\alpha \neq \{0\} \}.$$

Then, given $\alpha \in \mathbb{R}$, $\mathcal{F}_\alpha$ *is the vector subspace of* $C(T)$ *generated by*

$$\left\{ x_p \circ \psi \;\middle|\; p \in \mathbb{Z}^n, \quad 4\pi^2 \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 = \alpha \right\}$$

and

$$A := \left\{ 4\pi^2 \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 \;\middle|\; p \in \mathbb{Z}^n \right\}.$$

Let β be the matrix associated to φ. Then

$$\sum_{k=1}^{n} \beta_{ik} \omega_{jk} = (e_j)_i$$

for all $i, j \in \mathbb{N}_n$. Hence

$$\beta = (\omega^*)^{-1} = (\omega^{-1})^* = \omega'^*.$$

Take $p \in \mathbb{Z}^n$. Then for every $j \in \mathbb{N}_n$,

$$\frac{\partial^2(x_p \circ \psi)}{(\partial t_j)^2} = \sum_{k,\ell=1}^{n} \beta_{kj}\beta_{\ell j} \frac{\partial^2 x_p}{\partial s_k \partial s_\ell} \circ \psi =$$

$$= -4\pi^2 \left(\sum_{k,\ell=1}^{n} \omega'_{jk}\omega'_{j\ell} p_k p_\ell \right) x_p \circ \psi = -4\pi^2 \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 x_p \circ \psi ,$$

$$u(x_p \circ \psi) = \sum_{j=1}^{n} \frac{\partial^2(x_p \circ \psi)}{(\partial t_j)^2} = -4\pi^2 \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 x_p \circ \psi .$$

Take $\alpha \in \mathbb{R}$ and $x \in \mathcal{F}_\alpha$. Since x is a $\mathcal{C}^\infty$-function, there is a family $(\alpha_p)_{p\in\mathbb{Z}^n}$ in $\mathbb{C}$ such that

$$x = \sum_{p\in\mathbb{Z}^n} \alpha_p (x_p \circ \psi)$$

and

$$ux = \sum_{p\in\mathbb{Z}^n} \alpha_p u(x_p \circ \psi) = -4\pi^2 \sum_{p\in\mathbb{Z}^n} \alpha_p \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 x_p \circ \psi .$$

Thus

$$\sum_{p\in\mathbb{Z}^n} \alpha\, \alpha_p (x_p \circ \psi) = \alpha x = -ux = 4\pi^2 \sum_{p\in\mathbb{Z}^n} \alpha_p \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 x_p \circ \psi .$$

Hence, given $p \in \mathbb{Z}^n$, α_p vanishes whenever

$$4\pi^2 \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 \neq \alpha .$$

Thus $\mathcal{F}_\alpha$ is the vector subspace of $\mathcal{C}(T)$ generated by

$$\left\{ x_p \circ \psi \;\middle|\; p \in \mathbb{Z}^n ,\quad 4\pi^2 \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 = \alpha \right\}$$

and

$$A := \left\{ 4\pi^2 \sum_{j=1}^{n} \left(\sum_{k=1}^{n} \omega'_{jk} p_k \right)^2 \;\middle|\; p \in \mathbb{Z}^n \right\} . \qquad \blacksquare$$

Exercises

E 3.2.1 Consider the differential equation $x'' = 0$ on $[0,1]$ with the boundary condition

$$x(0) = x(1) = 0\,.$$

Find:

a) The Green function g for this boundary value problem.

b) The eigenvalues and the eigenvectors of the operator

$$C([0,1]) \longrightarrow C([0,1])\,, \quad x \longmapsto gx\,.$$

c) The twice continuously differentiable functions x on $[0,1]$, satisfying

$$x'' = \alpha x\,,$$

where $\alpha \in \mathbb{K}$.

E 3.2.2 Take $\alpha \in \mathbb{K}$, $\frac{\alpha}{2\pi} \notin \mathbb{Z}$. Consider the differential equation $x'' + \alpha x = 0$ on $[0,1]$ with the boundary conditions

$$x(0) = x(1) = 0\,.$$

Find:

a) The Green function g for this boundary value problem.

b) The eigenvalue and the eigenvectors of the operator

$$C([0,1]) \longrightarrow C([0,1])\,, \quad x \longmapsto gx\,.$$

c) The twice continuously differentiable functions x on $[0,1]$ satisfying

$$x'' + \alpha x = \beta x\,,$$

where $\beta \in \mathbb{K}$.

E 3.2.3 Consider the differential equation $x'' - x = 0$ on $[0,1]$ with boundary conditions

$$x(0) = x(1)\,, \quad x'(0) = x'(1)\,.$$

Find:

a) The Green function g for this boundary value problem.

b) The eigenvalue and the eigenvectors of the operator

$$\mathcal{C}([0,1]) \longrightarrow \mathcal{C}([0,1]), \quad x \longmapsto gx.$$

c) The twice continuously differentiable functions x on $[0,1]$ for which

$$x'' - x = \alpha x$$

where $\alpha \in \mathbb{K}$.

E 3.2.4 Let I be a finite set of cardinality n. Let T be the topological space obtained by identifying the points $((1,\iota))_{\iota \in I}$ in $[0,1] \times I$. Let $\mathcal{D}$ be the set of twice continuously differentiable functions $x \in \mathcal{C}(T)$ such that

$$\iota \in I \Longrightarrow x(0,\iota) = 0,$$

$$\sum_{\iota \in I} x'(1,\iota) = 0,$$

and g the function on $T \times T$ defined by

$$g(s,t) := \begin{cases} \frac{st}{n} & \text{if } s,t \text{ do not belong to the same interval} \\ \frac{st}{n} + s(1-t) & \text{if } s,t \text{ belong to the same interval and } s \leq t \\ \frac{st}{n} + t(1-s) & \text{if } s,t \text{ belong to the same interval and } s \geq t. \end{cases}$$

Prove the following:

a) If $x \in \mathcal{C}(T)$, then $gx \in \mathcal{D}$ and $(gx)'' = -x$.

b) If $x \in \mathcal{D}$, then $gx'' = -x$.

c) $\frac{\pi}{2} + \pi\mathbb{Z}$ is the spectrum of the operator

$$\mathcal{C}(T) \longrightarrow \mathcal{C}(T), \quad x \longmapsto gx$$

and for every $\alpha \in \frac{\pi}{2} + \pi\mathbb{Z}$, $\{\beta \sin \alpha t \mid \beta \in \mathbb{K}\}$ is the corresponding eigenspace.

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A^* 2.3.1.1

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$A \backslash B$ NT

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$\mathfrak{F}_I$ 1.1.6.1

$f|S$ NT

$f(a,\cdot)$ NT

$f(\cdot, b)$ NT

$f(A)$ NT

$f(x)$ NT

f^{-1} NT

$\overset{-1}{f}(B)$ NT

$\overset{-1}{f}(y)$ NT

$f : X \to Y$ NT

$f : X \to Y,\ x \mapsto T(x)$ NT

$F[s, t]$ NT

$F[t]$ NT

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$\{f = g\}$ NT

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$\mathbb{R}$ NT

$\overline{\mathbb{R}}$ NT

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Supp μ NT

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$\{x \in X \mid P(x)\}$ NT

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$]\alpha, \beta[\,,]\alpha, \beta]\,, [\alpha, \beta[\,, [\alpha, \beta]$ NT

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$+$ 1.2.4.1

$\times$ NT

$\setminus$ NT

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$\{\cdot | \cdot\}$ NT

$\{\cdot = \cdot\}$, $\{\cdot \neq \cdot\}$, $\{\cdot > \cdot\}$ NT

$\equiv (\mathrm{mod}\, p)$ NT

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$\forall$, $\exists$, $\exists!$ NT

$\circ$ NT, 1.5.2.1

$\oplus$ 1.2.5.3

$[\cdot,\cdot]$, $]\cdot,\cdot[$, $[\cdot,\cdot[$, $]\cdot,\cdot]$ NT